A stratified Whitney jet and a tempered-stratified ultradistribution

By

GIOVANNI MORANDO* and NAOFUMI HONDA**

§ 1. Introduction

A tempered distribution and a Whitney function appear in various aspects of analysis such as the Riemann-Hilbert correspondence [2], the theory of asymptotic analysis and etc. As they are defined by non-local properties, for example, a tempered distribution on an open set $U \subset X$ is, by definition, a distribution on $U$ which extends to the whole space $X$, tempered distributions and Whitney functions do not form sheaves in a usual topological space $X$. This is one of reasons why they were difficult to be managed. However they are still sheaves on the subanalytic site $X_{sa}$. The recent development of the theory for ind-sheaves and sheaves on subanalytic sites by M.Kashiwara-P.Schapira [5] and L.Prelli [11], etc. enables us to apply sheaf theoretic methods (Grothendieck’s six operations) to such an object. Therefore a tempered distribution and a Whitney function now can be managed well in a functorial way.

It is reasonable to expect tempered ultradistributions and Whitney functions of Gevrey class to form sheaves on $X_{sa}$ also. Unfortunately this expectation is not true as it is shown in Examples 2.3 and 2.4 of this note.

Our aim is to construct sheaves on $X_{sa}$ corresponding to tempered ultradistributions and Whitney functions of Gevrey class. We introduce, in the note, a tempered-stratified ultradistribution and a stratified Whitney jet of Gevrey class. They can be regarded as alternatives of a tempered ultradistribution and a Whitney jet of Gevrey class due to the fact that their sections on an open subanalytic subset with the smooth boundary coincide with those of a tempered ultradistribution and a Whitney jet of Gevrey class respectively.
The stratified Whitney jets of Gevrey class certainly form a sheaf on the subanalytic site $X_{sa}$. While, for a tempered ultradistribution, we can construct the corresponding sheaf on $X_{sa}$ only if $\dim X \leq 2$. We discuss, in the last section of the note, the possibility of its construction on a higher dimensional manifold.

This note is a short summary of our paper [10] in which the complete proofs for propositions and theorems here are given.

§ 2. Preparation

Let $X$ be a real analytic manifold. We always assume, in this note, a manifold to be countable at infinity. We denote by Mod($\mathbb{C}_X$) the category of sheaves on $X$ with values in $\mathbb{C}$-vector spaces, and by $\mathscr{C}^\infty$ the sheaf of infinitely differentiable functions on $X$.

Let $A$ be a locally closed subset in $X$, and $\mathcal{J}_A(X)$ designates the set of continuous sections over $A$ of the jet vector bundle $J_X$ of $X$. We have the canonical morphism $j_A : \mathscr{C}^\infty(X) \to \mathcal{J}_A(X)$ and the natural restriction map $J_{B,A} : \mathcal{J}_A(X) \to \mathcal{J}_B(X)$ for a locally closed subset $B \subset A$. If $X$ is the $n$-dimensional Euclidean space with coordinates $(x_1, \ldots, x_n)$, then an element in $\mathcal{J}_A(X)$ is identified with a family of continuous functions \(\{\varphi_\alpha(x)\}_{\alpha \in \{(0)\cup \mathbb{N}\}^n}\) on $A$, and $j_A$ is given by

\[j_A(\varphi) = \left\{ \frac{\partial^\alpha \varphi}{\partial x^\alpha} \right\}_{\alpha \in \{(0)\cup \mathbb{N}\}^n} \quad \text{for } \varphi \in \mathscr{C}^\infty(X).\]

In what follows, the symbol $*$ denotes $(s)$ or $\{s\}$ for some $s > 1$. Let us recall the definition of the sheaf $\mathscr{E}^*$ of ultradifferentiable functions of class $*\mathrm{on}X$. For an open subset $V$ in the $n$-dimensional Euclidean space with coordinates $(x_1, \ldots, x_n)$, an element $f(x) \in \mathscr{E}^{(s)}(V)$ (resp. $f(x) \in \mathscr{E}^{\{s\}}(V)$) is an infinitely differentiable function with the following growth condition. For any compact set $K \subset V$ and for any $h > 0$ (resp. some $h > 0$), there exists a constant $C$ such that

\[
\sup_{x \in K} \left| \frac{\partial^\alpha f(x)}{\partial x^\alpha} \right| \leq C h^{||\alpha||} s \quad \text{for any } \alpha \in \{(0) \cup \mathbb{N}\}^n,
\]

respectively. The definition naturally extends to that of an ultradifferentiable function on a real analytic manifold.

