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Semiclassical complex interactions at a non-analytic turning point

By

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Abstract

We continue a dominant WKB solution of the Schrödinger equation in the classically forbidden region to an outgoing WKB solution in the classically allowed region across a simple (multi-dimensional) turning point, without assuming the analyticity for the potential. This report explains briefly the method used in [FLM], where we computed the semiclassical asymptotics of the width of shape resonances for non-globally analytic potentials.

§ 1. Introduction

In this article, we discuss a local connection formula of a WKB solution for the Schrödinger equation in $\mathbb{R}^n$

$$Pu = Eu, \quad P = -h^2\Delta + V(x), \tag{1.1}$$

at a simple turning point of the non-analytic potential $V(x)$. Here h is the semiclassical parameter which tends to 0, and E is a spectral parameter which we will assume to be zero for simplicity although all the arguments also work when it depends on h: $E = E_0 + \mathcal{O}(h)$.

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We assume that $V(x)$ is smooth in a neighborhood Ω of the origin $x = 0$ and that $dV(0) \neq 0$. Then we can choose Euclidean coordinates $x = (x', x_n) = (x_1, \dots, x_{n-1}, x_n)$ such that

$$V(x) = -Cx_n + \mathcal{O}(|x|^2) \text{ as } x \to 0,$$

for some constant $C > 0$.

Let $p(x,\xi) = |\xi|^2 + V(x)$ be the classical Hamiltonian of P and let also $q(x,\xi) = |\xi|^2 - V(x)$. We consider the Hamilton flow $\exp tH_p$ and $\exp tH_q$ passing through the origin $(0,0)$ in the phase space $\mathbb{R}^n_x \times \mathbb{R}^n_\xi$, where H_p is the Hamilton vector field $H_p = \partial_\xi p \cdot \partial_x - \partial_x p \cdot \partial_\xi$. If we write $\exp tH_p(0,0) = (x(t), \xi(t))$, $\exp tH_q(0,0) = (y(t), \eta(t))$, they behave, as $t \to 0$, like

$$\begin{cases} x'(t) = \mathcal{O}(t^4), & \xi'(t) = \mathcal{O}(t^3), \\ x_n(t) = Ct^2 + \mathcal{O}(t^3), & \xi_n(t) = Ct + \mathcal{O}(t^2), \end{cases}$$

$$\begin{cases} y'(t) = \mathcal{O}(t^4), & \eta'(t) = \mathcal{O}(t^3), \\ y_n(t) = -Ct^2 + \mathcal{O}(t^3), & \eta_n(t) = -Ct + \mathcal{O}(t^2), \end{cases}$$

In particular, for small $|t|$, $\exp tH_p(0,0)$ is in the classically allowed region $\{x \in \Omega; V < 0\}$ and $\exp tH_q(0,0)$ is in the classically forbidden region $\{x \in \Omega; V > 0\}$.

Suppose we are given a solution u to (1.1) in Ω. We assume the following two conditions:

(C1) Let $x_0 = \exp t_0 H_q(0,0)$, with small $t_0 < 0$, be a point in Ω in the classically forbidden region. In a neighborhood of x_0, u has an asymptotic expansion of WKB form:

$$u(x,h) \sim e^{-\phi(x)/h} \sum_{k=0}^{\infty} a_k(x) h^k, \quad \phi(0) = 0 \tag{1.2}$$

whose associated Lagrangian manifold

$$\Lambda = \{(x,\xi); \xi = \partial_x \phi(x)\}$$

contains the Hamilton flow $\exp tH_q(0,0)$ for $t_0 < t < 0$.

Remark that, since $\frac{d}{dt}\phi(x(t)) = 2|\xi(t)|^2$, $\phi(x)$ is increasing along the flow $x(t)$, and in particular, $-\phi(x(t)) > 0$ for $t < 0$ i.e. u is exponentially large.

(C2) In the classically allowed region, u is *outgoing* in the sense that the incoming flow is not in the frequency set of u. More precisely, for any small negative t,

$$\exp tH_p(0,0) \notin \mathrm{FS}\,(u).$$

This means by definition that there exists a cut-off function χ identically 1 in a neighborhood of $\exp tH_p(0,0)$ such that the Bargmann transform of χu

$$T(\chi u) = \int e^{i(x-y)\cdot\xi/h-(x-y)^2/2h}\chi(y)u(y;h)dy$$

is $\mathcal{O}(h^\infty)$ as $h \to 0$.

The Lagrangian manifold Λ can be extended to a neighborhood of $(x,\xi) = (0,0)$ since H_q does not vanish there: $H_q|_{(0,0)} = -C\partial_{\xi_n}$. More precisely, there exists a smooth function $g(x', \xi_n)$ such that

$$\Lambda = \{(x,\xi); \xi' = \partial_{x'}g(x',\xi_n), x_n = -\partial_{\xi_n}g(x',\xi_n)\},$$

and there exist smooth functions $b(x') = \mathcal{O}(|x'|^2)$, $\delta(x')$ with $\delta(0) > 0$ and $\xi_n^c(x') = \mathcal{O}(|x'|^2)$ such that

$$\partial_{\xi_n}g(x',\xi_n) = b(x') + \delta(x')(\xi_n - \xi_n^c(x'))^2 + \mathcal{O}((\xi_n - \xi_n^c(x'))^3) \tag{1.3}$$

as $\xi_n \to \xi_n^c(x')$. The phase function $\phi(x)$, as well as the symbols $a_k(x)$, which satisfy the transport equations along the Hamilton flow on Λ, are well defined within the x-space projection of Λ, i.e. $\Omega_- := \{x \in \Omega; x_n + b(x') < 0\}$ and the boundary of this domain $\mathcal{C} := \{x \in \Omega; x_n + b(x') = 0\}$ is called *caustic set.*

(C3) Let $s(x)$ be such that $s(x) = \phi(x)$ where $\phi(x) < 0$ in Ω_-, and $s(x) = 0$ elsewhere in Ω. Then there exists $N_0 \in \mathbb{N}$ such that

$$||e^{s(x)/h}u||_{H^1(\Omega)} = \mathcal{O}(h^{-N_0}),$$

Our problem is

Problem Compute the asymptotic expansion of u in the domain $\Omega_+ := \{x \in \Omega; x_n + b(x') > 0\}$ under the conditions (C1), (C2) and (C3).

