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Two proofs for the convergence of formal solutions of singular first order nonlinear partial differential equations in complex domain

By

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Abstract

This paper is a summary of the papers which were written by M. Miyake and A. Shirai [7] "*Structure of Formal Solutions of Nonlinear First Order Singular Partial Differential Equations in Complex Domain*, Funkcial. Ekvac., **48**, (2005)" and by A. Shirai [10] "*Alternative Proof for the Convergence of Formal Solutions of Singular First Order Nonlinear Partial Differential Equations*, University Journal of Department of Education, Sugiyama Jogakuen University, **1**, (2008)" .

The purpose of this paper is to have already introduced two known proofs for the convergence of formal solutions of the equation $f(t,x,u,\partial_t u,\partial_x u)=0,\ u(0,x)\equiv 0$ where $(t,x)\in \mathbf{C}_t^d\times\mathbf{C}_x^n$. In [7], M. Miyake and A. Shirai proved the convergence of the formal solution from the viewpoint as evolution equation in t variables. On the other hand, in [10], A. Shirai gave the alternative proof of the result of Miyake and Shirai from the viewpoint that the roles of variables t and x are equivalent.

§ 1. Theorem

Let $\mathbf{C}$ be the set of complex numbers and $(t,x)=(t_1,\dots,t_d,x_1,\dots,x_n)\in\mathbf{C}_t^d\times\mathbf{C}_x^n$ be $(d+n)$-dimensional complex variables. We consider the following first order nonlinear partial differential equations:

$$f(t,x,u,\partial_t u,\partial_x u)=0 \qquad \text{with} \qquad u(0,x)\equiv 0, \tag{1.1}$$

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where $u = u(t,x)$ is an unknown function, $\partial_t u$ denotes $\partial_t u = (\partial_{t_1} u, \dots, \partial_{t_d} u)$ ($\partial_{t_j} = \partial/\partial t_j$) and ∂_x is similar to ∂_t.

Throughout this paper, we assume the following assumptions:

[**A1**] Let $\tau = (\tau_1, \dots, \tau_d) \in \mathbf{C}^d$, $\xi = (\xi_1, \dots, \xi_n) \in \mathbf{C}^n$. Then we assume that the function $f(t,x,u,\tau,\xi)$ is holomorphic in a neighborhood of the origin of $\mathbf{C}^{d+n+1+d+n}$. Moreover, $f(t,x,u,\tau,\xi)$ is an entire function in τ variables for any fixed t, x, u and ξ in the definite domain.

[**A2**] (Singular Equations). The equation (1.1) is *singular* in t variables in the sense that

$$f(0,x,0,\tau,0) \equiv 0 \quad \text{and} \quad \frac{\partial f}{\partial \xi_k}(0,x,0,\tau,0) \equiv 0 \ \ (k = 1,2,\dots,n). \tag{1.2}$$

In order to state our theorem we need to prepare some notations. We denote by $\mathcal{O}_x$ the ring of germs of holomorphic functions or the convergent power series in the variables x at $x = 0$. We denote by $\mathcal{O}_x[[t]]$ the ring of formal power series of t with coefficients in $\mathcal{O}_x$. Moreover, we set $\mathcal{M}_x[[t]] = \{u(t,x) \in \mathcal{O}_x[[t]]\,;\, u(0,x) \equiv 0\}$, that is,

$$u(t,x) \in \mathcal{M}_x[[t]] \ \Leftrightarrow \ u(t,x) = \sum_{|\alpha| \geq 1} u_\alpha(x) t^\alpha, \quad u_\alpha(x) \in \mathcal{O}_x, \tag{1.3}$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_d$ and $t^\alpha = (t_1)^{\alpha_1} \cdots (t_d)^{\alpha_d}$ for $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbf{N}^d$.

[**A3**] (Existence of Formal Solutions). The equation (1.1) has a formal solution $u(t,x) = \sum_{|\alpha|\geq 1} u_\alpha(x) t^\alpha \in \mathcal{M}_x[[t]]$.

Let $\varphi(x) = (\varphi_1(x), \dots, \varphi_d(x)) \in \mathcal{O}_x^d$ be the collection of coefficients of t_j. Then $\varphi_1(x), \dots, \varphi_d(x)$ satisfy the following system of functional equations:

$$\begin{aligned} &\frac{\partial}{\partial t_i} f(t,x,u(t,x),\partial_t u(t,x),\partial_x u(t,x))\Big|_{t=0} \\ &\qquad \equiv \frac{\partial f}{\partial t_i}(0,x,0,\varphi(x),0) + \frac{\partial f}{\partial u}(0,x,0,\varphi(x),0)\varphi_i(x) = 0, \end{aligned} \tag{1.4}$$

for $i = 1,2,\dots,d$.