**Definition 2.1.** For a locally closed subset $A$ in $X$, the set $j_A(\mathscr{E}^*(X)) \subset \mathcal{J}_A(X)$ is called Whitney jets of Gevrey class $*\mathrm{on}A$, and we denote it by $W^*_A(X)$.

For an open subset $V$ in $X$, the vector space $\mathscr{E}^*(V)$ can be endowed with a locally convex topology in a natural way. The set $\mathcal{D}b^*(V)$ of ultradistributions of class $*\mathrm{on}V$ is, by definition, the strong dual of $\mathscr{E}^*(V)$. It is well-known that a family \(\{\mathcal{D}b^*(V)\}_V\) forms the sheaf $\mathcal{D}b^*$ of ultradistributions of class $*\mathrm{on}X$. 


Given a closed set \( Z \subset X \) and \( F \in \text{Mod}(\mathbb{C}_X) \) we denote by \( \Gamma_Z(F) \) the subsheaf of \( F \) of sections supported by \( Z \).

**Definition 2.2.** Let \( U \) be an open set in \( X \). The set \( \mathcal{D}b_X^{*t}(U) \) of tempered ultradistributions of class * over \( U \) is defined by

\[
\mathcal{D}b_X^{*t}(U) := \frac{\Gamma(X; \mathcal{D}b^{*})}{\Gamma_{X \setminus U}(X; \mathcal{D}b^{*})}.
\]

Let \( \mathcal{F} \) be a presheaf on \( X \), and let \( U \) and \( V \) be open subanalytic subsets in \( X \) with \( U, V, U \cap V \) and \( U \cup V \) being locally cohomologically trivial. We now consider the following sequence of Abelian groups

\[
0 \to \mathcal{F}(U \cup V) \to \mathcal{F}(U) \oplus \mathcal{F}(V) \to \mathcal{F}(U \cap V).
\]

If \( \mathcal{F} \) is the presheaf of either tempered distributions or Whitney functions, it follows from the Lojasiewicz inequality [9] for subanalytic subsets that the sequence (2.1) is exact. The exactness of (2.1) implies that tempered distributions and Whitney functions form sheaves on the subanalytic site \( X_{sa} \). We refer the reader to [5] and [11] for the precise definitions of a subanalytic site and a sheaf defined on it.

While, for the presheaf either \( \mathcal{F}(U) := \mathcal{D}b_X^{*t}(U) \) or \( \mathcal{F}(U) := \mathcal{W}_{U}^{*}(X) \), the sequence (2.1) is not exact as the following examples show (see [10] for the details).

**Example 2.3.** Let \( l \geq 1 \). Set

\[
U_1 := \{(x, y) \in \mathbb{R}^2; y > x^{2l+1}\},
\]

\[
U_2 := \mathbb{R} \times \mathbb{R}_{<0}.
\]

Let \( L_{loc}(U) \) denote the set of locally integrable functions in an open subset \( U \). Define a function \( u_i(x, y) \in L_{loc}(U_i) \) \( (i = 1, 2) \) by

\[
u_1(x, y) := \begin{cases} \exp\left(\frac{1}{y-x^{2l+1}}\right) & \text{for } (x, y) \in U_1 \cap (\mathbb{R} \times \mathbb{R}_{>0}) \\ 0 & \text{for } (x, y) \in U_1 \cap (\mathbb{R} \times \mathbb{R}_{\leq 0}) \end{cases}\]

\[
u_2(x, y) := 0 \quad \text{for any } (x, y) \in U_2.
\]

Then we have \( u_1 \in \mathcal{D}b_X^{(2)t}(U_1) \). Clearly \( u_2 \in \mathcal{D}b_X^{(2)t}(U_2) \). As \( u_1|_{U_1 \cap U_2} = u_2|_{U_1 \cap U_2} \), there exists \( u \in L_{loc}(U_1 \cup U_2) \) such that \( u|_{U_1} = u_1 \) and \( u|_{U_2} = u_2 \), but we can show \( u \notin \mathcal{D}b_X^{(2)t}(U_1 \cup U_2) \).

**Example 2.4.** Let \( m \geq 2 \) and set

\[
U_1 := \{(x, y) \in \mathbb{R}^2; y < 0\},
\]

\[
U_2 := \{(x, y) \in \mathbb{R}^2; x < 0 \text{ or } y > x^m\}.
\]
Define the jets $G_1 = \{ g_{1, \alpha} \} \in \mathcal{J}_{U_1}(X)$ and $G_2 = \{ g_{2, \alpha} \} \in \mathcal{J}_{U_2}(X)$ by

$$
g_{1, \alpha}(x, y) := \begin{cases} 0 & (x, y) \in U_1 \cap \{ x \leq 0 \} \\ \frac{\partial^\alpha}{\partial x^\alpha} \exp \left( -\frac{1}{x} \right) & (x, y) \in U_1 \cap \{ x > 0 \} 
\end{cases}$$

$$g_{2, \alpha}(x, y) := 0 .$$

Then we have $G_k \in \mathcal{W}^{\{2\}}_{U_k}(X) (k = 1, 2)$. As $G_1 = G_2$ in $U_1 \cap U_2$, we find $G \in \mathcal{J}_{U_1 \cup U_2}(X)$.

