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On algebraic solutions of $G(3,2)$ and $G(5/2,1,1)$

By

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Abstract

In [8], H. Kimura considered degenerations of 2-dimensional Garnier system, and he obtained eight Hamiltonian systems. These Hamiltonian systems are called "degenerate Garnier systems". The author considered degenerations of each degenerate Garnier system, and obtained eight Hamiltonian systems in [6]. In this article[1], we consider degenerate Garnier systems $G(3,2)$ and $G(5/2,1,1)$ which are defined in [8] and [6], and give all "algebraic" solutions of each system.

§ 1. Introduction

In 19 century, there was an interest in finding new special functions defined by differential equations. (Probably) based on this, H. Kimura considered degenerations of linear differential equation $L(1,1,1,1,1)$ which gives the 2-dimensional Garnier system $G(1,1,1,1,1)$. He obtained eight differential equations

$$L(5), L(4,1), L(3,2), L(3,1,1), L(2,2,1), L(2,1,1,1), L(1,1,1,1,1), L(9/2) \ \cdots \ (\sharp)$$

by degenerations of $L(1,1,1,1,1)$, and defined 2-dimensional degenerate Garnier systems

$$G(5), G(4,1), G(3,2), G(3,1,1), G(2,2,1), G(2,1,1,1), G(1,1,1,1,1), G(9/2) \ \cdots \ (*_1)$$

by using equations ($\sharp$). (See [8].) In [6], the author considered differential equations

$$L(7/2,1),\ L(5/2,2),\ L(3/2,3),\ L(5/2,3/2),\ L(5/2,1,1),$$
$$L(3/2,2,1),\ L(3/2,3/2,1),\ L(3/2,1,1,1)$$

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[1]This article is a survey of [7].

which can be obtained by degeneration of each equation (♯). He considered monodromy preserving deformation of each equation and defined another 2-dimensional degenerate Garnier systems

$$G(7/2,1),\ G(5/2,2),\ G(3/2,3),\ G(5/2,3/2),\ G(5/2,1,1),\ G(3/2,2,1),\ G(3/2,3/2,1),\ G(3/2,1,1,1). \qquad \cdots (*_2)$$

Here $G(r_1, r_2, \cdots, r_m)$ stands for the monodromy preserving deformation equation of $L(r_1, r_2, \cdots, r_m)$, and $L(r_1, r_2, \cdots, r_m)$ stands for a second-order linear differential equation

$$\frac{d^2y}{dx^2} = R(x)\,y \tag{1.1}$$

which satisfies the following conditions:

(a) The equation (1.1) has m regular or irregular singular points at $\xi_1, \xi_2, \cdots, \xi_m$ and apparent singular points at $\lambda_1, \lambda_2, \cdots, \lambda_g$.

(b) The Poincaré rank of $x = \xi_i$ $(i = 1, 2, \cdots, m)$ is $r_i - 1$.

(c) The characteristic exponents at λ_i $(i = 1, 2, \cdots, g)$ are $-1/2$ and $3/2$.

(d) The number g (which is the number of apparent singular points) is given by

$$g = \big[r_1 + (1/2)\big] + \big[r_2 + (1/2)\big] + \cdots + \big[r_m + (1/2)\big] - 3,$$

where the symbol $[x]$ denotes the greatest integer not exceeding x.

In this article, we consider differential equations $G(3,2)$[2] and $G(5/2,1,1)$ which are given by

$$G(3,2)\ :\ \frac{\partial q_i}{\partial s_j} = \frac{\partial H_j}{\partial p_i}, \quad \frac{\partial p_i}{\partial s_j} = -\frac{\partial H_j}{\partial q_i}, \quad (i,j = 1,2), \tag{1.2}$$

$$\begin{aligned} s_1 H_1 &= -q_2\, p_1^2 + q_2^2\, p_2^2 + \frac{1}{2}\,(q_1 q_2 - 2\,s_1)p_1 \\ &\quad + \frac{1}{2}\big\{q_2^2 - 2\,s_1(q_1 - s_2) - 2(\kappa_0 - 1)q_2\big\}p_2 + \frac{\kappa_\infty}{2} q_2, \\ H_2 &= (q_1 - s_2)p_1^2 + 2\,q_2\,p_1 p_2 \\ &\quad - \frac{1}{2}\big\{q_1(q_1 - s_2) - q_2 + 2(\kappa_0 - 1)\big\}p_1 - \frac{1}{2}\,(q_1 q_2 - 2\,s_1)p_2 + \frac{\kappa_\infty}{2}\, q_1, \end{aligned}$$

[2]For convenience sake, we exchange s_1 and s_2 of equation $G(3,2)$ given in [8].