We set $\boldsymbol{a}(x) = (0,x,0,\varphi(x),0)$ for the simplicity of notation. We define holomorphic functions $a_{ij}(x)$ ($i,j = 1,2,\dots,d$) by

$$a_{ij}(x) = \frac{\partial^2 f}{\partial t_i \partial \tau_j}(\boldsymbol{a}(x)) + \frac{\partial^2 f}{\partial u \partial \tau_j}(\boldsymbol{a}(x))\varphi_i(x). \tag{1.5}$$

Under the above assumptions and notations, our theorem is stated as follows:

Theorem 1.1 (M. Miyake and A. Shirai [7], A. Shirai [10]). *Suppose the assumptions* [A1], [A2] *and* [A3]. *Let* $\{\lambda_1, \ldots, \lambda_d\}$ *be the eigenvalues of the matrix* $(a_{ij}(0))_{i,j=1,\ldots,d}$. *If* $\{\lambda_j\}$ *satisfies the condition* (1.6) *and* (1.7) *below which we call the Poincaré condition and the nonresonance condition respectively, then the formal solution* $u(t,x)$ *of the form* (1.3) *is holomorphic in a neighborhood of the origin:*

$$\mathrm{Ch}(\lambda_1, \ldots, \lambda_d) \not\ni 0 \qquad (\textit{Poincaré condition}) \tag{1.6}$$

where $\mathrm{Ch}(\lambda_1, \ldots, \lambda_d)$ *denotes the convex hull of* $\{\lambda_1, \ldots, \lambda_d\}$.

$$\sum_{j=1}^{d} \lambda_j \alpha_j + \frac{\partial f}{\partial u}(\boldsymbol{a}(0)) \neq 0 \qquad (\textit{Nonresonance condition}) \tag{1.7}$$

for all $\alpha \in \mathbf{N}^d$ *with* $|\alpha| \geq 2$.

§ 2. Related result and Examples

In [7], the following result, which is often called the Maillet type theorem, was also proved:

Theorem 2.1 (Divergent Case (M. Miyake and A. Shirai [7])). *Suppose that* $A(x) = (a_{ij}(x))_{i,j=1,\ldots,d}$ *is a nilpotent matrix, and take an integer* N *with* $1 \leq N \leq d$ *such that* $A(x)^N \equiv \mathbf{0}$, *but* $A(x)^j \not\equiv \mathbf{0}$ *for* $j = 0, 1, \ldots, N-1$, *where* $\mathbf{0}$ *denotes the null matrix. Then if* $f_u(\boldsymbol{a}(0)) \neq 0$, *the formal solution* $u(t,x) \in \mathcal{M}_x[[t]]$ *may diverge in general, and it belongs to the Gevrey class of order at most* $2N$ *in* t *variables, which means that the formal* $2N$*-Gevrey transformation* $\sum_{|\alpha|\geq 1} u_\alpha(x) t^\alpha / |\alpha|!^{2N-1}$ *of* $u(t,x)$ *is convergent in a neighborhood of the origin.*

Example 2.2. Let $(t,x) \in \mathbf{C}^2$ be the variables. We consider the following equation:

$$\{u - a(x)t\}u_t - uu_x = p(x)t^2, \qquad u(0,x) \equiv 0, \tag{2.1}$$

where $u_t = \partial u/\partial t$, $u_x = \partial u/\partial x$ and $a(x), p(x) \in \mathcal{O}_x$ with $a(0) \neq 0$. Let $u(t,x) = \sum_{n=1}^{\infty} u_n(x) t^n$ be a formal solution. Then $u_1(x)$ should satisfy

$$\{u_1(x) - a(x)\}u_1(x) \equiv 0.$$

Therefore, $u_1(x) \equiv 0$ or $u_1(x) = a(x)$, and after a choice of $u_1(x)$ we can see that the formal solution is determined uniquely.

- The case $u_1(x) \equiv 0$. The formal solution $u(t,x)$ satisfies

$$a(x)tu_t = -p(x)t^2 + uu_t - uu_x, \qquad u = O(t^2).$$

Since $a(0) \neq 0$, the Poincaré condition and the nonresonance condition are satisfied. Therefore by Theorem 1.1, the formal solution $u(t,x)$ is convergent in a neighborhood of the origin.

- The case $u_1(x) = a(x)$. Let $u(t,x) = a(x)t + v(t,x)$ $(v = O(t^2))$. Then $v(t,x)$ satisfies

$$a(x)v = (p(x) + a(x)a'(x))t^2 - vv_t + a'(x)tv + a(x)tv_x + vv_x, \qquad v = O(t^2).$$

This equation corresponds to the null matrix case for the matrix $A(x)$, therefore, by Theorem 2.1, the formal solution $v(t,x)$ belongs to a class of Gevrey order $2(= 2\times 1)$ in t variable, if $p(x) + a(x)a'(x) \neq 0$. On the other hand, if $p(x) + a(x)a'(x) = 0$, we have $v(t,x) \equiv 0$.

Example 2.3. Let $(t_1, t_2, x) \in \mathbf{C}^3$ be the variables. We consider the following first order nonlinear partial differential equation:

$$(1 - u_{t_1})u - \sqrt{2}\, t_2 u_{t_1} u_{t_2} + uu_x - a(x)t_1t_2 = 0, \qquad u(0,0,x) \equiv 0 \tag{2.2}$$

where $u = u(t_1, t_2, x)$, $u_{t_j} = \partial u/\partial t_j$ $(j = 1, 2)$, $u_x = \partial u/\partial x$, $a(x) \in \mathcal{O}_x$.

$$\text{Put} \begin{cases} f(t,x,u,\tau,\xi) = (1-\tau_1)u - \sqrt{2}\, t_2\tau_1\tau_2 + u\xi - a(x)t_1t_2 \\ u(t,x) = \varphi_1(x)t_1 + \varphi_2(x)t_2 + v(t,x) \quad (v = O(|t|^2)). \end{cases}$$

Then the coefficients $(\varphi_1(x), \varphi_2(x))$ of the linear part in t in the formal solution satisfy the following system of functional equations:

$$\begin{cases} \dfrac{\partial}{\partial t_1} f(t,x,u(t,x),\partial_t u(t,x),\partial_x u(t,x))\Big|_{t=0} = (1-\varphi_1(x))\varphi_1(x) = 0, \\ \dfrac{\partial}{\partial t_2} f(t,x,u(t,x),\partial_t u(t,x),\partial_x u(t,x))\Big|_{t=0} = -\sqrt{2}\,\varphi_1(x)\varphi_2(x) + (1-\varphi_1(x))\varphi_2(x) = 0. \end{cases}$$

The holomorphic solutions of the above system of functional equations are $(\varphi_1(x), \varphi_2(x)) \equiv (1,0), (0,0)$. We remark that after a choice of the pair $(\varphi_1(x), \varphi_2(x))$, the formal solution $v(t,x)$ is determined uniquely.