These examples indicate that the presheaves $\{ \mathcal{D}b_X^{\{t\}}(U) \}_U$ and $\{ \mathcal{W}_U^{\{r\}}(X) \}_U$ do not form sheaves on the subanalytic site $X_{sa}$.

At the end of the subsection, we give some definitions and results which are needed later on. Let $X$ be a real analytic manifold of dimension $n$ and $A$ a locally closed subanalytic subset of $X$. Throughout this paper, a subanalytic stratification $\{ A_\alpha \}_\alpha$ of $A$ is that of $X$ which is finer than the partition $\{ A, X \setminus A \}$.

**Definition 2.5.** We say that $A$ is 1-regular at $p \in X$ if there exist a neighborhood $U \subset X$ of $p$, a neighborhood $V \subset \mathbb{R}^n$ of the origin and an isomorphism $\psi : (U, p) \rightarrow (V, 0)$ satisfying the following condition. There exist a positive constant $\kappa > 0$ and a compact neighborhood $K \subset V$ of the origin such that for any $x_1, x_2 \in \psi(A \cap U) \cap K$ there exists a subanalytic curve $l$ in $\psi(A \cap U)$ joining $x_1$ and $x_2$ and satisfying the estimate

$$|l| \leq \kappa |x_1 - x_2| ,$$

where $|l|$ stands for the length of $l$.

The set $A$ is said to be 1-regular if it is 1-regular at any point $p \in X$.

Let $\{ A_\alpha \}_{\alpha \in \Lambda}$ be a stratification of $A$. Then $\{ A_\alpha \}_{\alpha \in \Lambda}$ is called a 1-regular stratification if each stratum is 1-regular, connected and relatively compact. A 1-regular stratification always exists thanks to the following proposition due to K. Kurdyka.

**Proposition 2.6 ([8]).**

1. Let $Z \subset X$ be a locally closed subanalytic subset and $\{ X_\alpha \}$ a stratification of $Z$. There exists a 1-regular stratification of $Z$ finer than $\{ X_\alpha \}$.

2. Let $U \subset X$ be a subanalytic open set. There exists an open covering $\{ U_j \}_{j \in J}$ of $U$ such that, for any $j \in J$, $U_j$ is a subanalytic 1-regular set and that, for any compact set $K$ in $X$, a finite number of $U_j$ only intersects $K$. 

§ 3. A stratified Whitney jet and a stratified ultradistribution

In this section, we first give the definitions of a stratified Whitney jet and a stratified ultradistribution. Then we investigate their properties and establish a relation between them.

§ 3.1. A stratified Whitney jet of Gevrey class

Let $A$ be a locally closed subanalytic subset.

**Definition 3.1.** We say that $F \in \mathcal{J}_A(X)$ is a stratified Whitney jet of class $*$ over $A$ if for any compact subanalytic set $K$ there exists a subanalytic stratification $\{A_\alpha\}_{\alpha \in \Lambda}$ of $A$ such that $j_{A_\alpha \cap K}, A(F) \in \mathcal{W}_{A_\alpha \cap K}^*(X)$ holds for any $\alpha \in \Lambda$ with $A_\alpha \subset A$.

We denote by $\mathcal{S}\mathcal{W}_{\mathcal{A}}^*(X)$ the set of stratified Whitney jets of class $*$ over $A$.

The following proposition is fundamental to understand a stratified Whitney jet.

**Proposition 3.2.** Let $A$ be a locally closed subanalytic set in $X$ and $\{A_\alpha\}$ a 1-regular stratification of $A$. Then we have $\mathcal{S}\mathcal{W}_A^*(X) = \mathcal{S}\mathcal{W}_{\{A_\alpha\}}^*(X)$. In particular, if $A$ is 1-regular, then $\mathcal{S}\mathcal{W}_A^*(X) = \mathcal{W}_A^*(X)$.

The reason why we introduce a stratified Whitney jet is explained by the following proposition.