$$(1.3)\quad G(5/2,1,1)\ :\ \frac{\partial q_i}{\partial s_j}=\frac{\partial H_j}{\partial p_i},\quad \frac{\partial p_i}{\partial s_j}=-\frac{\partial H_j}{\partial q_i},\quad (i,j=1,2),$$

$$\begin{aligned} s_1H_1 &= -2\,s_1^2q_2\,p_1^2+4\,s_1^2q_2\,p_1p_2+2(q_1+q_2-s_1^2+1)q_2\,p_2^2\\ &\quad +2(2\kappa_1-1)s_1^2\,p_1+2\big\{(2\kappa_1-1)(q_1+q_2-s_1^2+1)+(2\kappa_0-1)q_2\big\}p_2\\ &\quad +\frac{1}{2}\,s_1^2\,q_1q_2+\frac{1}{2}\,s_1^2(s_1^2+2s_2+1)q_2,\\ H_2 &= 2(q_1+1)p_1^2+4\,q_2\,p_1p_2-2\,q_2\,p_2^2\\ &\quad +4(\kappa_0+\kappa_1-1)p_1-2(2\kappa_1-1)p_2-\frac{1}{2}\,q_1^2-(s_2+1)q_1+\frac{1}{2}s_1^2\,q_2. \end{aligned}$$

These equations have two parameters and a fixed singularity at $s_1=0$ in $\mathbb{C}^2$. (Among the equations $(*_1)$ and $(*_2)$, only $G(3,2)$ and $G(5/2,1,1)$ satisfy this conditions.) We will give all solutions $(q_1(s_1,s_2),q_2(s_1,s_2),p_1(s_1,s_2),p_2(s_1,s_2))$ which have properties

(i) $q_1(\alpha,s_2),q_2(\alpha,s_2),p_1(\alpha,s_2),p_2(\alpha,s_2)$ (α is constant) are rational functions on $\mathbb{C}$,

(ii) $q_1(s_1,\alpha),q_2(s_1,\alpha),p_1(s_1,\alpha),p_2(s_1,\alpha)$ (α is constant) are rational functions on $\mathbb{C}\setminus\{0\}$,

and state that $G(5/2,1,1)$ cannot be transformed into $G(3,2)$ by algebraic transformation.[3]

§ 2. Main theorem

Let (q_i,p_i,H_i,s_i) be a Hamiltonian system

$$\frac{\partial q_i}{\partial s_j}=\frac{\partial H_j}{\partial p_i},\quad \frac{\partial p_i}{\partial s_j}=-\frac{\partial H_j}{\partial q_i}\quad (i,j=1,2)$$

which has two parameters (α_1,α_2). If a birational canonical transformation $(q_i,p_i,H_i,s_i)\to(Q_i,P_i,\bar{H}_i,z_i)$ satisfies the condition

$$\bar{H}_i\ =\ H_i|_{q_1=Q_1,\,q_2=Q_2,\,p_1=P_1,\,p_2=P_2,\,s_1=z_1,\,s_2=z_2,\,\alpha_1=\beta_1,\,\alpha_2=\beta_2}\qquad(i=1,2),$$

then we call this Bäcklund transformation changing (α_1,α_2) into (β_1,β_2). Using formulas written in [4], we have the following propositions.

Proposition 2.1. *There exist Bäcklund transformations of (1.2) changing* (κ_0,κ_∞) *into the following.*

1) $(\kappa_0,\kappa_\infty)\ \to\ (-\kappa_0,\kappa_\infty-\kappa_0)$,

[3] It is known that $G(5/2,1)$ can be transformed into $G(4)$, and $G(3/2,1,1)$ can be transformed into $G(2,2)$. (See [11].) Hence we note that this fact is not trivial.

2) $(\kappa_0, \kappa_\infty) \to (\kappa_0 \pm 2, \kappa_\infty)$,

3) $(\kappa_0, \kappa_\infty) \to (k_0, \kappa_\infty \pm 1)$.

Proposition 2.2. *There exist Bäcklund transformations of (1.3) changing* $(\kappa_0, \kappa_\infty)$ *into the following.*

1) $(\kappa_0, \kappa_1) \to (\kappa_1, \kappa_0)$,

2) $(\kappa_0, \kappa_1) \to (\kappa_0 \pm 1, \kappa_1)$,

3) $(\kappa_0, \kappa_1) \to (\kappa_0, \kappa_1 \pm 1)$.

We give in Appendix explicit formulas of these transformations.

Theorem 2.3. *The following facts hold for* $G(3,2)$.

1) There exists a solution satisfying (i) *and* (ii)*, if and only if* $\kappa_0 \in 2\mathbb{Z}$ *and* $\kappa_\infty + (1/2) \in \mathbb{Z}$.

