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A stratified Whitney jet and a tempered-stratified ultradistribution

By

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§ 1. Introduction

A tempered distribution and a Whitney function appear in various aspects of analysis such as the Riemann-Hilbert correspondence [2], the theory of asymptotic analysis and etc. As they are defined by non-local properties, for example, a tempered distribution on an open set $U \subset X$ is, by definition, a distribution on U which extends to the whole space X, tempered distributions and Whitney functions do not form sheaves in a usual topological space X. This is one of reasons why they were difficult to be managed. However they are still sheaves on the subanalytic site X_{sa}. The recent development of the theory for ind-sheaves and sheaves on subanalytic sites by M.Kashiwara-P.Schapira [5] and L.Prelli [11], etc. enables us to apply sheaf theoretic methods (Grothendieck's six operations) to such an object. Therefore a tempered distribution and a Whitney function now can be managed well in a functorial way.

It is reasonable to expect tempered ultradistributions and Whitney functions of Gevrey class to form sheaves on X_{sa} also. Unfortunately this expectation is not true as it is shown in Examples 2.3 and 2.4 of this note.

Our aim is to construct sheaves on X_{sa} corresponding to tempered ultradistributions and Whitney functions of Gevrey class. We introduce, in the note, *a tempered-stratified ultradistribution* and *a stratified Whitney jet of Gevrey class*. They can be regarded as alternatives of a tempered ultradistribution and a Whitney jet of Gevrey class due to the fact that their sections on an open subanalytic subset with the smooth boundary coincide with those of a tempered ultradistribution and a Whitney jet of Gevrey class respectively.

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The stratified Whitney jets of Gevrey class certainly form a sheaf on the subanalytic site X_{sa}. While, for a tempered ultradistribution, we can construct the corresponding sheaf on X_{sa} only if $\dim X \le 2$. We discuss, in the last section of the note, the possibility of its construction on a higher dimensional manifold.

This note is a short summary of our paper [10] in which the complete proofs for propositions and theorems here are given.

§ 2. Preparation

Let X be a real analytic manifold. We always assume, in this note, a manifold to be countable at infinity. We denote by $\mathrm{Mod}(\mathbb{C}_X)$ the category of sheaves on X with values in $\mathbb{C}$-vector spaces, and by $\mathscr{C}^\infty$ the sheaf of infinitely differentiable functions on X.

Let A be a locally closed subset in X, and $\mathcal{J}_A(X)$ designates the set of continuous sections over A of the jet vector bundle J_X of X. We have the canonical morphism $j_A : \mathscr{C}^\infty(X) \to \mathcal{J}_A(X)$ and the natural restriction map $J_{B,A} : \mathcal{J}_A(X) \to \mathcal{J}_B(X)$ for a locally closed subset $B \subset A$. If X is the n-dimensional Euclidean space with coordinates $(x_1, \dots, x_n)$, then an element in $\mathcal{J}_A(X)$ is identified with a family of continuous functions $\{\varphi_\alpha(x)\}_{\alpha \in (\{0\} \cup \mathbb{N})^n}$ on A, and j_A is given by

$$j_A(\varphi) = \left\{ \left. \frac{\partial^\alpha \varphi}{\partial x^\alpha} \right|_A \right\}_{\alpha \in (\{0\} \cup \mathbb{N})^n} \qquad \text{for } \varphi \in \mathscr{C}^\infty(X).$$

In what follows, the symbol $*$ denotes (s) or $\{s\}$ for some $s > 1$. Let us recall the definition of the sheaf $\mathscr{C}^*$ of ultradifferentiable functions of class $*$ in X. For an open subset V in the n-dimensional Euclidean space with coordinates $(x_1, \dots, x_n)$, an element $f(x) \in \mathscr{C}^{(s)}(V)$ (resp. $f(x) \in \mathscr{C}^{\{s\}}(V)$) is an infinitely differentiable function with the following growth condition. For any compact set $K \subset V$ and for any $h > 0$ (resp. some $h > 0$), there exists a constant C such that

$$\sup_{x \in K} \left| \frac{\partial^\alpha}{\partial x^\alpha} f(x) \right| \le C h^{|\alpha|} (|\alpha|!)^s \qquad \text{for any } \alpha \in (\{0\} \cup \mathbb{N})^n,$$

respectively. The definition naturally extends to that of an ultradifferentiable function on a real analytic manifold.