This is a localized version of the problem considered for the study of the asymptotic expansion of the width of shape resonances created by a *well in an island*, first in [HeSj] in the analytic case and recently in [FLM] in the C^∞ case.

Following Maslov's idea in [HeSj], we write u in a neighborhood of $x = 0$, as Laplace transform in ξ_n of a WKB solution $e^{-g(x',\xi_n)/h}c(x',\xi_n;h)$ in x', ξ_n:

$$I(x,h) := h^{-1/2}\int_{\gamma(x)} e^{-(x_n\xi_n+g(x',\xi_n))/h}c(x',\xi_n;h)d\xi, \tag{1.4}$$

where the contour $\gamma(x)$ will be suitably chosen to be a steepest descent path depending on x.

It turns out from (1.3) that, for x in Ω_-, the critical momentums ξ_n of the phase $x_n\xi_n + g(x', \xi_n)$ are real. In Ω_+, however, they are imaginary if g is analytic, and they are not defined if g is only C^∞, which is the case in general when the potential is not analytic.

The purpose of this report is to explain the techniques used in [FLM]. Because of the non-analyticity, we can extend u to Ω_+ only up to a distance of order $(h \ln \frac{1}{h})^{2/3}$ from the caustic.

In order to simplify the argument, we suppose that the dimension is 1, and after that we make some remarks for the multi-dimensional case.

§ 2. Connection of WKB solutions

In this and the next sections, we assume that $n = 1$ and $C = 1$, i.e. we consider the equation

$$Pu := -h^2 \frac{d^2u}{dx^2} + V(x)u = 0, \quad V(x) = -x + \mathcal{O}(x^2). \tag{2.1}$$

Notice that the caustic $\mathcal{C}$ is a simple turning point $x = 0$. If $\Omega = (-c, c')$ for some positive small c, c', then $\Omega_- = (-c, 0)$, $\Omega_+ = (0, c')$. The phase function $\phi(x)$ is given by

$$\phi(x) = \int_0^x \sqrt{V(x)}dx, \text{ for } -c < x < 0.$$

Remark that $-\phi(x)$ is the *Agmon distance* $d(x)$ from 0 to x and

$$-\phi(x) = d(x) \sim \frac{2}{3}|x|^{3/2}. \tag{2.2}$$

The principal symbol $a_0(x)$ is known to be constant times $(-x)^{-1/4}$. We take here

$$a_0(x) = \frac{1}{(-x)^{1/4}}.$$

Let us denote

$$k = h \ln \frac{1}{h}.$$

We will show step by step the following theorem:

Theorem 2.1. *There exist a function $\tilde{\phi}(x, h)$ verifying*

$$\tilde{\phi}(x,h) = \phi(x) \qquad \text{in } (-c, 0),$$

$$\left.\begin{array}{l} \operatorname{Re}\tilde{\phi}(x,h) \geq \mathcal{O}(h^\infty) \\ \operatorname{Im}\partial_x\tilde{\phi}(x,h) = -\sqrt{x} + \mathcal{O}(x) + \mathcal{O}(h^\infty) \end{array}\right\} \text{ in } (0, (Nk)^{2/3}),$$

and, for any large N, a smooth function $w_N(x, h) \in C^\infty((-c, (Nk)^{2/3}))$ such that the following properties hold:

(i) *There exist* m_j $(j\in\mathbb{N})$ *and* $\delta>0$*, both independent of* N*, such that*

$$\left(\tfrac{d}{dx}\right)^j w_N(x,h)=\mathcal{O}(h^{-m_j}e^{-\operatorname{Re}\tilde\phi(x,h)/h}),$$

$$Pw_N(x,h)=\mathcal{O}(h^{\delta N}e^{-\operatorname{Re}\tilde\phi(x,h)/h}).$$

(ii) *In* $(-c,0)$*,* w_N *has the same asymptotic expansion as (1.2): for any* $L\in\mathbb{N}$*, one has*

$$w_N(x,h)=e^{-\phi(x)/h}\sum_{l=0}^{L}a_l(x)h^l+\mathcal{O}(h^{L+1}). \tag{2.3}$$

The coefficients $a_l(x)$ *are of the form*

$$a_l(x)=\frac{f_l(\sqrt{-x})}{(-x)^{\frac14+\frac{3l}{2}}}$$

for some functions $f_l(y)$ *smooth in a neighborhood of* $y=0$ *and in particular* $f_0(y)=1$.

(iii) *Let* $x=(Nk)^{2/3}\tilde x$ *and suppose* $0<\tilde x<1$ *is independent of* h*. Then there exists a family of functions* $\{b_{l,m,j}(\tilde x)\}_{l,m,j}$ *smooth in* $\sqrt{\tilde x}$ *in* $[0,1)$ *and in particular* $b_{0,0,0}(\tilde x)=e^{\pi i/4}$ *such that* w_N *has the following asymptotic expansion: for any* $L\in\mathbb{N}$*, there exists* $N\in\mathbb{N}$ *such that*

$$\begin{aligned} w_N(x,h)&=e^{-\tilde\phi(x)/h}\\ &\times\left\{\sum_{l=0}^{L}\sum_{m=0}^{3L}\sum_{j=0}^{[3L\ln\frac1h]}\frac{b_{l,m,j}(\tilde x)}{\tilde x^{\frac32 j+\frac14}}h^l(Nk)^{\frac{m}{3}-\frac16}(N\ln\frac1h)^{-j}+\mathcal{O}(k^{L+1/3})\right\}. \end{aligned}\tag{2.4}$$

(In terms of $\tilde x$ *and* $\kappa=\frac{1}{N\ln\frac1h}$*,* $-\frac{\tilde\phi(x)}{h}=\frac{2i\tilde x^{3/2}}{3\kappa}(1+\mathcal{O}((Nk)^{1/3}))$*.)*