2) If $\kappa_0 = 0$ *and* $\kappa_\infty = -1/2$*, the equation* $G(3,2)$ *has no solution satisfying* (i) *and* (ii) *except for*

$$
\begin{aligned}
&q_1 = \frac{2s_2}{3}, \quad q_2 = \frac{s_1}{\zeta}, \quad p_1 = \frac{-3\zeta + s_2}{6}, \\
&p_2 = \frac{\zeta(3\zeta^2 - \zeta s_2 - 1)}{6\, s_1}, \quad \left(\zeta = (-s_1/2)^{1/3}\right).
\end{aligned} \tag{2.1}
$$

3) Every solution satisfying (i) *and* (ii) *can be transformed into (2.1) by Bäcklund transformation of Proposition 2.1.*

Theorem 2.4. *The following facts hold for* $G(5/2,1,1)$.

1) There exists a solution satisfying (i) *and* (ii)*, if and only if* $\kappa_0 + (1/4)$, $\kappa_0 - (1/4)$, $\kappa_1 + (1/4)$ *or* $\kappa_1 - (1/4)$ *is integer.*

2) If $\kappa_0 = 1/4$*, the equation* $G(5/2,1,1)$ *has no solution satisfying* (i) *and* (ii) *except for*

$$
q_1 = -s_2 - 1, \quad q_2 = \pm\frac{-2\kappa_1 + 1}{s_1}, \quad p_1 = 0, \quad p_2 = \pm\frac{s_1}{2}. \tag{2.2}
$$

3) Every solution satisfying (i) *and* (ii) *can be transformed into (2.2) by Bäcklund transformation of Proposition 2.2.*

Since (2.2) has a parameter κ_1, we have the following.

Corollary 2.5. *Let* (q_1, q_2, p_1, p_2) *be a general solution of* $G(5/2,1,1)$ *and let* $\bar{G}(3,2)$ *be differential system which is obtained by changing variables with* $q_i = Q_i, p_i = P_1, s_i = z_i$ $(i = 1,2)$ *in* $G(3,2)$. *There is no rational number* m *and no algebraic functions* $\varphi_i(q_1, q_2, p_1, p_2, s_1^m, s_2), \psi_i(q_1, q_2, p_1, p_2, s_1^m, s_2)$ $(i = 1,2)$ *such that*

$$Q_i = \varphi_i(q_1, q_2, p_1, p_2, s_1^m, s_2),\ P_i = \psi_i(q_1, q_2, p_1, p_2, s_1^m, s_2),\ z_1 = s_1^m,\ z_2 = s_2$$

satisfy $\bar{G}(3,2)$.

§ 3. Outline of a proof

We outline a proof of Theorem 2.4. (Theorem 2.3 can be proved in a similar way.)

Proof. Let $(q_1(s_1,s_2), q_2(s_1,s_2), p_1(s_1,s_2), p_2(s_1,s_2))$ be a solution of $G(5/2,1,1)$ and we put

$$\bar{q}_i(s_2) = q_i(\alpha, s_2), \quad \bar{p}_i(s_2) = p_i(\alpha, s_2) \quad (i = 1,2\ ;\ \alpha \text{ is a constant}).$$

Since $(q_1(s_1,s_2), q_2(s_1,s_2), p_1(s_1,s_2), p_2(s_1,s_2))$ satisfies the equations

$$\begin{aligned}
\frac{\partial q_1}{\partial s_2} &= 4(q_1+1)p_1 + 4q_2p_2 + 4(\kappa_0 + \kappa_1 - 1),\\
\frac{\partial q_2}{\partial s_2} &= 4q_2p_1 - 4q_2p_2 - 4\kappa_1 + 2,\\
\frac{\partial p_1}{\partial s_2} &= -2p_1^2 + q_1 + s_2 + 1,\\
\frac{\partial p_2}{\partial s_2} &= -4p_1p_2 + 2p_2^2 - \frac{1}{2}s_1^2,
\end{aligned}$$

$(\bar{q}_1(s_2), \bar{q}_2(s_2), \bar{p}_1(s_2), \bar{p}_2(s_2))$ is a solution of

$$\begin{aligned}
\frac{d\bar{q}_1}{ds_2} &= 4(\bar{q}_1+1)\bar{p}_1 + 4\bar{q}_2\bar{p}_2 + 4(\kappa_0 + \kappa_1 - 1),\\
\frac{d\bar{q}_2}{ds_2} &= 4\bar{q}_2\bar{p}_1 - 4\bar{q}_2\bar{p}_2 - 4\kappa_1 + 2,\\
\frac{d\bar{p}_1}{ds_2} &= -2\bar{p}_1^2 + \bar{q}_1 + s_2 + 1,\\
\frac{d\bar{p}_2}{ds_2} &= -4\bar{p}_1\bar{p}_2 + 2\bar{p}_2^2 - \frac{1}{2}\alpha^2.
\end{aligned} \tag{3.1}$$

We can find all rational solutions of above system in a similar way as in [10]. By using these solutions, we can construct solutions of $G(5/2,1,1)$ satisfying (i) and (ii). (See in detail [7].) □