Definition 2.1. For a locally closed subset A in X, the set $j_A(\mathscr{C}^*(X)) \subset \mathcal{J}_A(X)$ is called Whitney jets of Gevrey class $*$ over A, and we denote it by $\mathcal{W}^*_A(X)$.

For an open subset V in X, the vector space $\mathscr{C}^*(V)$ can be endowed with a locally convex topology in a natural way. The set $\mathcal{D}b^*(V)$ of ultradistributions of class $*$ on V is, by definition, the strong dual of $\mathscr{C}^*(V)$. It is well-known that a family $\{\mathcal{D}b^*(V)\}_V$ forms the sheaf $\mathcal{D}b^*$ of ultradistributions of class $*$ on X.

Given a closed set $Z \subset X$ and $F \in \mathrm{Mod}(\mathbb{C}_X)$ we denote by $\Gamma_Z(F)$ the subsheaf of F of sections supported by Z.

Definition 2.2. Let U be an open set in X. The set $\mathcal{D}b_X^{*t}(U)$ of tempered ultradistributions of class $*$ over U is defined by

$$\mathcal{D}b_X^{*t}(U) := \frac{\Gamma(X; \mathcal{D}b^*)}{\Gamma_{X \setminus U}(X; \mathcal{D}b^*)}.$$

Let $\mathcal{F}$ be a presheaf on X, and let U and V be open subanalytic subsets in X with U, V, $U \cap V$ and $U \cup V$ being locally cohomologically trivial. We now consider the following sequence of Abelian groups

$$0 \to \mathcal{F}(U \cup V) \to \mathcal{F}(U) \oplus \mathcal{F}(V) \to \mathcal{F}(U \cap V)\ . \tag{2.1}$$

If $\mathcal{F}$ is the presheaf of either tempered distributions or Whitney functions, it follows from the Lojasiewicz inequality [9] for subanalytic subsets that the sequence (2.1) is exact. The exactness of (2.1) implies that tempered distributions and Whitney functions form sheaves on the subanalytic site X_{sa}. We refer the reader to [5] and [11] for the precise definitions of a subanalytic site and a sheaf defined on it.

While, for the presheaf either $\mathcal{F}(U) := \mathcal{D}b_X^{*t}(U)$ or $\mathcal{F}(U) := \mathcal{W}_U^*(X)$, the sequence (2.1) is not exact as the following examples show (see [10] for the details).

Example 2.3. Let $l \geq 1$. Set

$$U_1 := \{(x, y) \in \mathbb{R}^2;\ y > x^{2l+1}\}\ ,$$

$$U_2 := \mathbb{R} \times \mathbb{R}_{<0}\ .$$

Let $L_{loc}(U)$ denote the set of locally integrable functions in an open subset U. Define a function $u_i(x, y) \in L_{loc}(U_i)$ $(i = 1, 2)$ by

$$u_1(x, y) := \begin{cases} \exp\left(\dfrac{1}{y - x^{2l+1}}\right) & \text{for } (x, y) \in U_1 \cap (\mathbb{R} \times \mathbb{R}_{>0}) \\ 0 & \text{for } (x, y) \in U_1 \cap (\mathbb{R} \times \mathbb{R}_{\leq 0}) \end{cases}$$

$$u_2(x, y) := 0 \quad \text{for any } (x, y) \in U_2\ .$$

Then we have $u_1 \in \mathcal{D}b_X^{(2)t}(U_1)$. Clearly $u_2 \in \mathcal{D}b_X^{(2)t}(U_2)$. As $u_1|_{U_1 \cap U_2} = u_2|_{U_1 \cap U_2}$, there exists $u \in L_{loc}(U_1 \cup U_2)$ such that $u|_{U_1} = u_1$ and $u|_{U_2} = u_2$, but we can show $u \notin \mathcal{D}b_X^{(2)t}(U_1 \cup U_2)$.