§ 2.1. Airy type integral representation

We look for a solution to (2.1) near the origin $x=0$ in the form of (1.4). Suppose $x<0$. Then $g(\xi)$ is the Legendre transform of $\phi(x)$, i.e.

$$g(\xi)=\sup_x(-x\xi+\phi(x)).$$

By (2.2), the critical point $x_c(\xi)$ satisfies $x_c(\xi)=-\xi^2+\mathcal{O}(\xi^3)$, and the critical value $g(\xi)$ behaves like

$$g(\xi)=\frac13\xi^3+\mathcal{O}(\xi^4)\quad\text{as}\quad\xi\to0.$$

Conversely, $\phi(x)$ is the inverse Legendre transform in the sense:

$$\phi(x) = \inf_{\xi} (x\xi + g(\xi)).$$

Since $g'(\xi) = \xi^2 + \mathcal{O}(\xi^3)$, there exist, for negative small x, two real critical points $\xi_l^+(x), \xi_l^-(x)$ smooth with respect to $\sqrt{-x}$ satisfying

$$\xi_l^{\pm}(x) = \pm\sqrt{-x} + \mathcal{O}(x),$$

and $x\xi + g(\xi)$ takes a local minimum at $\xi = \xi_l^+(x)$ and a local maximum at $\xi = \xi_l^-(x)$ as a function of ξ on the real line. Hence

$$\phi(x) = x\xi_l^+(x) + g(\xi_l^+(x)) \sim -\frac{2}{3}(-x)^{3/2}.$$

Let us define the integral contour $\gamma(x)$ as a real interval containing $\xi_l^+(x)$ inside as the only non degenerate minimum.

By the usual Laplace's method, we have an asymptotic expansion of $I(x,h)$ of the form (2.3):

Lemma 2.2. *For each $x < 0$ close to 0, one has*

$$I(x,h) \sim e^{-\phi(x)/h} \sum_{j=0}^{\infty} \tilde{a}_j(x) h^j, \quad \tilde{a}_j(x) = \frac{f_j(y)}{y^{1/2+3j}}|_{y=\sqrt{-x}},$$

for some functions $f_j(y)$ smooth in a neighborhood of $y = 0$ with $f_0(0) = \sqrt{\pi}$.

The coefficients $\{\tilde{a}_j(x)\}_j$ for negative small x are determined in a bijective way by the coefficients $\{c_k(\xi)\}_k$ of the asymptotic expansion $c(\xi;h) \sim \sum_{k=0}^{\infty} c_k(\xi)h^k$, for ξ near $\xi_l^+(x) \sim \sqrt{-x}$. Thus we can define $\{c_k(\xi)\}_k$ from $\{a_j(x)\}_j$ given in (1.2). These $\{c_k(\xi)\}_k$ are defined near $\xi = \xi_l^+(x) > 0$, and extended to a full real neighborhood of $\xi = 0$ by the transport equations in momentum variable.

§ 2.2. Holomorphic approximation

Definition 2.3. Let $I \subset \mathbb{R}$ be an open interval and f a smooth function on I. We call the family of functions $\{f_\delta\}_{\delta>0}$ *holomorphic δ-approximation* of f on I if each f_δ is holomorphic in $\Gamma_\delta = \{z \in \mathbb{C}; \mathrm{dist}(z, I) < \delta\}$ and

$$\frac{d^k}{dx^k}(f - f_\delta) = \mathcal{O}(\delta^\infty) \text{ as } \delta \to 0 \tag{2.5}$$

uniformly on I for each $k \in \mathbb{N}$.

A holomorphic δ-approximation can be constructed as follows: Let $\tilde{f}$ be an almost analytic extension of f, which can be constructed, for example, by

$$\tilde{f}(x,y)=\sum_{k=0}^{\infty}\frac{(iy)^k}{k!}f^{(k)}(x)\chi\left(\frac{|y|}{\epsilon_k}\right),$$

where ϵ_k tends to 0 rapidly enough and $\chi\in C_0^\infty(\mathbb{R}^+)$ is a cut-off function identically 1 near 0. Then

$$f_\delta(z)=\frac{1}{2i\pi}\int_{\gamma(\delta)}\frac{\tilde{f}(\operatorname{Re}\zeta,\operatorname{Im}\zeta)}{z-\zeta}d\zeta$$

with $\gamma(\delta)=\{\zeta\in\mathbb{C};\operatorname{dist}(\zeta,I)=2\delta\}$ is a holomorphic δ-approximation in Γ_δ. In fact, it is obvoiusly holomorphic in Γ_δ, and we have for all N and k

$$\sup_{x+iy\in\Gamma_\delta}|\frac{d^k}{dx^k}(\tilde{f}-f_\delta)|\leq C(k,N)\delta^N.$$

In particular, this implies (2.5).

Let us define, for any large $N\in\mathbb{N}$,

$$\tilde{I}_N(x,h)=h^{-1/2}\int_{\tilde{\gamma}_N(x,h)}e^{-(x\xi+\tilde{g}_N(\xi,h))/h}\tilde{c}_N(\xi,h)d\xi,$$

where $\tilde{g}_N(\xi,h)$, $\tilde{c}_N(\xi,h)$ are holomorphic $(Nh\ln\frac{1}{h})^{1/3}$-approximation of $g(\xi)$, $c(\xi,h)$ respectively.

Lemma 2.4. *For $x\in(0,(Nk)^{2/3})$, the equation*

$$x+\tilde{g}_N'(\xi;h)=0$$

has two complex roots $\xi_r^+(x;h)$, $\xi_r^-(x;h)$, holomorphic with respect to $\sqrt{x}$ satisfying

$$\xi_r^\pm(x;h)=\pm i\sqrt{x}+\mathcal{O}(x)+\mathcal{O}(h^\infty).$$

Moreover, the critical value at $\xi=\xi_r^-(x;h)$

$$\tilde{\phi}(x;h):=x\xi_r^-(x;h)+\tilde{g}_N(\xi_r^-(x;h))=-i\frac{2}{3}x^{3/2}+\mathcal{O}(x^2)+\mathcal{O}(h^\infty)$$

is a holomorphic function of $\sqrt{x}$ and

$$\operatorname{Re}\tilde{\phi}(x,h)\geq\mathcal{O}(h^\infty). \tag{2.6}$$

Now, as integration contour, we define $\tilde{\gamma}_N(x,h)$ for $x\in(-\epsilon,(Nk)^{2/3})$ (ϵ is a sufficiently small constant) such that

- $\tilde{\gamma}_N(x;h)\subset\{\xi\in\mathbb{C};|\operatorname{Im}\xi|<(Nk)^{1/3}\}$.