Next we calculate a matrix $A(x) = \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix}$.

By the formula

$$a_{ij}(x) = \frac{\partial^2 f}{\partial t_i \partial \tau_j}(0,x,0,\varphi(x),0) + \frac{\partial^2 f}{\partial u \partial \tau_j}(0,x,0,\varphi(x),0)\varphi_i(x) \qquad (i,j = 1,2),$$

we have

$$A(x) = \begin{pmatrix} -\varphi_1(x) & 0 \\ -\sqrt{2}\,\varphi_2(x) & -\sqrt{2}\,\varphi_1(x) \end{pmatrix}.$$

- The case $(\varphi_1(x), \varphi_2(x)) \equiv (1, 0)$. $A(x) \equiv A(0) = \begin{pmatrix} -1 & 0 \\ 0 & -\sqrt{2} \end{pmatrix}$. Since $\lambda_1 = -1$, $\lambda_2 = -\sqrt{2}$, the Poincaré condition is satisfied. Moreover, by $f_u(\boldsymbol{a}(0)) = 1 - \varphi_1(0) = 0$, the nonresonance condition is also satisfied. Therefore, the formal solution $v(t, x)$ is uniquely detemined and it is convergent in a neighborhood of the origin by Theorem 1.1.
- The case $(\varphi_1(x), \varphi_2(x)) \equiv (0, 0)$. $A(x) \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $f_u(\boldsymbol{a}(0)) = 1 - \varphi_1(0) = 1 \neq 0$. Therefore, the formal solution $v(t, x)$ is determined uniquely, and it belongs to the Gevrey class of order at most $2N = 2$ by Theorem 2.1.

§ 3. Sketch of the Proof of Theorem 1.1.

In this section, we shall give two proofs of Theorem 1.1. Firstly, in the subsection 3.1, we shall give a reduction of the equation (1.1).

Secondly, in the subsection 3.2, we shall introduce the sketch of the proof in [7], which is from the viewpoint as evolution equation in t variables.

Finally, in the subsection 3.3, we shall introduce the sketch of the proof in [10], which is from the viewpoint that the roles of variables t and x are equivalent.

§ 3.1. Reduction of the Equation

We can prove the followig inequality by the Poincaré condition and the nonresonance condition:

"For all $\alpha \in \mathbf{N}^d$ with $|\alpha| \geq 2$, there exists a positive constant $C_0 > 0$ independent of α such that

$$\left| \sum_{j=1}^{d} \lambda_j \alpha_j + \frac{\partial f}{\partial u}(\boldsymbol{a}(0)) \right| \geq C_0 |\alpha| \quad \text{(Nonresonance-Poincaré condition)."} \tag{3.1}$$

We put $v(t, x) = u(t, x) - \sum_{j=1}^{d} \varphi_j(x) t_j \; (= O(|t|^2))$ as a new unknown function. By substituting this into the equation (1.1), we see that $v(t, x)$ satisfies the following singular nonlinear partial differential equation:

$$\left(\sum_{i,j=1}^{d} a_{ij}(x) t_i \partial_{t_j} + \frac{\partial f}{\partial u}(\boldsymbol{a}(x)) \right) v(t, x) = \sum_{|\alpha|=2} d_\alpha(x) t^\alpha + f_3(t, x, v, \partial_t v, \partial_x v) \tag{3.2}$$

with $v(t,x) = O(|t|^2)$, where $d_\alpha(x) \in \mathcal{O}_x$ and $f_3(t,x,v,\tau,\xi)$ is holomorphic in a neighborhood of the origin with the Taylor expansion

$$f_3(t,x,v,\tau,\xi) = \sum_{|\alpha|+2p+|q|+2|r|\geq 3} f_{\alpha pqr}(x) t^\alpha v^p \tau^q \xi^r \tag{3.3}$$

where $(\alpha, p, q, r) \in \mathbf{N}^d \times \mathbf{N} \times \mathbf{N}^d \times \mathbf{N}^n$ and $f_{\alpha pqr}(x) \in \mathcal{O}_x$.

Remark. The equation (3.2) is similar to the one which was studied by Gérard and Tahara in their joint works (cf. [4]). However, their equation is not general than ours. Indeed, they assume that the vector field $\sum_{i,j=1}^d a_{ij}(x) t_i \partial_{t_j}$ on the left hand side is triangular, that is, $a_{ij}(x) \equiv 0$ $(i > j)$. Moreover, they assume that the t derivatives appearing in the nonlinear part f_3 are of the form $\{t_i \partial_{t_j} v\}$ instead of $\partial_t v$.