**Proposition 3.3.** Let $A_1$ and $A_2$ be locally closed subanalytic subsets in $X$ and set $A = A_1 \cup A_2$. Assume that both $A_1$ and $A_2$ are either closed or open in $A$.

1. The sequence of Abelian groups

\[
0 \rightarrow \mathcal{S}\mathcal{W}_A^*(X) \rightarrow \mathcal{S}\mathcal{W}_{A_1}^*(X) \oplus \mathcal{S}\mathcal{W}_{A_2}^*(X) \rightarrow \mathcal{S}\mathcal{W}_{A_1 \cap A_2}^*(X)
\]

is exact.

2. If $A_1 \cap A_2$ is 1-regular or if $\dim X \leq 2$ and $A$ is closed, then the third arrow of (3.1.1) is surjective.

§ 3.2. A stratified ultradistribution

The vector space $\mathcal{S}\mathcal{W}_A^*(X)$ of stratified Whitney jets has also a natural locally convex topology. We introduce, in the subsection, the set of stratified ultradistributions which is the strong dual of $\mathcal{S}\mathcal{W}_A^*(X)$ for a compact subanalytic set $A$. 
Let $X$ be a real analytic manifold and $A$ a closed subanalytic subset in $X$. For a stratification $\{A_\alpha\}$ of $A$, let us define stratified ultradistributions along $\{A_\alpha\}$.

**Definition 3.4.** An ultradistribution $u \in \mathcal{D}b^*(X)$ is said to be stratified along $\{A_\alpha\}$ if $u$ can be written in the form:

$$u = \sum_{\alpha \text{ with } A_\alpha \subset A} u_\alpha, \quad u_\alpha \in \Gamma_{A_\alpha}(X; \mathcal{D}b^*).$$

We set

$$\mathcal{S} \mathcal{D}b^*_{\{A_\alpha\}}(X) := \{ u \in \mathcal{D}b^*(X); u \text{ is stratified along } \{A_\alpha\} \} \subset \Gamma_A(X; \mathcal{D}b^*).$$

We define the set $\mathcal{S} \mathcal{D}b^*_{\{A\}}(X)$ of stratified ultradistributions of class $\ast$ along $A$ by

$$\mathcal{S} \mathcal{D}b^*_{\{A\}}(X) := \lim_{\leftarrow} \mathcal{S} \mathcal{D}b^*_{\{A_\alpha\}}(X).$$

Note that, since for any stratification $\{A_\alpha\}$ there exists a 1-regular stratification finer than $\{A_\alpha\}$, we have

$$\mathcal{S} \mathcal{D}b^*_A(X) = \lim_{1\text{-regular stratification}} \mathcal{S} \mathcal{D}b^*_{\{A_\alpha\}}(X).$$

We have the similar propositions as those for a stratified Whitney jet.

**Proposition 3.5.** Let $A$ be a closed subanalytic subset in $X$ and $\{A_\alpha\}$ a 1-regular stratification of $A$. Then we have $\mathcal{S} \mathcal{D}b^*_{\{A\}}(X) = \mathcal{S} \mathcal{D}b^*_{\{A_\alpha\}}(X)$. In particular, if $A$ is 1-regular, we have $\mathcal{S} \mathcal{D}b^*_{\{A\}}(X) = \Gamma_A(X; \mathcal{D}b^*)$.

**Proposition 3.6.** Let $A_1$ and $A_2$ be closed subanalytic sets. If $A_1 \cap A_2$ is 1-regular or if $\dim X \leq 2$ holds, then the sequence of Abelian groups

$$0 \to \mathcal{S} \mathcal{D}b^*_{\{A_1 \cap A_2\}}(X) \to \mathcal{S} \mathcal{D}b^*_{\{A_1\}}(X) \oplus \mathcal{S} \mathcal{D}b^*_{\{A_2\}}(X) \to \mathcal{S} \mathcal{D}b^*_{\{A_1 \cup A_2\}}(X) \to 0$$

is exact.

These results come from the corresponding ones in the previous subsection by the following duality.

**Theorem 3.7.** Let $X$ be an orientable real analytic manifold, and $A \subset X$ a compact subanalytic set. Then, algebraically, we have

$$\mathcal{S} \mathcal{D}b^*_{\{A\}}(X) \simeq (\mathcal{S} \mathcal{W}^*_A(X))'$$

where $(\mathcal{S} \mathcal{W}^*_A(X))'$ denotes the topological dual space of $\mathcal{S} \mathcal{W}^*_A(X)$.
§ 4. Sheaves of Gevrey classes on subanalytic sites

§ 4.1. The sheaf of the stratified Whitney jets on $X_{sa}$

Let $X$ be a real analytic manifold. The presheaf of stratified Whitney jets of class $*$ is defined by

$$SW^*_X(U) := SW^*_U(X),$$

where $U$ is a subanalytic open subset of $X$. Note that, since $\Gamma(\overline{U}; D_X)$ acts on $SW^*_U(X)$, $SW^*_X$ is a $\rho D_X$-module where $D_X$ is the sheaf of linear differential operators in $X$ and $\rho : X \to X_{sa}$ is the canonical morphism (see [5] and [11]).