- If $x \in (-\epsilon, 0)$, then $\tilde{\gamma}_N(x;h)$ contains ξ_l^+ and there exists $\delta > 0$ such that

$$\operatorname{Re}(x\xi + \tilde{g}) - \phi \geq \delta(|x|^{1/2} + |\xi - \xi_l^+(x)|)|\xi - \xi_l^+|^2 \quad \text{on} \quad \tilde{\gamma}_N,$$
$$|\xi - \xi_l^+| \geq \delta(Nk)^{1/3} \quad \text{at the extremities.}$$

- If $x \in (0, (Nk)^{2/3})$, then $\tilde{\gamma}_N(x;h)$ contains ξ_r^- and there exists $\delta > 0$ such that

$$\operatorname{Re}(x\xi + \tilde{g} - \tilde{\phi}) \geq \delta(|x|^{1/2} + |\xi - \xi_r^-|)|\xi - \xi_r^-|^2 \quad \text{on} \quad \tilde{\gamma}_N, \tag{2.7}$$
$$|\xi - \xi_r^-| \geq \delta(Nk)^{1/3} \quad \text{at the extremities.}$$

We have the following propositions:

Proposition 2.5. *There exists $\delta' > 0$ such that for any $N \in \mathbb{N}$ and $k \in \mathbb{N}$, one has*

$$\frac{d^k}{dx^k}\left(I(x,h) - \tilde{I}_N(x,h)\right) = \mathcal{O}\left(h^{\delta' N - 1/2 - k} e^{-\phi(x)/h}\right)$$

for $x \in (-\epsilon, 0)$.

Proposition 2.6. *There exists $\delta' > 0$ such that for any N, one has*

$$P\tilde{I}_N(x,h) = \mathcal{O}\left(h^{\delta' N} e^{-\operatorname{Re}\tilde{\phi}(x;h)/h}\right)$$

uniformly in $x \in (-\epsilon, (Nk)^{2/3})$.

Thus we obtain our global approximate solution w_N in Theorem 2.1 by connecting $u(x,h)$, $I(x,h)$ and $\tilde{I}_N(x,h)$ with a suitable partition of unity.

§ 2.3. Asymptotic expansion for positive x

Here we compute the asymptotic expansion of our approximate solution $\tilde{I}_N(x,h)$ for positive but small x of order $k^{2/3}$. We do it by the Laplace's method as for $x < 0$, but we should be careful because, as $h \to 0$, x tends to 0 where the critical point degenerates.

Proposition 2.7. *Let $x = (Nk)^{1/3}\tilde{x}$. There exists a family of smooth functions $\{b_{l,m,j}(\tilde{x})\}_{l,m,j}$ in a neighborhood of $\tilde{x} = 0$ with*

$$b_{0,0,0}(\tilde{x}) = e^{\pi i/4}\sqrt{\pi}\tilde{c}_N(0,0),$$

such that for any positive integer L, there exist $N(L) \in \mathbb{N}$ and $\epsilon_L > 0$ such that if $\tilde{x}$ is in an interval (c_1, c_2), $0 < c_1 < c_2 \leq 1$, then

$$\tilde{I}_N(x,h) = e^{-\tilde{\phi}/h}\left\{\sum_{l=0}^{L}\sum_{m=0}^{3L}\sum_{j=0}^{[3L\ln\frac{1}{h}]} \frac{b_{l,m,j}(\tilde{x})}{\tilde{x}^{\frac{3}{2}j+\frac{1}{4}}} h^l (Nk)^{\frac{m}{3}-\frac{1}{6}}\kappa^j + \mathcal{O}(k^{L+\frac{1}{3}})\right\}, \tag{2.8}$$

where $k = h\ln\frac{1}{h}$ and $\kappa = h/(Nk) = 1/(N\ln\frac{1}{h})$.

Remark. The formula $b_{0,0,0}(\tilde{x}) = e^{\pi i/4}$ in Theorem 2.1 (iii) follows from

$$\tilde{c}_N(0,0) = c(0,0) = a(0,0)/\sqrt{\pi} = 1.$$

Proof Set

$$r(x;h) := \frac{1}{2}\tilde{g}''(\xi_r^-(x;h)) = -i\sqrt{x}\left(1 + \mathcal{O}(\sqrt{x}) + \mathcal{O}(h^\infty)\right). \tag{2.9}$$

The Taylor expansion of the phase at $\eta := \xi - \xi_r^-(x) = 0$ is

$$x\xi + \tilde{g}(\xi) = \tilde{\phi}(x) + r(x)\eta^2 + G(x,\eta)\eta^3,$$

where

$$G(x,\eta) = \frac{1}{2}\int_0^1 (1-t)^2 \tilde{g}^{(3)}(\xi_r^-(x) + \eta t)dt$$

is holomorphic both in $\sqrt{x}$ and in η in a disc centered at 0 with radius of order $(Nk)^{1/3}$.