Remark. It is known that $G(1,1,1,1,1)$, $G(2,1,1,1)$, $G(3,1,1)$, $G(2,2,1)$ and $G(4,1)$ can be regarded as an extension of P_{VI}, P_{V}, P_{IV}, P_{III} and P_{II}, respectively. (See [14].) Also, the equation $G(5/2,1,1)$ can be regareded as an extension of P_{II}. (In fact, substituting $\bar{q}_2 = 0$ into (3.1), we get $\kappa_1 = 1/2$ and

$$\frac{d\bar{q}_1}{ds_2} = 4(\bar{q}_1 + 1)\bar{p}_1 + 2(2\kappa_0 - 1), \tag{3.2}$$

$$\frac{d\bar{p}_1}{ds_2} = -2\bar{p}_1^2 + \bar{q}_1 + s_2 + 1, \tag{3.3}$$

$$\frac{d\bar{p}_2}{ds_2} = -4\bar{p}_1\bar{p}_2 + 2\bar{p}_2^2 - \frac{1}{2}\alpha^2.$$

By changing variables

$$\bar{q}_1 = \frac{2\mu}{c} - 1, \quad \bar{p}_1 = -\frac{c\lambda}{2}, \quad s_2 = -\frac{t}{c}, \quad (c = 2^{2/3}),$$

equations (3.2) and (3.3) are written

$$\frac{d\lambda}{dt} = \frac{\partial H}{\partial \mu}, \quad \frac{d\mu}{dt} = -\frac{\partial H}{\partial \lambda}, \quad \left(H = \frac{\mu^2}{2} - \left(\lambda^2 + \frac{t}{2}\right)\mu + (2\kappa_0 - 1)\lambda\right).$$

This Hamiltonian system is equivalent to the second Painlevé equation P_{II}.) Hence we see that none of the equations $G(1,1,1,1,1)$, $G(2,1,1,1)$, $G(3,1,1)$, $G(2,2,1)$ can be transformed into $G(5/2,1,1)$ by birational transformations. Moreover, using the following three facts, we find that $G(4,1)$ cannot be transformed into $G(5/2,1,1)$ by birational transformations.

1) Every solution of $G(4,1)$:

$$\frac{\partial u_i}{\partial z_j} = \frac{\partial L_j}{\partial v_i}, \quad \frac{\partial v_i}{\partial z_j} = -\frac{\partial L_j}{\partial u_i}, \quad (i,j = 1,2)$$

$$L_1 = v_1^2 - u_2 v_2^2 + (u_2 - u_1^2 - z_1)v_1 + (-u_1 u_2 + z_2 u_2 + \kappa_0)v_2 + \kappa_\infty u_1,$$

$$L_2 = -2u_2 v_1 v_2 - u_2(u_1 + z_2)v_2^2 + (-u_1 u_2 + z_2 u_2 + \kappa_0)v_1 + (-u_2^2 + z_2^2 u_2 + z_1 u_2 + \kappa_0 u_1 + \kappa_0 z_2)v_2 + \kappa_\infty u_2,$$

is meromorphic on $\mathbb{C}^2$.

2) If there exists a rational solution of $G(4,1)$, the parameters κ_0 and κ_∞ must satisfy the following conditions:

- κ_0, κ_∞ are integers,
- $\kappa_\infty \geq 0$, $\kappa_\infty \leq \kappa_0 - 1$ or $\kappa_\infty \leq -1$, $\kappa_\infty \leq \kappa_0$.

3) If κ_0 and κ_∞ satisfy above conditions, there exists a unique rational solution of $G(4,1)$.

(See [9], [5] in detail.)

Remark. It is known that $G(5)$, $G(9/2)$ are holomorphic on $\mathbb{C}^2$ and have no rational solutions. Hence we see that neither of the equations can be transformed into $G(5/2,1,1)$ by birational transformation. Using above Remark, Corollay 2.5 and this fact, we find that none of the equations of $(*_1)$ can be transformed into $G(5/2,1,1)$ by birational transformation.

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§ 4. Appendix

We give Bäcklund transformations of Proposition 2.1 and 2.2.

- **Bäcklund transformation of (1.2) changing** $(\kappa_0, \kappa_\infty)$ **into** $(-\kappa_0, \kappa_\infty - \kappa_0)$

$$Q_1 = q_1, \quad Q_2 = q_2, \quad P_1 = p_1 + \frac{\eta s_1}{q_2}, \quad P_2 = p_2 - \frac{\eta s_1(q_1 - s_2)}{q_2^2} - \frac{\kappa_0}{q_2},$$
$$\bar{H}_1 = -H_1 - \frac{\eta s_2}{2} + \frac{\kappa_0}{s_1} + \frac{\eta(q_1 - s_2)}{q_2}, \quad \bar{H}_2 = H_2 + \frac{\kappa_0 s_2}{2} + \frac{\eta s_1}{2} + \frac{\eta s_1}{q_2},$$
$$\bar{s}_1 = -s_1, \quad \bar{s}_2 = s_2.$$