Propositions 3.2 and 3.3 can be rephrased in terms of sheaves on subanalytic sites as follows.

**Proposition 4.1.** The presheaf $SW^*_X$ of stratified Whitney jets is a sheaf on the subanalytic site $X_{sa}$. If $U \subset X$ is a 1-regular open subanalytic set, then we have

$$SW^*_X(U) = W^*_U(X) \simeq \frac{C^*(X)}{I_{X,\overline{U}}^*(X)},$$

where $I_{X,\overline{U}}^*$ denotes the subsheaf of $C^*$ consisting of functions vanishing on $\overline{U}$ up to infinite order.

We define the presheaf $W^*_X$ by $W^*_X(U) := W^*_U(X)$ for a subanalytic open subset $U$.

**Corollary 4.2.** We have $W^*_{X_{sa}} \simeq SW^*_X$ where $W^*_{X_{sa}}$ denotes the sheafication of the presheaf $W^*_X$ on the subanalytic site $X_{sa}$.

§ 4.2. A tempered-stratified ultradistribution

For $U \subset X$ a subanalytic open set, we define the set of tempered-stratified ultradistributions on $U$ as

$$Db^*_{X_{sa}}(U) := \frac{SDb^*_{[X]}(X)}{SDb^*_{[X \setminus U]}(X)} = \frac{Db^*(X)}{SDb^*_{[X \setminus U]}(X)}.$$

**Theorem 4.3.** Let $U$ be an open subanalytic subset of $X$.

1. The ring $\Gamma(\overline{U}; D_X)$ acts on $Db^*_{X_{sa}}(U)$.

2. Let $V$ be an open subanalytic subset of $X$. Then we have the following exact sequence.

$$Db^*_{X_{sa}}(U \cup V) \to Db^*_{X_{sa}}(U) \oplus Db^*_{X_{sa}}(V) \to Db^*_{X_{sa}}(U \cap V) \to 0.$$

Further, if $\dim X \leq 2$, then the first morphism of the above sequence is injective.
3. If $X \setminus U$ is 1-regular, then $\mathcal{D}b_{X_{\mathrm{sa}}}^{*\text{ts}}(U)$ coincides with the sections of tempered ultradistributions of class $\ast$ on $U$, that is,

$$
\mathcal{D}b_{X_{\mathrm{sa}}}^{*\text{ts}}(U) = \mathcal{D}b_X^{*\text{t}}(U).
$$

As an immediate corollary of the above theorem, we have

**Corollary 4.4.** If $\dim X \leq 2$, then $\mathcal{D}b_{X_{\mathrm{sa}}}^{*\text{ts}}$ is a sheaf on the subanalytic site $X_{\text{sa}}$.

§5. **Higher dimensional case**

Let $\mathcal{G}$ be a sheaf on the subanalytic site $X_{\text{sa}}$, and let $r_{V,U}$ denote the restriction morphism of $\mathcal{G}$ for $V \subset U$ open subanalytic subsets. Assume that $\mathcal{G}$ satisfies the following conditions.

1. If an open subanalytic subset $U$ has smooth boundary, then $\mathcal{G}(U) \simeq \mathcal{D}b^{\ast t}(U)$.

2. For any open subanalytic subset $U$, the restriction morphism $r_{U,X} : \mathcal{G}(X) \rightarrow \mathcal{G}(U)$ is surjective, i.e. $\mathcal{G}$ is quasi-injective.

If $\dim X \leq 2$, our $\mathcal{D}b_{X_{\text{sa}}}^{*\text{ts}}$ satisfies Conditions 1. and 2. above. Moreover the sheaf of tempered distributions on $X_{\text{sa}}$ also satisfies Condition 2. Note that, as tempered distributions themselves form a sheaf on the subanalytic site $X_{\text{sa}}$, it does not make sense to ask for Condition 1. For the possibility of a general construction of a sheaf of tempered ultradistributions on $X_{\text{sa}}$, we have the following result.

**Proposition 5.1.** If $\dim X > 2$, then there exists no sheaf $\mathcal{G}$ on the subanalytic site $X_{\text{sa}}$ that satisfies the above conditions 1. and 2.

**References**


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