Put

$$r(x)\eta^2 + G(x,\eta)\eta^3 =: r(x)\zeta^2$$

which can be rewritten as

$$\hat{\eta}^2 + G(x, r\hat{\eta})\hat{\eta}^3 = \hat{\zeta}^2$$

for $\hat{\eta} = \eta/r$, $\hat{\zeta} = \zeta/r$. This can be solved with respect to $\hat{\eta}$ and $\hat{\eta}(x,\hat{\zeta})$ is holomorphic in a h-independent neighborhood of $\hat{\zeta} = 0$, $d\hat{\eta}/d\hat{\zeta}|_{\hat{\zeta}=0} = 1$. Since $d\eta/d\zeta = d\hat{\eta}/d\hat{\zeta}$, we get

$$\tilde{I}_N(x;h) = h^{-1/2} r e^{-\tilde{\phi}/h}\int e^{-r^3\hat{\zeta}^2/h} F(x,\hat{\zeta};h)d\hat{\zeta}$$

where

$$F(x,\hat{\zeta};h) := \frac{d\hat{\eta}}{d\hat{\zeta}}\cdot \tilde{c}_N(\xi_r^-(x) + r\hat{\eta}(x,\hat{\zeta});h)$$

is holomorphic in $\hat{\zeta}$ in a h-independent neighborhood of $\hat{\zeta} = 0$, and the integration contour is included in a h-independent neighborhood of $\hat{\zeta} = 0$ and passes through the origin $\hat{\zeta} = 0$ as the steepest descent path.

Set

$$\tilde{r}(\tilde{x}) := (Nk)^{-1/3} r(x) = -i\sqrt{\tilde{x}}(1 + \mathcal{O}((Nk)^{1/3})), \quad t := \tilde{r}^{3/2}\hat{\zeta}.$$

Remark that t is real on the steepest descent path, and that $e^{-\delta^2/\kappa} = h^{\delta^2 N}$. Then we have, for some positive small constants δ and ϵ,

$$\tilde{I}_N(x;h) = \frac{e^{-\tilde{\phi}/h}}{(\kappa r)^{1/2}}\left\{\int_{-\delta}^{\delta} e^{-t^2/\kappa} F(x, \tilde{r}^{-3/2}t; h)dt + \mathcal{O}(h^{\epsilon N})\right\}.$$

The function F has a classical asymptotic expansion in h: for each $L \in \mathbb{N}$,

$$F\left(x,\hat{\zeta};h\right) = \sum_{l=0}^{L} f_l(x,\hat{\zeta})h^l + \mathcal{O}(h^{L+1}),$$

and hence

$$\tilde{I}_N(x;h) = \frac{e^{-\tilde{\phi}/h}}{r^{1/2}} \left\{ \sum_{l=0}^{L} h^l \kappa^{-1/2} \int_{-\delta}^{\delta} e^{-t^2/\kappa} f_l(x,t)dt + \mathcal{O}(h^{\epsilon N}) + \mathcal{O}(h^{L+1}) \right\}.$$

Recall that each f_l is holomorphic in $\hat{\zeta}$. Then Lemma 2.8 below says that there exists a positive constant ϵ_L small enough such that

$$\begin{aligned} & r^{1/2} e^{\tilde{\phi}/h} \tilde{I}_N(x;h) \\ &= \sum_{l=0}^{L} h^l \sum_{j=0}^{[\epsilon_L/\kappa]} \Gamma\left(j+\frac{1}{2}\right) f_{l,2j}(x) \left(\frac{\kappa}{\tilde{r}^3}\right)^j + R_L(x,\kappa) + \mathcal{O}(h^{\epsilon N}) + \mathcal{O}(h^{L+1}), \end{aligned}$$

with

$$|R_L(x,\kappa)| \le \frac{2}{\epsilon_L} e^{-\epsilon_L/\kappa} = \frac{2}{\epsilon_L} h^{\epsilon_L N}, \tag{2.10}$$

uniformly in a neighborhood of $x=0$. Here we used $\sum_{l=0}^{L} h^l \le 2$.

We finish the proof by taking for example $N = 3L/\epsilon_L \ge \frac{L+1}{\min(\epsilon,\epsilon_L/2)}$ and expanding $f_{l,2j}(x)$ and $r(x)$ in Taylor series in $\sqrt{x} = (Nk)^{1/3}\sqrt{\tilde{x}}$. $\square$

Lemma 2.8. *Let $f(t) = \sum_{m=0}^{\infty} f_m t^m$ be a holomorphic function of t at the origin. Then there exists $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$, any $\delta > 0$ and any small enough $\kappa > 0$, one has*

$$\kappa^{-1/2} \int_{-\delta}^{\delta} e^{-t^2/\kappa} f(t)dt = \sum_{j=0}^{[\epsilon/\kappa]} \Gamma\left(j+\frac{1}{2}\right) f_{2j}\kappa^j + R_\epsilon(\kappa),$$

where

$$R_\epsilon(\kappa) \le \frac{1}{\epsilon} e^{-\epsilon/\kappa}.$$

Proof There exists $C > 0$ such that for each M

$$f(t) = \sum_{m=0}^{M} f_m t^m + r_M(t) \tag{2.11}$$

with

$$|f_m| \le C^{m+1}, \quad |r_M(t)| \le C^{M+2}|t|^{M+1}.$$

We first show that

$$I(f;\kappa) := \kappa^{-1/2} \int_{-\delta}^{\delta} e^{-t^2/\kappa} f(t)dt$$

satisfies

$$I(f;\kappa) = \sum_{j=0}^{[M/2]} \Gamma\left(j+\frac{1}{2}\right) f_{2j}\kappa^j + R_M(\kappa)$$

with

$$|R_M(\kappa)| \le 2Ce^{-\delta^2/2\kappa} \sum_{m=0}^{M} (C^2 m\kappa)^{m/2} + C^2\kappa^{1/2}(C^2 M\kappa)^{M/2}, \tag{2.12}$$

and then, taking $M = [1/(eC^2\kappa)]$, in particular, that the error R_M is exponentially small:

$$|R_M(\kappa)| \le \frac{2\sqrt{e}}{\sqrt{e}-1} Ce^{-\delta^2/2\kappa} + \sqrt{e}C^2\kappa^{1/2}e^{-1/(2eC^2\kappa)}. \tag{2.13}$$

By (2.11),

$$\begin{aligned} I(f;\kappa) &= \kappa^{-1/2}\sum_{m=0}^{M} f_m \int_{-\delta}^{\delta} e^{-t^2/\kappa}t^m dt + \kappa^{-1/2}\int_{-\delta}^{\delta} e^{-t^2/\kappa} r_M(t)dt \\ &= \sum_{j=0}^{[M/2]} \Gamma\left(j+\frac{1}{2}\right) f_{2j}\kappa^j + R_M(\kappa) \end{aligned}$$

with

$$R_M(\kappa) = \kappa^{-1/2}\sum_{m=0}^{M} f_m \int_{|t|>\delta} e^{-t^2/\kappa}t^m dt + \kappa^{-1/2}\int_{-\delta}^{\delta} e^{-t^2/\kappa} r_M(t)dt.$$