- **Bäcklund transformation of (1.2) changing** $(\kappa_0, \kappa_\infty)$ **into** $(\kappa_0 + 2, \kappa_\infty + 1)$

$$Q_1 = \frac{q_1 + (2q_2p_1 - q_1q_2 + 2s_1)q_2}{2\,q_2^2\,p_2 + q_2^2 - 2s_1q_1 - 2\kappa_0q_2 + 2s_1s_2} + \frac{q_2(q_2p_1 + s_1)}{q_2^2\,p_2 - s_1q_1 - \kappa_0q_2 + s_1s_2},$$

$$\begin{aligned} Q_2 = -q_2 &+ \frac{2\,q_2^2\,p_1^2 - (q_1q_2 - 4s_1)q_2p_1 - s_1q_1q_2 - (\kappa_0 - \kappa_\infty)q_2^2 + 2s_1^2}{q_2^2\,p_2 - s_1q_1 - \kappa_0q_2 + s_1s_2} \\ &- \frac{2\{2\,q_2^2\,p_1^2 - (q_1q_2 - 4s_1)q_2\,p_1 - s_1q_1q_2 + (\kappa_\infty + 1)q_2^2 + 2s_1^2\}}{2\,q_2^2\,p_2 + q_2^2 - 2s_1q_1 - 2\kappa_0q_2 + 2s_1s_2}, \end{aligned}$$

$$P_1 = (q_2p_1 + s_1)\left(\frac{1}{q_2} + \frac{1}{2}\frac{q_2}{q_2^2\,p_2 - s_1q_1 - \kappa_0q_2 + s_1s_2}\right),$$

$$P_2 = -p_2 - \frac{1}{2} + \frac{\kappa_0}{q_2} + \frac{s_1(q_1 - s_2)}{q_2^2},$$

$$\omega_1 = -\frac{q_1 - s_2}{q_2} - \frac{\kappa_0}{s_1}, \quad \omega_2 = \frac{s_1}{q_2} - \frac{s_2}{2}, \quad \bar{H}_1 = H_1 + \omega_1, \quad \bar{H}_2 = H_2 + \omega_2.$$

• **Bäcklund transformation of (1.2) changing** $(\kappa_0, \kappa_\infty)$ **into** $(\kappa_0 - 2, \kappa_\infty - 1)$

$$Q_1 = q_1 + \frac{2\,p_1 - q_1}{2\,p_2 + 1} + \frac{p_1}{p_2},$$

$$Q_2 = -q_2 - \frac{2(2\,p_1^2 - q_1\,p_1 - \kappa_0 + \kappa_\infty + 1)}{2\,p_2 + 1} + \frac{2\,p_1^2 - q_1\,p_1 + \kappa_\infty}{p_2},$$

$$P_1 = (2\,p_2 + 1)\left\{\frac{p_1}{2\,p_2} - \frac{s_1 p_2}{2\,p_1^2 - 2\,q_2\,p_2^2 - q_1 p_1 - (q_2 - 2\kappa_0 + 2)p_2 + \kappa_\infty}\right\},$$

$$\begin{aligned} P_2 = -p_2 - \frac{1}{2} + &\frac{2\,p_2 + 1}{\{2\,p_1^2 - 2\,q_2\,p_2^2 - q_1\,p_1 - (q_2 - 2\kappa_0 + 2)p_2 + \kappa_\infty\}^2} \\ &\times \Big[\big\{4\,p_1 p_2 + 2(q_1 - s_2)p_2^2 + p_1 - s_2 p_2\big\} s_1 p_2 \\ &\quad + (\kappa_0 - 2)\big\{2\,p_1^2 - 2\,q_2\,p_2^2 - q_1\,p_1 - (q_2 - 2\kappa_0 + 2)p_2 + \kappa_\infty\big\} p_2\Big], \end{aligned}$$

$$\omega_1 = \frac{4\,p_1 p_2 + 2(q_1 - s_2)p_2^2 + p_1 - s_2\,p_2}{2\,p_1^2 - 2\,q_2\,p_2^2 - q_1\,p_1 - (q_2 - 2\kappa_0 + 2)p_2 + \kappa_\infty} + \frac{\kappa_0 - 2}{s_1},$$

$$\omega_2 = -\frac{s_1\,p_2(2\,p_2 + 1)}{2\,p_1^2 - 2\,q_2\,p_2^2 - q_1\,p_1 - (q_2 - 2\kappa_0 + 2)p_2 + \kappa_\infty} + \frac{s_2}{2},$$

$$\bar{H}_1 = H_1 + \omega_1, \quad \bar{H}_2 = H_2 + \omega_2.$$

• **Bäcklund transformation of (1.2) changing** $(\kappa_0, \kappa_\infty)$ **into** $(\kappa_0, \kappa_\infty + 1)$