Next, we take a regular matrix Q such that

$$Q(a_{ij}(0))Q^{-1} = \begin{pmatrix} \lambda_1 & & & \\ \delta_1 & \lambda_2 & & \\ & \ddots & \ddots & \\ & & \delta_{d-1} & \lambda_d \end{pmatrix} \qquad \text{(Jordan canonical form)}.$$

By a linear change of variables $(\tau_1, \dots, \tau_d) = (t_1, \dots, t_d)Q$, the equation (3.2) is reduced to the following:

$$(\Lambda_0 + \Delta - L_1 - L_2)v(t,x) = \sum_{|\alpha|=2} \zeta_\alpha(x) t^\alpha + g_3(t,x,v,\partial_t v, \partial_x v) \tag{3.4}$$

with $v = O(|t|^2)$, where we rewrite the variables τ by t again, and the operators Λ_0, Δ, L_1 and L_2 are given by

$$\begin{cases} \Lambda_0 = \displaystyle\sum_{j=1}^d \lambda_j t_j \partial_{t_j} + \frac{\partial f}{\partial u}(\boldsymbol{a}(0)), & \Delta = \displaystyle\sum_{j=1}^{d-1} \delta_j t_{j+1} \partial_{t_j}, \\ L_1 = \displaystyle\sum_{i,j=1}^d \alpha_{ij}(x) t_i \partial_{t_j}, & L_2 = \dfrac{\partial f}{\partial u}(\boldsymbol{a}(0)) - \dfrac{\partial f}{\partial u}(\boldsymbol{a}(x)) =: \eta(x). \end{cases} \tag{3.5}$$

Moreover, the functions $\alpha_{ij}(x)$ and $\eta(x)$ vanish at $x = 0$, that is, $\alpha_{ij}(x) = O(|x|)$, $\eta(x) = O(|x|)$, $\zeta_\alpha(x) \in \mathcal{O}_x$ and $g_3(t,x,v,\tau,\xi)$ is holomorphic in a neighborhood of the origin with the same Taylor expansion as f_3.

In the following subsections 3.2 and 3.3, we give the convergence of formal solution of the equation (3.4) by the two methods. If we can prove the convergence of formal solution $v(t,x)$ of (3.4), we give the convergence of the formal solution $u(t,x)$ of (1.1), because $u(t,x) = v(t,x) + \sum_{j=1}^d \varphi_j(x) t_j$ and $\varphi(x) = (\varphi_1(x), \dots, \varphi_d(x)) \in \mathcal{O}_x^d$.

§ 3.2. Sketch of the Proof in [7] (Original Proof)

In [7], the proof of convergence of formal solution is done from the viewpoint as evolution equation in t variables. Let $P = \Lambda_0 + \Delta - L_1 - L_2$.

Let

$$\mathcal{O}_x(R)[t]_L = \left\{ \sum_{|\alpha|=L} u_\alpha(x)t^\alpha \,;\, u_\alpha(x) \in \mathcal{O}_x \text{ on } |x| \le R \right\}$$

be the set of homogeneous polynomials of degree L in t variables with holomorphic coefficients.

Here we define a majorant series and a majorant operator. For two formal power series $f(t) = \sum_{|\alpha|\ge 0} f_\alpha t^\alpha \in \mathbf{C}[[t]]$ and $F(t) = \sum_{|\alpha|\ge 0} F_\alpha t^\alpha \in \mathbf{C}[[t]]$, we say that $F(t)$ is a majorant series of $f(t)$, if $|f_\alpha| \le F_\alpha$ hold for all $\alpha \in \mathbf{N}^d$ and we write this relation $f(t) \ll F(t)$.

Next, for two formal power series $f(t,x) = \sum_{|\alpha|\ge 0} f_\alpha(x)t^\alpha \in \mathcal{O}_x(R)[[t]]$ and $F(t) = \sum_{|\alpha|\ge 0} F_\alpha(x)t^\alpha \in \mathcal{O}_x(R)[[t]]$, we say that $F(t,x)$ is a majorant series of $f(t,x)$, if $f_\alpha(x) \ll F_\alpha(x)$ hold for all $\alpha \in \mathbf{N}^d$ and we write this relation $f(t,x) \ll F(t,x)$.

Moreover, for two formal power series $f(t,x)$ and $F(t,x)$ such that $f(t,x) \ll F(t,x)$ and for two operators P_1 and P_2 , we say that P_2 is a majorant operator of P_1 (we write this relation $P_1 \ll P_2$), if $P_1 f(t,x) \ll P_2 F(t,x)$ holds.

Lemma 3.1. (i) *For all $L \ge 2$, the mapping $P : \mathcal{O}_x(R)[t]_L \to \mathcal{O}_x(R)[t]_L$ is invertible for sufficiently small $R > 0$.*

(ii) *For $u(t,x) \in \mathcal{O}_x(R)[t]_L$, we suppose a majorant relation*

$$u(t,x) \ll W(x)(t_1 + \cdots + t_d)^L$$

does hold by a function $W(x)$ with non-negative Taylor coefficients. Then for sufficiently small $R > 0$, there exists a positive constant $F > 0$ independent of L such that

$$P^{-1}u(t,x) \ll \frac{1}{L}\frac{F}{R-X} W(x)(t_1 + \cdots + t_d)^L = (T\partial_T)^{-1}\frac{F}{R-X} W(x)T^L, \tag{3.6}$$

where $T = t_1 + \cdots + t_d$ and $X = x_1 + \cdots + x_n$.

You can find the proof of Lemma 3.1 in [7, Proposition 6.2]. So we omit the proof.

Let $U(t,x) = Pv(t,x)$ be a new unknown function. Then $U(t,x)$ satisfies the following equation:

$$U = \sum_{|\alpha|=2} \zeta_\alpha(x)t^\alpha + g_3(t,x,P^{-1}U, \partial_t P^{-1}U, \partial_x P^{-1}U) \tag{3.7}$$

with $U = O(|t|^2)$.