Since $e^t > t^{m-1}/(m-1)!$, one has

$$\int_{|t|>\delta} e^{-t^2/\kappa}|t|^m dt \le 2\sqrt{(m-1)!}\kappa^{\frac{m+1}{2}}e^{-\delta^2/2\kappa},$$

$$\int_{-\delta}^{\delta} e^{-t^2/\kappa}|t|^{M+1}dt \le \Gamma\left(\frac{M}{2}+1\right)\kappa^{\frac{M}{2}+1}.$$

Hence, one gets

$$|R_M(\kappa)| \le 2e^{-\delta^2/2\kappa}\sum_{m=0}^{M} C^{m+1}\sqrt{(m-1)!}\kappa^{\frac{m}{2}} + C^{M+2}\Gamma\left(\frac{M}{2}+1\right)\kappa^{\frac{M+1}{2}},$$

and we have the estimate (2.12) using $(m-1)! \le m^m$, $\Gamma(M/2+1) \le M^{M/2}$.

If we take $M = [1/(eC^2\kappa)]$, we obtain (2.13) since $C^2 m\kappa \le C^2 M\kappa \le 1/e$, $M > 1/(eC^2\kappa) - 1$. □

§ 3. Accuracy of the approximation

We estimate $v_N := u - w_{CN}$ in $(-(Nk)^{2/3}, (Nk)^{2/3})$ for sufficiently large C. We do this in two steps; first we show, by using the Agmon estimate, that v_N is at most of polynomial order of h^{-1}.

Proposition 3.1. *There exists N_0 such that for all N*

$$||v_N||_{H^1((-(Nk)^{2/3},(Nk)^{2/3}))} = \mathcal{O}(h^{-N_0}).$$

Then we show, using this and the propagation theorem of singularity that

Theorem 3.2. *For any $L \in \mathbb{N}$ large enough, there exists $N \in \mathbb{N}$ such that*

$$||v_N||_{H^1((-(Nk)^{2/3},(Nk)^{2/3}))} = \mathcal{O}(h^L).$$

§ 3.1. A priori estimate

The estimate in Proposition 3.1 is obvious in $(0, (Nk)^{2/3})$ because both u and w_{CN} satisfy it (see (C3), (2.7) and (2.6)). Hence it suffices to estimate v_N in $(-(Nk)^{2/3}, 0)$.

One sees that there exists N_0 such that

$$||e^{\phi(x)/h} v_N||_{H^1(\Omega_-)} = \mathcal{O}(h^{-N_0}),$$

since this holds for u and w_{CN} instead of v. In particular, denoting by $x_{d=d_0}$ the point $x < 0$ such that $d(x) = d_0$ (recall $d(x) = -\phi(x)$ is the Agmon distance),

$$||v_N||_{H^1([x_{d=2k},0])} = \mathcal{O}(h^{-N_0-2}).$$

In order to prove this estimate in $(-(Nk)^{2/3}, x_{d=2k})$, we use the so called *Agmon esitmate*:

Lemma 3.3. *For any $h > 0$, $V \in L^\infty(\mathbb{R}^n)$ real-valued, $E \in \mathbb{R}$, $f \in H^1(\mathbb{R}^n)$, and ψ real-valued and Lipschitz on $\mathbb{R}^n$, one has*

$$\begin{aligned}&\mathrm{Re}\left\langle e^{\psi/h}(-h^2\Delta + V)f, e^{\psi/h} f\right\rangle \\ &= ||h\nabla(e^{\psi/h} f)||^2 + \left\langle (V - |\nabla\psi(x)|^2) e^{\psi/h} f, e^{\psi/h} f\right\rangle.\end{aligned}$$

Let χ_h be a cut-off function with support in $[x_0, x_{d=k}]$, which is identically 1 on $[x_0 + \epsilon, x_{d=2k}]$.

Recall that $d(x) = -\int_0^x \sqrt{V} dx \le C|x|^{3/2}$ for some constant $C > 0$. With this C, we define

$$\psi_N(x) = \min(-d(x) + CNk + kd(x)^{1/3}, (1 - k^{1/3})d(x)).$$

This function ψ_N is non-negative in $(-(Nk)^{2/3}, 0)$ since $d(x) < CNk$. Hence it suffices to show that

$$||e^{\psi_N/h}\chi_h v_N||_{H^1} = \mathcal{O}(h^{-N_0}). \tag{3.1}$$

We apply Lemma 3.3 with $\psi = \psi_N$, $f = \chi_h v_N$. We check the following facts.

First, there exists a positive constant C_1 such that

$$V - |\psi_N'(x)|^2 \geq \frac{k}{C_1}.$$

Next, the support of χ_h' is included in $[x_0, x_0 + \epsilon] \cup [x_{d=2k}, x_{d=k}]$. On $[x_0, x_0 + \epsilon]$, $e^{\phi/h}v_N$ and its derivative are of $\mathcal{O}(h^\infty)$ and hence so is $e^{\psi_N/h}v_N$. On $[x_{d=2k}, x_{d=k}]$, we have $e^{\psi_N/h} = \mathcal{O}(h^{-2})$.

Then we obtain

$$h^2||(e^{\psi_N/h}\chi_h v_N)'||^2 + k||e^{\psi_N/h}\chi_h v_N||^2 = \mathcal{O}(h^\infty + h^{-N_0}||e^{\psi_N/h}\chi_h v_N||)$$

for some N_0. Thus we proved (3.1).