$$\begin{aligned} Q_1 = &\; 2\,p_1 - q_1 + s_2 \\ &+ \frac{4\,q_2\,p_1 p_2 + 2(q_2 - 2\kappa_\infty + 2)p_1 - 2(q_1 q_2 - 2s_1)p_2 - q_1 q_2 + 2(\kappa_\infty - 1)q_1 + 2s_1}{2(q_1 - s_2)p_1 + 2\,q_2\,p_2 - q_1^2 + s_2 q_1 + q_2 - 2\kappa_0 + 2\kappa_\infty + 2}, \end{aligned}$$

$$Q_2 = \frac{(2\,p_1 - q_1)(2\,q_2\,p_1 - q_1 q_2 + 2s_1)}{2(q_1 - s_2)p_1 + 2q_2\,p_2 - q_1^2 + s_2 q_1 + q_2 - 2\kappa_0 + 2\kappa_\infty + 2},$$

$$P_1 = \frac{1}{2}\left(-q_1 + s_2 - \frac{2\,q_2\,p_2 + q_2 - 2\kappa_0 + 2\kappa_\infty + 2}{2\,p_1 - q_1}\right),$$

$$P_2 = \frac{2\,p_2 + 1}{2} \cdot \frac{2(q_1 - s_2)p_1 + 2\,q_2\,p_2 - q_1^2 + s_2 q_1 + q_2 - 2\kappa_0 + 2\kappa_\infty + 2}{(2\,p_1 - q_1)^2},$$

$$\omega_1 = -\frac{2\,p_2 + 1}{2\,p_1 - q_1} - \frac{(2\,p_1 - q_1)(2\,q_2\,p_1 - q_1 q_2 + 2s_1)}{\big\{2(q_1 - s_2)p_1 + 2\,q_2\,p_2 - q_1^2 + s_2 q_1 + q_2 - 2\kappa_0 + 2\kappa_\infty + 2\big\} s_1},$$

$$\omega_2 = 3\,p_1 - q_1 + \frac{4\,q_2\,p_1 p_2 + 2(q_2 - 2\kappa_\infty + 2)p_1 - 2(q_1 q_2 - 2s_1)p_2 + 2s_1 - q_1 q_2 + 2(\kappa_\infty - 1)q_1}{2(q_1 - s_2)p_1 + 2\,q_2\,p_2 - q_1^2 + s_2 q_1 + q_2 - 2\kappa_0 + 2\kappa_\infty + 2} + \frac{1}{2}\frac{2\,q_2\,p_2 + q_2 - 2\kappa_0 + 2\kappa_\infty + 2}{2\,p_1 - q_1},$$

$$\bar{H}_1 = H_1 + \omega_1, \quad \bar{H}_2 = H_2 + \omega_2.$$

- **Bäcklund transformation of (1.2) changing** $(\kappa_0, \kappa_\infty)$ **into** $(\kappa_0, \kappa_\infty - 1)$

$$Q_1 = -\frac{2(q_1 - s_2)p_1^2 + 4\,q_2\,p_1 p_2 - (s_2 q_1 - s_2^2 + 2\kappa_0)p_1 - (s_2 q_2 - 2s_1)p_2 + \kappa_\infty s_2}{q_1 p_1 + q_2 p_2 - s_2 p_1 - \kappa_\infty},$$

$$Q_2 = -\frac{2(q_2\,p_1 + s_1)p_1}{q_1\,p_1 + q_2\,p_2 - s_2 p_1 - \kappa_\infty},$$

$$P_1 = -\big\{2(q_1 - s_2)p_1^3 + 4\,q_2\,p_1^2 p_2 - (q_1^2 - s_2 q_1 + 2\kappa_0)p_1^2 - (2\,q_1 q_2 - s_2 q_2 - 2s_1)p_1 p_2 - q_2^2\,p_2^2 + \kappa_\infty(2q_1 - s_2)p_1 + 2\kappa_\infty q_2 p_2 - \kappa_\infty^2\big\}/\big\{2(q_1 p_1 + q_2 p_2 - p_1 s_2 - \kappa_\infty)p_1\big\},$$

$$P_2 = -\frac{p_1^2 + (q_1 - s_2)p_1 p_2 + q_2\,p_2^2 - \kappa_\infty p_2}{2p_1^2},$$

$$\omega_1 = \frac{p_2}{p_1}, \quad \omega_2 = -p_1 + \frac{q_1}{2} + \frac{1}{2}\frac{q_2 p_2 - \kappa_\infty}{p_1}, \quad \bar{H}_1 = H_1 + \omega_1, \quad \bar{H}_2 = H_2 + \omega_2.$$

Remark. The Bäcklund transformation of (1.2) changing $(\kappa_0, \kappa_\infty)$ into $(\kappa_0 \pm 2, \kappa_\infty)$ can be obtained from above transformations.