In order to prove the convergence of formal solution $U(t,x)$, we prepare convergent majorant functions by some positive constants $R > 0$, $A > 0$ and $G_{\alpha pqr} \geq 0$ as follows.

$$\begin{aligned}
\sum_{|\alpha|=2} \zeta_\alpha(x) t^\alpha &\ll \frac{A}{(R-X)^2} T^2 \qquad (T = t_1 + \cdots + t_d,\ X = x_1 + \cdots + x_n), \\
g_3(t,x,u,\tau,\xi) &\ll \sum_{|\alpha|+2p+|q|+2|r|\geq 3} \frac{G_{\alpha pqr}}{(R-X)^{|\alpha|+p+|q|+|r|}} T^{|\alpha|} u^p \tau^q \xi^r \\
&=: G_3(T,X,u,\tau,\xi).
\end{aligned}$$

We consider the following equation:

$$W(T,X) = \frac{A}{(R-X)^2} T^2 + G_3\left(T, X, \frac{F}{R-X} W, \left\{\frac{F}{R-X}\frac{W}{T}\right\}_{j=1}^{d}, \left\{\partial_X (T\partial_T)^{-1} \frac{F}{R-X} W\right\}_{k=1}^{n}\right) \tag{3.8}$$

with $W = O(T^2)$. We put a formal solution $W(T,X)$ by $W(T,X) = \sum_{K\geq 2} W_K(X) T^K$. By substituting $W(T,X)$ into (3.8), we have the following majorant relations: $W_2(X) = A/(R-X)^2$ and for $K \geq 3$,

$$\begin{aligned}
W_K(X) = \sum_{|\alpha|+2p+|q|+2|r|\geq 3} &\Bigg\{ \frac{G_{\alpha pqr}}{(R-X)^{|\alpha|+p+|q|+|r|}} \\
&\times {\sum}^{*} \prod_{\ell=1}^{p} \frac{FW_{K_\ell}(X)}{R-X} \prod_{j=1}^{d}\prod_{\ell=1}^{q_j} \frac{FW_{L_{j\ell}}(X)}{R-X} \prod_{j=1}^{n}\prod_{\ell=1}^{r_j} \partial_X \frac{FW_{M_{j\ell}}(X)}{M_{j\ell}(R-X)} \Bigg\},
\end{aligned}$$

where $\sum^*$ is a summation which is taken over

$$|\alpha| + \sum_{\ell=1}^{p} K_\ell + \sum_{j=1}^{d}\sum_{\ell=1}^{q_j} (L_{j\ell} - 1) + \sum_{j=1}^{d}\sum_{\ell=1}^{r_j} M_{j\ell} = K.$$

By the above recurrence formulas, we can see that the formal solution $W(T,X)$ of (3.8) exists uniquely, and $W(T,X)$ satisfies $U(t,x) \ll W(T,X)$, because the functions are replaced by majorant functions and $P^{-1} \ll (T\partial_T)^{-1} \frac{F}{R-X} \ll \frac{F}{R-X}$ hold by Lemma 3.1.

The following lemma is obtained by estimating the powers of $1/(R-X)$ in the above recurrence formulas.

Lemma 3.2. *The coefficients $\{W_K(X)\}_{K\geq 2}$ are given by*

$$W_K(X) = \sum_{\ell=2}^{10K-18} \frac{W_{K\ell}}{(R-X)^\ell}, \quad \text{by some } W_{K\ell} \geq 0. \tag{3.9}$$

We omit the details of the proof of Lemma 3.2. (see [7, Lemma 6.6]).

By using Lemma 3.2, we have

$$\partial_X (T\partial_T)^{-1}\frac{F}{R-X}W(T,X) = \sum_{K\geq 2}\sum_{\ell=2}^{10K-18}\frac{\ell+1}{K}\frac{FW_{K\ell}}{(R-X)^{\ell+2}}T^K \ll \frac{10F}{(R-X)^2}W(T,X). \tag{3.10}$$

Now we consider the following equation:

$$V(T,X) = \frac{A}{(R-X)^2}T^2 + G_3\left(T, X, \frac{F}{R-X}V, \left\{\frac{F}{R-X}\frac{V}{T}\right\}_{j=1}^{d}, \left\{\frac{10F}{(R-X)^2}V\right\}_{k=1}^{n}\right) \tag{3.11}$$

with $V = O(T^2)$. By setting $V(T,X) = T^2\hat{V}(T,X)$, the above equation is reduced to that for $\hat{V}$ for which the existence of unique formal solution $\hat{V}$ which is convergent follows from the classical implicit function theorem. The above considerations show that $U(t,x) \ll W(T,X) \ll V(T,X)$ which proves the convergence of $U(t,x)$.

This implies that

$$v(t,x) = P^{-1}U(t,x) \ll (T\partial_T)^{-1}\frac{F}{R-X}W(T,X) \ll \frac{F}{R-X}W(T,X) \ll \frac{F}{R-X}V(T,X) \in \mathbf{C}\{t,x\},$$

which proves the convergence of $v(t,x) = u(t,x) - \sum_{j=1}^{d}\varphi_j(x)t_j$.