§ 3.2. Propagation of microsupport

The condition (C2) says that, for any h-independent small negative t, the point $(x(t), \xi(t)) = \exp tH_p(0,0)$ does not belong to the frequency set of u. We can show, furthermore, that this is true even for h-dependent small negative $t = t_N := -\delta^{-1}(Nk)^{1/3}$. More precisely, denoting by T_μ the scaled Bargman transform;

$$T_\mu(f) = \int e^{i(x-y)\cdot\xi - \mu(x-y)^2/2h} f(y;h)dy,$$

we have

Lemma 3.4. *There exists $\delta_0 > 0$ such that for any $0 < \delta < \delta_0$, for any N large enough, and for $t_N := -\delta^{-1}(Nk)^{1/3}$, one has*

$$T_\mu(\mathbf{1}_{(0,c')}u) = \mathcal{O}(h^{\delta N}), \quad \mu = (Nk)^{-1/3}, \tag{3.2}$$

uniformly in

$$\mathcal{W}_h(t_N) := \{(x,\xi); |x - x(t_N)| < \delta(Nk)^{2/3}, |\xi - \xi(t_N)| < \delta(Nk)^{1/3}\}.$$

Here, $\mathbf{1}_{(0,c')}$ is the characteristic function of the interval $(0, c')$.

This can be proved by using an improved microlocal exponential estimate with h-dependent weight (see [FLM] §8.2).

The same estimate holds for the WKB solution w_{CN}:

Lemma 3.5. *For any $L \in \mathbb{N}$ large enough, there exists $\delta_L > 0$ such that for any $\delta \in (0, \delta_L]$, for any $N \geq L/\delta_L$ and for $t_N := -\delta^{-1}(Nk)^{1/3}$, one has*

$$T_\mu(\mathbf{1}_{(\epsilon(Nk)^{2/3},(Nk)^{2/3})} w_{CN}) = \mathcal{O}(h^{\delta N} + h^L), \tag{3.3}$$

uniformly in $\mathcal{W}_h(t_N)$.

These two lemmas, combined with Proposition 3.1, imply that the difference $v_N := \chi_N(x)(u - w_{CN})$, cut-off by $\chi_N(x) = \chi_0(x/(Nk)^{2/3})$ where $\chi_0(x) \in C_0^\infty(\mathbb{R})$ is identically equal to 1 in a sufficiently large neighborhood of the origin, satisfies

$$T_\mu(v_N) = \mathcal{O}(h^L) \text{ uniformly in } \mathcal{W}_h(t_N), \tag{3.4}$$

for $N = L/\delta_L$.

Let us introduce a scale change of the variables (x, ξ) as well as the semiclassical parameter h:

$$\begin{cases} \tilde{x} = x/(Nk)^{2/3}, \quad \tilde{\xi} = \xi/(Nk)^{1/3}, \\ \tilde{h} = h/(Nk) = (N \ln \frac{1}{h})^{-1} \quad \text{i.e.} \quad h = e^{-1/(N\tilde{h})}. \end{cases} \tag{3.5}$$

Setting $(x, \xi) = A_N(\tilde{x}, \tilde{\xi})$ and $\tilde{p}(\tilde{x}, \tilde{\xi}) = (Nk)^{-2/3} p \circ A_N(\tilde{x}, \tilde{\xi})$, one sees that for all $\tilde{t} \in \mathbb{R}$, one has

$$\exp \tilde{t} H_{\tilde{p}} = A_N^{-1} \circ (\exp t H_p) \circ A_N. \tag{3.6}$$

with $\tilde{t} = (Nk)^{-1/3} t$.

We define

$$\tilde{v}_N(\tilde{x}) := (Nk)^{1/3} v_N((Nk)^{2/3} \tilde{x}),$$

so that $||v_N||_{L^2(\mathbb{R}_x)} = ||\tilde{v}_N||_{L^2(\mathbb{R}_{\tilde{x}})}$. Then

$$T(\tilde{v}_N)(\tilde{x}, \tilde{\xi}; \tilde{h}) = (Nk)^{-1/3} T_\mu(v_N)(x, \xi; h), \tag{3.7}$$

with $\mu = (Nk)^{-1/3}$, and (3.4) means, for some $0 < \delta_L' < \delta_L$,

$$T(\tilde{v}_N) = \mathcal{O}(e^{-\delta_L'/\tilde{h}}), \tag{3.8}$$

uniformly in $\mathcal{W}_h(t_N)$, which corresponds, in the $(\tilde{x}, \tilde{\xi})$-space, to

$$\tilde{\mathcal{W}}_h := \{(\tilde{x}, \tilde{\xi}); |\tilde{x} - \tilde{x}(-\delta^{-1})| < \delta, |\tilde{\xi} - \tilde{\xi}(-\delta^{-1})| < \delta\},$$

where $(\tilde{x}(\tilde{t}), \tilde{\xi}(\tilde{t})) = \exp \tilde{t} H_{\tilde{p}}(0, 0)$, and we used (3.6).

Recall $V(x) = -x + W(x)$, $W(x) = \mathcal{O}(x^2)$. Let $W_N(x)$ be a holomorphic $C(Nk)^{2/3}$-approximation of W with sufficiently large constant C, and put $V_N(x) = -x + W_N(x)$, $P_N = -h^2 \frac{d^2}{dx^2} + V_N(x)$.

By the above scale change, P_N becomes

$$P_N = (Nk)^{2/3}\tilde{P}_N, \quad \tilde{P}_N = -\tilde{h}^2\frac{d^2}{d\tilde{x}^2} - \tilde{x} + (Nk)^{-2/3}W_N((Nk)^{2/3}\tilde{x}).$$

It follows from Proposition 2.6 and (3.7) that there exists a positive constant ϵ such that

$$T\tilde{P}_N\tilde{v}_N = \mathcal{O}(e^{-\epsilon/\tilde{h}}). \tag{3.9}$$

Now we can apply the usual propagation theorem of *microsupport* (see for example [Ma]): (3.8) implies that the microsupport of $\tilde{v}_N$ is disjoint with the set $\tilde{\mathcal{W}}_h$. Notice that the Hamilton flow of the principal symbol $\tilde{p}_0 := \tilde{\xi}^2 - \tilde{x}$ of $\tilde{P}_N$;

$$\exp\tilde{t}H_{\tilde{p}_0}(0,0) = (\tilde{t}^{\,2},\tilde{t})$$

passes through this set $\tilde{\mathcal{W}}_h$ for small enough h. Under the conditions of Proposition 3.1 and (3.9), the theorem says that the origin $(\tilde{x},\tilde{\xi}) = (0,0)$ does not belong to the microsupport of $\tilde{v}_N$:

Proposition 3.6. *There exists a positive constant δ such that*

$$T\tilde{v}_N = \mathcal{O}(e^{-\delta/\tilde{h}})$$

in $\{(\tilde{x},\tilde{\xi}); |\tilde{x}| < \delta, |\tilde{\xi}| < \delta\}$.