- **Bäcklund transformation of (1.3) changing** (κ_0, κ_1) **into** (κ_1, κ_0)

$$Q_1 = q_1, \quad Q_2 = -q_1 - q_2 - 1, \quad P_1 = p_1 - p_2, \quad P_2 = -p_2,$$

$$\bar{H}_1 = \sqrt{-1}H_1 + \bar{s}_1 H_2 - \frac{\bar{s}_1(\bar{s}_1^2 + 2\bar{s}_2 + 1)}{2}, \quad \bar{H}_2 = H_2 - \frac{\bar{s}_1^2}{2},$$

$$\bar{s}_1 = -\sqrt{-1}s_1, \quad \bar{s}_2 = \frac{s_1^2}{2} + s_2.$$

- **Bäcklund transformation of (1.3) changing** (κ_0, κ_1) **into** $(\kappa_0 + 1, \kappa_1)$

$$Q_1 = q_1 + \frac{16\kappa_0^2 s_1^4}{D_{0+}^2} - \frac{16\kappa_0 s_1^2 p_1}{D_{0+}},$$

$$Q_2 = q_2 + \frac{64\kappa_0^2(q_2 p_2 + 2\kappa_1 - 1)p_2}{D_{0+}^2} + \frac{8\kappa_0(2q_2 p_2 + 2\kappa_1 - 1)}{D_{0+}},$$

$$P_1 = p_1 - \frac{2\kappa_0+1}{Q_1+Q_2+1} - \frac{2\kappa_0 s_1^2}{D_{0+}},$$
$$P_2 = p_2 - \frac{2\kappa_0+1}{Q_1+Q_2+1} - \frac{8\kappa_0 p_2^2}{D_{0+}+8\kappa_0 p_2},$$

$$\bar{H}_1 = H_1 - \frac{2(2\kappa_0+2\kappa_1-1)}{s_1} - \frac{4\kappa_0 s_1(4p_1^2-q_1-2s_2-1)}{D_{0+}}, \quad \bar{H}_2 = H_2 + \frac{4\kappa_0 s_1^2}{D_{0+}},$$

$$\begin{aligned} D_{0+} &= -4s_1^2p_1^2 + 4q_2p_2^2 + 4(2\kappa_1-1)p_2 + s_1^2(q_1+2s_2+1) \\ &= -4s_1^2P_1^2 + 4Q_2P_2^2 + 4(2\kappa_1-1)P_2 + s_1^2(Q_1+2s_2+1) \\ &\quad + \frac{4(2\kappa_0+1)^2(Q_2-s_1^2)}{(Q_1+Q_2+1)^2} - \frac{4(2\kappa_0+1)(2s_1^2P_1-2Q_2P_2-2\kappa_1+1)}{Q_1+Q_2+1}. \end{aligned}$$

- **Bäcklund transformation of (1.3) changing** (κ_0,κ_1) **into** (κ_0-1,κ_1)

$$q_1 = Q_1 + \frac{16(\kappa_0-1)^2 s_1^4}{D_{0-}^2} - \frac{16(\kappa_0-1)s_1^2P_1}{D_{0-}},$$
$$q_2 = Q_2 + \frac{64(\kappa_0-1)^2(Q_2P_2+2\kappa_1-1)P_2}{D_{0-}^2} + \frac{8(\kappa_0-1)(2Q_2P_2+2\kappa_1-1)}{D_{0-}},$$
$$p_1 = P_1 - \frac{2\kappa_0-1}{q_1+q_2+1} - \frac{2(\kappa_0-1)s_1^2}{D_{0-}},$$
$$p_2 = P_2 - \frac{2\kappa_0-1}{q_1+q_2+1} - \frac{8(\kappa_0-1)P_2^2}{D_{0-}+8(\kappa_0-1)P_2},$$

$$\bar{H}_1 = H_1 + \frac{2(2\kappa_0+2\kappa_1-3)}{s_1} + \frac{4(\kappa_0-1)s_1(4P_1^2-Q_1-2s_2-1)}{D_{0-}},$$
$$\bar{H}_2 = H_2 - \frac{4(\kappa_0-1)s_1^2}{D_{0-}},$$

$$\begin{aligned} D_{0-} &= -4s_1^2p_1^2 + 4q_2p_2^2 + 4(2\kappa_1-1)p_2 + s_1^2(q_1+2s_2+1) \\ &\quad + \frac{4(2\kappa_0-1)^2(q_2-s_1^2)}{(q_1+q_2+1)^2} - \frac{4(2\kappa_0-1)(2s_1^2p_1-2q_2p_2-2\kappa_1+1)}{q_1+q_2+1} \\ &= -4s_1^2P_1^2 + 4Q_2P_2^2 + 4(2\kappa_1-1)P_2 + s_1^2(Q_1+2s_2+1). \end{aligned}$$