Example 2.4. Let $m \geq 2$ and set

$$U_1 := \{(x, y) \in \mathbb{R}^2;\ y < 0\},$$
$$U_2 := \{(x, y) \in \mathbb{R}^2;\ x < 0 \text{ or } y > x^m\}.$$

Define the jets $G_1 = \{g_{1,\alpha}\} \in \mathcal{J}_{U_1}(X)$ and $G_2 = \{g_{2,\alpha}\} \in \mathcal{J}_{U_2}(X)$ by

$$g_{1,\alpha}(x,y) := \begin{cases} 0 & (x,y) \in U_1 \cap \{x \leq 0\} \\ \dfrac{\partial^\alpha}{\partial x^\alpha} \exp\left(-\dfrac{1}{x}\right) & (x,y) \in U_1 \cap \{x > 0\} \end{cases}$$

$$g_{2,\alpha}(x,y) := 0 \ .$$

Then we have $G_k \in \mathcal{W}^{\{2\}}_{U_k}(X)$ $(k = 1, 2)$. As $G_1 = G_2$ in $U_1 \cap U_2$, we find $G \in \mathcal{J}_{U_1 \cup U_2}(X)$ with $G = G_k$ in U_k for $k = 1, 2$. However $G \notin \mathcal{W}^{\{2\}}_{U_1 \cup U_2}(X)$.

These examples indicate that the presheaves $\{\mathcal{D}b^{*t}_X(U)\}_U$ and $\{\mathcal{W}^*_U(X)\}_U$ do not form sheaves on the subanalytic site X_{sa}.

At the end of the subsection, we give some definitions and results which are needed later on. Let X be a real analytic manifold of dimension n and A a locally closed subanalytic subset of X. Throughout this paper, a subanalytic stratification $\{A_\alpha\}_\alpha$ of A is that of X which is finer than the partition $\{A, X \setminus A\}$.

Definition 2.5. We say that A is 1-regular at $p \in X$ if there exist a neighborhood $U \subset X$ of p, a neighborhood $V \subset \mathbb{R}^n$ of the origin and an isomorphism $\psi : (U, p) \to (V, 0)$ satisfying the following condition. There exist a positive constant $\kappa > 0$ and a compact neighborhood $K \subset V$ of the origin such that for any $x_1, x_2 \in \psi(A \cap U) \cap K$ there exists a subanalytic curve l in $\psi(A \cap U)$ joining x_1 and x_2 and satisfying the estimate

$$|l| \leq \kappa |x_1 - x_2|,$$

where $|l|$ stands for the length of l.

The set A is said to be 1-regular if it is 1-regular at any point $p \in X$.

Let $\{A_\alpha\}_{\alpha \in \Lambda}$ be a stratification of A. Then $\{A_\alpha\}_{\alpha \in \Lambda}$ is called a 1-*regular stratification* if each stratum is 1-regular, connected and relatively compact. A 1-regular stratification always exists thanks to the following proposition due to K. Kurdyka.

Proposition 2.6 ([8]).

1. *Let $Z \subset X$ be a locally closed subanalytic subset and $\{X_\alpha\}$ a stratification of Z. There exists a* 1*-regular stratification of Z finer than $\{X_\alpha\}$.*
2. *Let $U \subset X$ be a subanalytic open set. There exists an open covering $\{U_j\}_{j \in J}$ of U such that, for any $j \in J$, U_j is a subanalytic* 1*-regular set and that, for any compact set K in X, a finite number of U_j only intersects K.*

§ 3. A stratified Whitney jet and a stratified ultradistribution

In this section, we first give the definitions of a stratified Whitney jet and a stratified ultradistribution. Then we investigate their properties and establish a relation between them.