On the other hand, $\tilde{p}_0$ is microlocally elliptic at $(0,\tilde{\xi})$ except at the origin. Thus combining the above fact with this ellipticity, we conclude that $T\tilde{v}_N$ is uniformly exponentially small with respect to $\tilde{h}$ in a neighborhood of $\{0\}\times\mathbb{R}_{\tilde{\xi}}$, and hence $\tilde{v}_N(\tilde{x},h)$ is exponentially small with respect to $\tilde{h}$ in a fixed neighborhood of $\tilde{x} = 0$. Returning to the variable x and paremeter h, this means Theorem 3.2.

§ 4. Remark in the multi-dimensional case

As for the connection of the WKB solution w_N, essentially the same argument as in §1 works in the multi-dimensional case $n \geq 2$, replacing x by $x_n + b(x')$, which defines the caustic $\mathcal{C}$.

As for the accuracy, it is important to see to which distance in x' and ξ' our estimates are valid. Roughly speaking, $|x'|$ and $|\xi'|$ should be of order $(Nk)^{1/3}$ in the classically allowed region, in order that $\xi_n^2 - x_n$ be the "principal term". On the other hand, in a neighborhood of the origin, $|x'|$ should be of order $(Nk)^{1/2}$ since the phase $-\operatorname{Re}\tilde{\phi}(x)$ increases with order (at most) $|x'|^2$ in the directions x'.

The latter reasoning suggests to make a scale change

$$\begin{cases} \tilde{x}_n = x_n/(Nk)^{2/3}, \quad \tilde{\xi}_n = \xi_n/(Nk)^{1/3}, \\ \tilde{x}' = x'/(Nk)^{1/2}, \quad \tilde{\xi}' = \xi'/(Nk)^{1/2}, \end{cases} \tag{4.1}$$

corresponding to (3.5). Then defining

$$\tilde{v}_N(\tilde{x}) = (Nk)^{\frac{n}{4}+\frac{1}{12}} v((Nk)^{1/2}\tilde{x}', (Nk)^{2/3}\tilde{x}_n),$$

one has

$$T\tilde{v}_N = (Nk)^{-(\frac{n}{4}+\frac{1}{12})}\mathbf{T}_N v_N,$$

where $\mathbf{T}_N$ is a partially rescaled Bargmann transform:

$$\mathbf{T}_N v(x,\xi;h) := \int e^{i(x-y)\cdot\xi/h-\{\mu(x'-y')^2+(x_n-y_n)^2\}/2h} v(y)dy, \;\; \mu = (Nk)^{-1/3}.$$

The scale change leads us to an operator $\tilde{P}_N$ whose $\tilde{h}$-semiclassical symbol is given by

$$\tilde{p}_N(\tilde{x},\tilde{\xi}) = \tilde{\xi}_n^2 - \tilde{x}_n + (Nk)^{1/3}(\tilde{\xi}')^2 + (Nk)^{1/3}\mathcal{O}(|\tilde{x}|^2 + (N\ln\frac{1}{h})^{-1}).$$

Notice that, for $|\tilde{x}|$ small enough, it is elliptic only for $(Nk)^{1/6}|\tilde{\xi}'| + |\tilde{\xi}_n| > \delta$ for some positive δ.

Corresponding to (3.4), we can show

$$\mathbf{T}_N v_N = \mathcal{O}(h^{\delta N}) \tag{4.2}$$

in a h-dependent neighborhood of $\exp t_N H_p(0,0)$

$$\begin{aligned} \mathcal{W}_h(t_N) = \{ &|x_n - x_n(t_N)| \le \delta(Nk)^{2/3}, |\xi_n - \xi_n(t_N)| \le \delta(Nk)^{1/3}, \\ &|x' - x'(t_N)| \le \delta(Nk)^{1/3}, \;\; |\xi' - \xi'(t_N)| \le \delta(Nk)^{1/3} \}. \end{aligned}$$

By the above scale change, this can be rewritten as

$$T\tilde{v}_N = \mathcal{O}(e^{-\delta/\tilde{h}}) \tag{4.3}$$

in the tubular domain

$$\begin{aligned} \tilde{\mathcal{W}} = \{ &|\tilde{x}_n - \tilde{x}_n(-\delta^{-1})| \le \delta, \qquad\qquad |\tilde{\xi}_n - \tilde{\xi}_n(-\delta^{-1})| \le \delta, \\ &|\tilde{x}' - \tilde{x}'(-\delta^{-1})| \le \delta(Nk)^{-1/6}, |\tilde{\xi}' - \tilde{\xi}'(-\delta^{-1})| \le \delta(Nk)^{-1/6} \}. \end{aligned}$$

Then by a (modified) analytic propagation theorem of microsupport, we can show, corresponding to Proposition 3.6,

Proposition 4.1. *There exists a positive constant δ such that*

$$T\tilde{v}_N = \mathcal{O}(e^{-\delta/\tilde{h}})$$

in $V(\delta) := \{(\tilde{x}, \tilde{\xi}); |\tilde{x}| < \delta, (Nk)^{1/6}|\tilde{\xi}'| + |\tilde{\xi}_n| < \delta\}$.

In $(\{|\tilde{x}| < \delta'\} \times \mathbb{R}^n_{\tilde{\xi}}) \backslash V(\delta)$, $\tilde{p}_N$ is elliptic for small enough $\delta' > 0$, and we conclude that $\tilde{v}_N$ is exponentially small with respect to $\tilde{h}$ and that v_N is of $\mathcal{O}(h^{\delta N})$ for some positive δ.

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