- **Bäcklund transformation of (1.3) changing** (κ_0,κ_1) **into** (κ_0,κ_1+1)

$$Q_1 = q_1 + \frac{16\kappa_1^2 s_1^4}{D_{1+}^2} + \frac{16\kappa_1 s_1^2(p_1-p_2)}{D_{1+}},$$

$$Q_2 = q_2 + \frac{16\kappa_1^2\{4(q_1+q_2+1)p_2^2 + 4(2\kappa_0-1)p_2 - s_1^4\}}{D_{1+}^2} - \frac{8\kappa_1\{2s_1^2p_1 + 2(q_1+q_2-s_1^2+1)p_2 + 2\kappa_0 - 1\}}{D_{1+}},$$

$$P_1 = p_1 + \frac{8\kappa_1 p_2^2}{D_{1+} - 8\kappa_1 p_2} + \frac{2\kappa_1 s_1^2}{D_{1+}},$$

$$P_2 = p_2 + \frac{8\kappa_1 p_2^2}{D_{1+} - 8\kappa_1 p_2} - \frac{2\kappa_1+1}{Q_2},$$

$$\bar{H}_1 = H_1 - \frac{2(2\kappa_0+2\kappa_1-1)}{s_1} + \frac{4\kappa_1 s_1(4p_1^2 - 8p_1p_2 + 4p_2^2 - q_1 - 2s_1^2 - 2s_2 - 1)}{D_{1+}},$$

$$\bar{H}_2 = H_2 - \frac{4\kappa_1 s_1^2}{D_{1+}},$$

$$\begin{aligned} D_{1+} &= 4s_1^2p_1^2 - 8s_1^2p_1p_2 - 4(q_1+q_2-s_1^2+1)p_2^2 \\ &\quad - 4(2\kappa_0-1)p_2 - s_1^2q_1 - s_1^2(s_1^2+2s_2+1) \\ &= 4s_1^2P_1^2 - 8s_1^2P_1P_2 - 4(Q_1+Q_2-s_1^2+1)P_2^2 - 4(2\kappa_0+4\kappa_1+1)P_2 \\ &\quad - (Q_1+s_1^2+2s_2+1)s_1^2 - \frac{4(2\kappa_1+1)^2(Q_1-s_1^2+1)}{Q_2^2} \\ &\quad - \frac{8(2\kappa_1+1)\{s_1^2P_1 + (Q_1-s_1^2+1)P_2 + \kappa_0+\kappa_1\}}{Q_2}. \end{aligned}$$

- **Bäcklund transformation of (1.3) changing (κ_0, κ_1) into $(\kappa_0, \kappa_1 - 1)$**

$$q_1 = Q_1 + \frac{16(\kappa_1-1)^2 s_1^4}{D_{1-}^2} + \frac{16(\kappa_1-1)s_1^2(P_1-P_2)}{D_{1-}},$$

$$q_2 = Q_2 + \frac{16(\kappa_1-1)^2\{4(Q_1+Q_2+1)P_2^2 + 4(2\kappa_0-1)P_2 - s_1^4\}}{D_{1-}^2} - \frac{8(\kappa_1-1)\{2s_1^2P_1 + 2(Q_1+Q_2-s_1^2+1)P_2 + 2\kappa_0 - 1\}}{D_{1-}},$$

$$p_1 = P_1 + \frac{8(\kappa_1-1)P_2^2}{D_{1-} - 8(\kappa_1-1)P_2} + \frac{2(\kappa_1-1)s_1^2}{D_{1-}},$$

$$p_2 = P_2 + \frac{8(\kappa_1-1)P_2^2}{D_{1-} - 8(\kappa_1-1)P_2} - \frac{2\kappa_1-1}{q_2},$$

$$\begin{aligned}\bar{H}_1 = H_1 &+ \frac{2(2\kappa_0 + 2\kappa_1 - 3)}{s_1} \\ &- \frac{4(\kappa_1 - 1)s_1(4P_1^2 - 8P_1P_2 + 4P_2^2 - Q_1 - 2s_1^2 - 2s_2 - 1)}{D_{1-}},\end{aligned}$$

$$\bar{H}_2 = H_2 + \frac{4(\kappa_1 - 1)s_1^2}{D_{1-}},$$

$$\begin{aligned}D_{1-} &= 4s_1^2P_1^2 - 8s_1^2P_1P_2 - 4(Q_1 + Q_2 - s_1^2 + 1)P_2^2 \\ &\qquad - 4(2\kappa_0 - 1)P_2 - s_1^2Q_1 - s_1^2(s_1^2 + 2s_2 + 1) \\ &= 4s_1^2p_1^2 - 8s_1^2p_1p_2 - 4(q_1 + q_2 - s_1^2 + 1)p_2^2 - 4(2\kappa_0 + 4\kappa_1 - 3)p_2 \\ &\quad - (q_1 + s_1^2 + 2s_2 + 1)s_1^2 - \frac{4(2\kappa_1 - 1)^2(q_1 - s_1^2 + 1)}{q_2^2} \\ &\quad - \frac{8(2\kappa_1 - 1)\{s_1^2p_1 + (q_1 - s_1^2 + 1)p_2 + \kappa_0 + \kappa_1 - 1\}}{q_2}.\end{aligned}$$

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