§ 3.1. A stratified Whitney jet of Gevrey class

Let A be a locally closed subanalytic subset.

Definition 3.1. We say that $F \in \mathcal{J}_A(X)$ is a stratified Whitney jet of class $*$ over A if for any compact subanalytic set K there exists a subanalytic stratification $\{A_\alpha\}_{\alpha\in\Lambda}$ of A such that $j_{A_\alpha\cap K,\,A}(F) \in \mathcal{W}^*_{A_\alpha\cap K}(X)$ holds for any $\alpha \in \Lambda$ with $A_\alpha \subset A$.

We denote by $\mathcal{SW}^*_A(X)$ the set of stratified Whitney jets of class $*$ over A. If $\{A_\alpha\}_{\alpha\in\Lambda}$ is a stratification of A, then we denote by $\mathcal{SW}^*_{\{A_\alpha\}}(X)$ the subset of $\mathcal{SW}^*_A(X)$ defined by:

$$\mathcal{SW}^*_{\{A_\alpha\}}(X) = \left\{F \in \mathcal{J}_A(X);\, j_{A_\alpha,\,A}(F) \in \mathcal{W}^*_{A_\alpha}(X) \text{ for any } \alpha \in \Lambda \text{ with } A_\alpha \subset A\right\}.$$

The following proposition is fundamental to understand a stratified Whitney jet.

Proposition 3.2. *Let A be a locally closed subanalytic set in X and $\{A_\alpha\}$ a 1-regular stratification of A. Then we have $\mathcal{SW}^*_A(X) = \mathcal{SW}^*_{\{A_\alpha\}}(X)$. In particular, if A is 1-regular, then $\mathcal{SW}^*_A(X) = \mathcal{W}^*_A(X)$.*

The reason why we introduce a stratified Whitney jet is explained by the following proposition.

Proposition 3.3. *Let A_1 and A_2 be locally closed subanalytic subsets in X and set $A = A_1 \cup A_2$. Assume that both A_1 and A_2 are either closed or open in A.*

1. *The sequence of Abelian groups*

   $$0 \to \mathcal{SW}^*_A(X) \to \mathcal{SW}^*_{A_1}(X) \oplus \mathcal{SW}^*_{A_2}(X) \to \mathcal{SW}^*_{A_1\cap A_2}(X) \tag{3.1.1}$$

   is exact.

2. *If $A_1 \cap A_2$ is 1-regular or if $\dim X \leq 2$ and A is closed, then the third arrow of (3.1.1) is surjective.*

§ 3.2. A stratified ultradistribution

The vector space $\mathcal{SW}^*_A(X)$ of stratified Whitney jets has also a natural locally convex topology. We introduce, in the subsection, the set of stratified ultradistributions which is the strong dual of $\mathcal{SW}^*_A(X)$ for a compact subanalytic set A.

Let X be a real analytic manifold and A a closed subanalytic subset in X. For a stratification $\{A_\alpha\}$ of A, let us define stratified ultradistributions along $\{A_\alpha\}$.