§ 3.3. Sketch of the Proof in [10] (Alternative Proof)

The equation (3.4) is rewritten by

$$(\Lambda_0 + \Delta)v(t,x) = \sum_{i,j=1}^{d}\alpha_{ij}(x)t_i\partial_{t_j}v(t,x) + \eta(x)v(t,x) + \sum_{|\alpha|=2}\zeta_\alpha(x)t^\alpha + g_3(t,x,v,\partial_t v,\partial_x v) \tag{3.12}$$

with $v = O(|t|^2)$ where

$$\Lambda_0 = \sum_{j=1}^{d}\lambda_j t_j\partial_{t_j} + \frac{\partial f}{\partial u}(\boldsymbol{a}(0)), \qquad \Delta = \sum_{j=1}^{d-1}\delta_j t_{j+1}\partial_{t_j}. \tag{3.13}$$

We recall that the functions $\alpha_{ij}(x)$ and $\eta(x)$ vanish at $x=0$, that is, $\alpha_{ij}(x) = O(|x|)$, $\eta(x) = O(|x|)$, $\zeta_\alpha(x) \in \mathcal{O}_x$ and $g_3(t,x,v,\tau,\xi)$ is holomorphic in a neighborhood of the origin with the same Taylor expansion as f_3.

3.3.1. Existence of Formal solution. Let $\mathbf{C}[t]_L[x]_M$ be the set of quasi-homogeneous polynomials of degree L in t and of degree M in x, that is,

$$\mathbf{C}[t]_L[x]_M = \left\{ v_{LM}(t,x) = \sum_{|\alpha|=L,|\beta|=M} v_{\alpha\beta} t^\alpha x^\beta \,;\, v_{\alpha\beta} \in \mathbf{C} \right\}.$$

We define as $\mathbf{C}[[x]][t]_L$ the set of homogeneous polynomials of degree L in t with coefficients of formal power series in x, that is,

$$\begin{aligned}\mathbf{C}[[x]][t]_L &= \left\{ v_L(t,x) = \sum_{M\ge 0} v_{LM}(t,x)\,;\, v_{LM}(t,x) \in \mathbf{C}[t]_L[x]_M \right\} \\ &= \left\{ v_L(t,x) = \sum_{|\alpha|=L} v_\alpha(x) t^\alpha \,;\, v_\alpha(x) \in \mathbf{C}[[x]] \right\}.\end{aligned}$$

By substituting $v(t,x) = \sum_{L\ge 2} v_L(t,x) \in \mathbf{C}[[t,x]]$ $(v_L(t,x) \in \mathbf{C}[[x]][t]_L)$ into (3.12), we obtain the following recurrence formulas:

$$\begin{aligned}(3.14)\quad (\Lambda_0 + \Delta) v_L(t,x) &= \sum_{i,j=1}^{d} \alpha_{ij}(x) t_i \partial_{t_j} v_L(t,x) + \eta(x) v_L(t,x) \\ &\quad + H_L(t,x,\{v_{L'}\}_{L'<L}, \{\partial_t v_{L''}\}_{L''<L}, \{\partial_x v_{L'''}\}_{L'''<L}),\end{aligned}$$

where H_L denotes a homogeneous polynomial of degree L in t which is determined from the nonlinear part g_3. Especially, if $L=2$, we put $H_2 = \sum_{|\alpha|=2} \zeta_\alpha(x) t^\alpha$.

Next, by subsutituting $v_L(t,x) = \sum_{M\ge 0} v_{LM}(t,x) \in \mathbf{C}[[x]][t]_L$ with $v_{LM}(t,x) \in \mathbf{C}[t]_L[x]_M$ into the recurrence formulas (3.14), we obtain the following recurrence formulas for every L and M:

$$(3.15)\qquad (\Lambda_0 + \Delta) v_{LM}(t,x) = H_{LM}(t,x,\{v_{L'M'}(t,x) \text{ and its derivatives}\}),$$

where H_{LM} is a homogeneous polynomials of degree L in t and of degree M in x which is determined from the right hand side of (3.14). Moreover, L' and M' satisfy

$$L' < L,\ M' \le M \quad \text{or} \quad L' = L,\ M' < M.$$

Here the following lemma plays an important role:

Lemma 3.3. *Let $P_0 = \Lambda_0 + \Delta$. Then we have:*

(i) *For all $L \geq 2$ and $M \geq 0$, the mapping $P_0 : \mathbf{C}[t]_L[x]_M \to \mathbf{C}[t]_L[x]_M$ is invertible.*

(ii) *For $v_{LM}(t,x) \in \mathbf{C}[t]_L[x]_M$, we suppose that a majorant relation*

$$v_{LM}(t,x) \ll W_{LM}T^L X^M \qquad (W_{LM} \geq 0)$$

does hold, then there exists a positive constant $C_1 > 0$ independet of L and M such that

$$\begin{aligned} P_0^{-1} v_{LM}(t,x) &\ll \frac{C_1}{L} W_{LM} T^L X^M \ (= C_1 W_{LM} \times (T\partial_T)^{-1} T^L X^M) \\ &\ll C_1 W_{LM} T^L X^M. \end{aligned} \tag{3.16}$$

Remark. By the above lemma, majorant operators of P_0^{-1} on $\mathbf{C}[[t,x]]$ are obtained by

$$P_0^{-1} \ll C_1 (T\partial_T)^{-1} \ll C_1. \tag{3.17}$$

Moreover, in this lemma, we obtain the majorant relations on the constant coefficients, not functional coefficients. In this situation, it is easier to prove Lemma 3.3 than proving Lemma 3.1.

The proof of Lemma 3.3 is found in [10, Lemma 1], so we omit the proof.

By Lemma 3.3 and the recurrence formula (3.15), $v_{LM}(t,x)$ are uniquely determined by the induction. Therefore, the formal solution exists uniquely.