Definition 3.4. An ultradistribution $u \in \mathcal{D}b^*(X)$ is said to be stratified along $\{A_\alpha\}$ if u can be written in the form:

$$u = \sum_{\alpha \text{ with } A_\alpha \subset A} u_\alpha, \qquad u_\alpha \in \Gamma_{\overline{A_\alpha}}(X;\, \mathcal{D}b^*).$$

We set

$$\mathcal{SD}b^*_{\{A_\alpha\}}(X) := \{u \in \mathcal{D}b^*(X);\, u \text{ is stratified along } \{A_\alpha\}\ \} \subset \Gamma_A(X;\, \mathcal{D}b^*)\ .$$

We define the set $\mathcal{SD}b^*_{[A]}(X)$ of *stratified ultradistributions of class* $*$ *along* A by

$$\mathcal{SD}b^*_{[A]}(X) := \varprojlim_{\substack{\text{stratification} \\ \{A_\alpha\} \text{ of } A}} \mathcal{SD}b^*_{\{A_\alpha\}}(X).$$

Note that, since for any stratification $\{A_\alpha\}$ there exists a 1-regular stratification finer than $\{A_\alpha\}$, we have

$$\mathcal{SD}b^*_{[A]}(X) = \varprojlim_{\substack{\text{1-regular stratification} \\ \{A_\alpha\} \text{ of } A}} \mathcal{SD}b^*_{\{A_\alpha\}}(X)\ .$$

We have the similar propositions as those for a stratified Whitney jet.

Proposition 3.5. *Let A be a closed subanalytic subset in X and $\{A_\alpha\}$ a 1-regular stratification of A. Then we have $\mathcal{SD}b^*_{[A]}(X) = \mathcal{SD}b^*_{\{A_\alpha\}}(X)$. In particular, if A is 1-regular, we have $\mathcal{SD}b^*_{[A]}(X) = \Gamma_A(X;\, \mathcal{D}b^*)$.*

Proposition 3.6. *Let A_1 and A_2 be closed subanalytic sets. If $A_1 \cap A_2$ is 1-regular or if* $\dim X \leq 2$ *holds, then the sequence of Abelian groups*

$$0 \to \mathcal{SD}b^*_{[A_1 \cap A_2]}(X) \to \mathcal{SD}b^*_{[A_1]}(X) \oplus \mathcal{SD}b^*_{[A_2]}(X) \to \mathcal{SD}b^*_{[A_1 \cup A_2]}(X) \to 0$$

is exact.

These results come from the corresponding ones in the previous subsection by the following duality.

Theorem 3.7. *Let X be an orientable real analytic manifold, and $A \subset X$ a compact subanalytic set. Then, algebraically, we have*

$$\mathcal{SD}b^*_{[A]}(X) \simeq (\mathcal{SW}^*_A(X))'$$

where $(\mathcal{SW}^*_A(X))'$ denotes the topological dual space of $\mathcal{SW}^*_A(X)$.

§ 4. Sheaves of Gevrey classes on subanalytic sites

§ 4.1. The sheaf of the stratified Whitney jets on X_{sa}

Let X be a real analytic manifold. The presheaf of stratified Whitney jets of class $*$ is defined by

$$\mathcal{SW}^*_{X_{sa}}(U) := \mathcal{SW}^*_U(X),$$

where U is a subanalytic open subset of X. Note that, since $\Gamma(\overline{U};\,\mathcal{D}_X)$ acts on $\mathcal{SW}^*_U(X)$, $\mathcal{SW}^*_{X_{sa}}$ is a $\varrho_!\mathcal{D}_X$-module where $\mathcal{D}_X$ is the sheaf of linear differential operators in X and $\varrho : X \to X_{sa}$ is the canonical morphism (see [5] and [11]).

Propositions 3.2 and 3.3 can be rephrased in terms of sheaves on subanalytic sites as follows.

Proposition 4.1. *The presheaf $\mathcal{SW}^*_{X_{sa}}$ of stratified Whitney jets is a sheaf on the subanalytic site X_{sa}. If $U \subset X$ is a 1-regular open subanalytic set, then we have*

$$\mathcal{SW}^*_{X_{sa}}(U) = \mathcal{W}^*_U(X) \simeq \mathcal{W}^*_{\overline{U}}(X) \simeq \frac{\mathscr{C}^*(X)}{\mathcal{I}^*_{X,\overline{U}}(X)}\ ,$$

where $\mathcal{I}^*_{X,\overline{U}}$ denotes the subsheaf of $\mathscr{C}^*$ consisting of functions vanishing on $\overline{U}$ up to infinite order.

We define the presheaf $\mathcal{W}^*_{X_{sa}}$ by $\mathcal{W}^*_{X_{sa}}(U) := \mathcal{W}^*_U(X)$ for a subanalytic open subset U.

Corollary 4.2. *We have $\mathcal{W}^{*a}_{X_{sa}} \simeq \mathcal{SW}^*_{X_{sa}}$ where $\mathcal{W}^{*a}_{X_{sa}}$ denotes the sheafication of the presheaf $\mathcal{W}^*_{X_{sa}}$ on the subanalytic site X_{sa}.*

§ 4.2. A tempered-stratified ultradistribution

For $U \subset X$ a subanalytic open set, we define the set of *tempered-stratified ultradistributions* on U as

$$\mathcal{D}b^{*ts}_{X_{sa}}(U) := \frac{\mathcal{SD}b^*_{[X]}(X)}{\mathcal{SD}b^*_{[X\setminus U]}(X)} = \frac{\mathcal{D}b^*(X)}{\mathcal{SD}b^*_{[X\setminus U]}(X)}\ .$$

Theorem 4.3. *Let U be an open subanalytic subset of X.*

1. *The ring $\Gamma(\overline{U};\,\mathcal{D}_X)$ acts on $\mathcal{D}b^{*ts}_{X_{sa}}(U)$.*
2. *Let V be an open subanalytic subset of X. Then we have the following exact sequence.*

$$\mathcal{D}b^{*ts}_{X_{sa}}(U\cup V) \to \mathcal{D}b^{*ts}_{X_{sa}}(U) \oplus \mathcal{D}b^{*ts}_{X_{sa}}(V) \to \mathcal{D}b^{*ts}_{X_{sa}}(U\cap V) \to 0.$$

Further, if $\dim X \leq 2$, then the first morphism of the above sequence is injective.

3. *If $X \setminus U$ is 1-regular, then $\mathcal{D}b^{*ts}_{X_{sa}}(U)$ coincides with the sections of tempered ultradistributions of class $*$ on U, that is,*

$$\mathcal{D}b^{*ts}_{X_{sa}}(U) = \mathcal{D}b^{*t}_{X}(U) \ .$$

As an immediate corollary of the above theorem, we have

Corollary 4.4. *If $\dim X \leq 2$, then $\mathcal{D}b^{*ts}_{X_{sa}}$ is a sheaf on the subanalytic site X_{sa}.*

§ 5. Higher dimensional case

Let $\mathcal{G}$ be a sheaf on the subanalytic site X_{sa}, and let $r_{V,U}$ denote the restriction morphism of $\mathcal{G}$ for $V \subset U$ open subanalytic subsets. Assume that $\mathcal{G}$ satisfies the following conditions.

1. If an open subanalytic subset U has smooth boundary, then $\mathcal{G}(U) \simeq \mathcal{D}b^{*t}(U)$.
2. For any open subanalytic subset U, the restriction morphism $r_{U,X} : \mathcal{G}(X) \to \mathcal{G}(U)$ is surjective, i.e. $\mathcal{G}$ is quasi-injective.

If $\dim X \leq 2$, our $\mathcal{D}b^{*ts}_{X_{sa}}$ satisfies Conditions 1. and 2. above. Moreover the sheaf of tempered distributions on X_{sa} also satisfies Condition 2. Note that, as tempered distributions themselves form a sheaf on the subanalytic site X_{sa}, it does not make sense to ask for Condition 1. For the possibility of a general construction of a sheaf of tempered ultradistributions on X_{sa}, we have the following result.

Proposition 5.1. *If $\dim X > 2$, then there exists no sheaf $\mathcal{G}$ on the subanalytic site X_{sa} that satisfies the above conditions 1. and 2.*

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