3.3.2. Convergence of Formal Solution. We put $U(t,x) = P_0 v(t,x)$ as a new unknown function. Then $U(t,x)$ satisfies the following equation:

$$\begin{aligned} U(t,x) = &\sum_{i,j=1}^{d} \alpha_{ij}(x) t_i \partial_{t_j} P_0^{-1} U + \eta(x) P_0^{-1} U \\ &+ \sum_{|\alpha|=2} \zeta_\alpha(x) t^\alpha + g_3(t,x,P_0^{-1}U, \partial_t P_0^{-1}U, \partial_x P_0^{-1}U) \end{aligned} \tag{3.18}$$

with $U(t,x) = O(|t|^2)$.

In order to construct a majorant equation for (3.18), we prepare some notations. For a formal power series $f(x) = \sum f_\beta x^\beta$, we define $|f|(x)$ by $|f|(x) = \sum |f_\beta| x^\beta$, and for $g_3(t,x,u,\tau,\xi) = \sum g_{\alpha pqr}(x) t^\alpha u^p \tau^q \xi^r$, we define $|g_3|(t,x,u,\tau,\xi)$ by

$$|g_3|(t,x,u,\tau,\xi) = \sum |g_{\alpha pqr}|(x) t^\alpha u^p \tau^q \xi^r.$$

We put $\mathbf{T} = (T, \ldots, T) \in \mathbf{C}^d$ and $\mathbf{X} = (X, \ldots, X) \in \mathbf{C}^n$. Then the majorant relations $\alpha_{ij}(x) \ll |\alpha_{ij}|(x) \ll |\alpha_{ij}|(\mathbf{X})$ and

$$g_3(t, x, u, \tau, \xi) \ll |g_3|(t, x, u, \tau, \xi) \ll |g_3|(\mathbf{T}, \mathbf{X}, u, \tau, \xi)$$

hold clearly.

By Lemma 3.3 (ii), if a majorant relation $U(t, x) \ll W(T, X)$ holds, then we have the following majorant relations:

- $\displaystyle\sum_{i,j=1}^{d} \alpha_{ij}(x) t_i \partial_{t_j} P_0^{-1} U \ll C_1 \left(\sum_{i,j=1}^{d} |\alpha_{ij}|(\mathbf{X}) \right) W,$
- $\eta(x) P_0^{-1} U \ll C_1 |\eta|(\mathbf{X}) W,$
- $\displaystyle\sum_{|\alpha|=2} \zeta_\alpha(x) t^\alpha \ll \left(\sum_{|\alpha|=2} |\zeta_\alpha|(\mathbf{X}) \right) T^2,$
- $g_3(t, x, P_0^{-1}U, \partial_t P_0^{-1}U, \partial_x P_0^{-1}U)$
  $\qquad \ll |g_3|(\mathbf{T}, \mathbf{X}, C_1 W, \{C_1 W/T\}, \{C_1 \partial_X (T\partial_T)^{-1} W\}).$

Let us consider the following equation which is a majorant equation of (3.18):

$$\begin{aligned}(3.19)\qquad P(X)W(T, X) = &\left(\sum_{|\alpha|=2} |\zeta_\alpha|(\mathbf{X}) \right) T^2 \\ &+ |g_3| \Big(\mathbf{T}, \mathbf{X}, C_1 W, \{C_1 W/T\}, \{C_1 \partial_X (T\partial_T)^{-1} W\} \Big)\end{aligned}$$

with $W = O(T^2)$ where $P(X)$ is a holomorphic function at $X = 0$ given by

$$P(X) = 1 - C_1 \sum_{i,j=1} |\alpha_{ij}|(\mathbf{X}) - C_1 |\eta|(\mathbf{X}).$$

If (3.19) has a formal solution $W(T, X)$, then we can obtain the majorant relation $U(t, x) \ll W(T, X)$, because the functions are replaced by the majorant functions and the operators are also replaced by the majorant operators.

Here $1/P(X)$ is holomorphic in a neighborhood of $X = 0$, because $P(0) \neq 0$ by $|\alpha_{ij}|(\mathbf{X}) = O(X)$ and $|\eta|(\mathbf{X}) = O(X)$. Therefore, by dividing (3.19) by $P(X)$, the equation (3.19) is reduced to the following:

$$(3.20)\qquad W(T, X) = Z(X) T^2 + G_3(T, X, C_1 W, \{C_1 W/T\}, \{C_1 \partial_X (T\partial_T)^{-1} W\})$$

with $W = O(T^2)$ where $Z(X)$ and $G_3(T, X, u, \tau, \xi)$ are holomorphic functions given by

$$Z(X) = \sum_{|\alpha|=2} \frac{|\zeta_\alpha|(\mathbf{X})}{P(X)}, \qquad G_3(T, X, u, \tau, \xi) = \frac{|g_3|(\mathbf{T}, \mathbf{X}, u, \tau, \xi)}{P(X)}.$$

We put $W(T,X)=\sum_{K\geq 2} W_K(X)T^K$. By substituting this into (3.20), we have the following recurrence formula: $W_2(X)=Z(X)$ and for $K\geq 3$

$$\begin{aligned} W_K(X) = \sum_{|\alpha|+2p+|q|+2|r|\geq 3} &\Bigg\{ \frac{|g_{\alpha pqr}|(\mathbf{X})}{P(X)} \\ &\times {\sum}^{**} \prod_{\ell=1}^{p} C_1 W_{K_\ell}(X) \prod_{j=1}^{d}\prod_{\ell=1}^{q_j} C_1 W_{L_{j\ell}} \prod_{j=1}^{n}\prod_{\ell=1}^{r_j} \partial_X \frac{C_1 W_{M_{j\ell}}}{M_{j\ell}} \Bigg\}, \end{aligned}$$

where $\sum^{**}$ is a summation which is taken over

$$|\alpha| + \sum_{\ell=1}^{p} K_\ell + \sum_{j=1}^{d}\sum_{\ell=1}^{q_j}(L_{j\ell}-1) + \sum_{j=1}^{d}\sum_{\ell=1}^{r_j} M_{j\ell} = K.$$

By this recurrence formula, we can see that the formal solution $W(T,X)$ exists uniquely.

We take majorant functions of $Z(X)$ and $G_3(T,X,u,\tau,\xi)$ by

$$Z(X) \ll \frac{A}{(R-X)^2} =: Q(X), \tag{3.21}$$

$$\begin{aligned} G_3(T,X,u,\tau,\xi) &\ll \sum_{|\alpha|+2p+|q|+2|r|\geq 3} \frac{G_{\alpha pqr}}{(R-X)^{|\alpha|+p+|q|+|r|}} T^{|\alpha|} u^p \tau^q \xi^r \\ &=: R_3(T,X,u,\tau,\xi) \end{aligned} \tag{3.22}$$

where A and $G_{\alpha pqr}$ are non-negative constants and R is a positive constant sufficiently small.

We consider the following equation:

$$V(T,X) = Q(X)T^2 + R_3(T,X,C_1V,\{C_1V/T\},\{C_1\partial_X(T\partial_T)^{-1}V\}) \tag{3.23}$$

with $V=O(T^2)$.

We put $V(T,X)=\sum_{K\geq 2} V_K(X)T^K$. By substituting $V(T,X)$ into (3.23), the coefficients $V_K(X)$ $(K=2,3,\dots)$ satisfy the following recurrence formula: $V_2(X) = A/(R-X)^2$, and for $K\geq 3$

$$\begin{aligned} V_K(X) = \sum_{|\alpha|+2p+|q|+2|r|\geq 3} &\Bigg\{ \frac{G_{\alpha pqr}}{(R-X)^{|\alpha|+p+|q|+|r|}} \\ &\times {\sum}^{**} \prod_{\ell=1}^{p} C_1 V_{K_\ell}(X) \prod_{j=1}^{d}\prod_{\ell=1}^{q_j} C_1 V_{L_{j\ell}} \prod_{j=1}^{n}\prod_{\ell=1}^{r_j} \partial_X \frac{C_1 V_{M_{j\ell}}}{M_{j\ell}} \Bigg\}, \end{aligned}$$

where $\sum^{**}$ is the same summation as the one in the recurrence formula for $\{W_K(X)\}$.

By the construction of majorant equation (3.23) and the recurrence formula, the formal solution $V(T,X)$ exists uniquely, and it satisfies the majorant relations

$$U(t,x) \ll W(T,X) \ll V(T,X).$$

Moreover, by calculating the upper bound estimate of the power of $1/(R-X)$ in the recurrence formula, we have the following lemma:

Lemma 3.4. *The coefficients* $\{V_K(X)\}_{K\geq 2}$ *are given by*

$$V_K(X) = \sum_{\ell=2}^{7K-12} \frac{V_{K\ell}}{(R-X)^{\ell}}, \quad \textit{by some} \;\; V_{K\ell} \geq 0. \tag{3.24}$$

We can easily prove Lemma 3.4 by the same calculation as the one in the proof of Lemma 3.2. (See [10, Lemma 2]).

By (3.24), we have the following majorant relation:

$$\partial_X (T\partial_T)^{-1} V(T,X) \ll \sum_{K\geq 2} \sum_{\ell=2}^{7K-12} \frac{\ell}{K} \frac{V_{K\ell}}{(R-X)^{\ell+1}} T^K \ll \frac{7}{R-X} V(T,X). \tag{3.25}$$

We consider the following functional equation:

$$Y(T,X) = Q(X)T^2 + R_3(T,X,C_1Y,\{C_1Y/T\},\{7C_1Y/(R-X)\}) \tag{3.26}$$

with $Y = O(T^2)$. The equation (3.26) has a unique formal solution $Y(T,X)$. Moreover, by the construction of (3.26), $Y(T,X)$ is a majorant function of $V(T,X)$, that is, the following majorant relations hold:

$$U(t,x) \ll W(T,X) \ll V(T,X) \ll Y(T,X).$$

We put $Y(T,X) = T^2\hat{Y}(T,X)$. Then $\hat{Y}$ satisfies

$$\hat{Y}(T,X) = Q(X) + \frac{1}{T^2} R_3(T,X,C_1T^2\hat{Y},\{C_1T\hat{Y}\},\{7C_1T^2\hat{Y}/(R-X)\}) \tag{3.27}$$

with $\hat{Y}(0,X) = Q(X)$. Since R_3 has a vanishing order in T at least 3, therefore R_3/T^2 is holomorphic in a neighborhood of $T=0$ and vanish at $T=0$.

For the equation (3.27), we can prove that the formal solution $\hat{Y}$ is convergent in a neighborhood of the origin by the classical implicit function theorem. Therefore, we obtain that

$$U(t,x) \ll W(T,X) \ll V(T,X) \ll Y(T,X) = T^2\hat{Y}(T,X) \in \mathbf{C}\{t,x\}.$$

Therefore, we have

$$v(t,x) = P_0^{-1}U(t,x) \ll C_1(T\partial_T)^{-1}Y(T,X) \ll C_1 Y(T,X) \in \mathbf{C}\{t,x\},$$

which proves the convergence of $v(t,x) = u(t,x) - \sum_{j=1}^{d} \varphi_j(x)t_j$.

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