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Coupling of two singular partial differential equations and its application

By

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Abstract

In this note, we will consider a reduction of a singular Briot-Bouquet type partial differential equation (A) $t\partial u/\partial t = F(t,x,u,\partial u/\partial x)$ to a simple form (B) $t\partial w/\partial t = \lambda(x)w$ with $\lambda(x) = (\partial F/\partial u)(0,x,0,0)$ in the complex domain under the assumption that (A) satisfies certain Poincaré condition. The reduction is done by considering the coupling of two equations (A) and (B), and by solving their coupling equation. The result is applied to the problem of finding all the singular solutions of (A). This is an announcement of [5], and the details will be published in [5]. In the case of non-singular partial differential equations, its reduction to a normal form is done in [4].

§ 1. Introduction

Let (t,x) be the variables in $\mathbb{C}_t \times \mathbb{C}_x$, and let $F(t,x,u,v)$ be a holomorphic function defined in a polydisk Δ centered at the origin of $\mathbb{C}_t \times \mathbb{C}_x \times \mathbb{C}_u \times \mathbb{C}_v$. In the paper [4], we have established the equivalence of the following two partial differential equations

$$\frac{\partial u}{\partial t} = F\Big(t,x,u,\frac{\partial u}{\partial x}\Big) \quad \text{and} \quad \frac{\partial w}{\partial t} = 0$$

by considering the coupling of these two equations and by solving their coupling equations.

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In this note, we will consider the following nonlinear singular partial differential equation

$$t\frac{\partial u}{\partial t} = F\Big(t, x, u, \frac{\partial u}{\partial x}\Big) \tag{1.1}$$

under the assumptions

A$_1$) $F(t, x, u, v)$ is a holomorphic function on Δ,
A$_2$) $F(0, x, 0, 0) \equiv 0$ on $\Delta_0 = \Delta \cap \{t = 0, u = 0, v = 0\}$, and
A$_3$) $\frac{\partial F}{\partial v}(0, x, 0, 0) \equiv 0$ on Δ_0.

In the book of Gérard-Tahara [3], the equation (1.1) is called *a Briot-Bouquet type* partial differential equation with respect to t if it satisfies the conditions A$_1$), A$_2$) and A$_3$); the function

$$\lambda(x) = \frac{\partial F}{\partial u}(0, x, 0, 0) \tag{1.2}$$

is called *the characteristic exponent* (or *the characteristic exponent function*) of (1.1). About the structure of holomorphic and singular solutions of (1.1) in a neighborhood of $(0, 0) \in \mathbb{C}_t \times \mathbb{C}_x$, one can refer to Gérard-Tahara [2] and [3]. Among them, the following theorem is the most fundamental result:

Theorem 1.1 ([2]). *If* $\lambda(0) \notin \{1, 2, \dots\}$ *holds, the equation* (1.1) *has a unique holomorphic solution* $u_0(t, x)$ *in a neighborhood of* $(0, 0) \in \mathbb{C}_t \times \mathbb{C}_x$ *satisfying* $u_0(0, x) \equiv 0$ *near* $x = 0$.

The main theme of this note is to consider the following problem:

Problem 1.2. Find a canonical form of the equastion (1.1) by considering the coupling of two partial differential equations.

This is an announcement of [5], and the details will be published in [5].

§ 2. Analysis of the coupling equations

As is seen in the case of Briot-Bouquet's ordinary differential equations (in chapter 4 of [3]), it will be reasonable to treat the following equation

$$t\frac{\partial w}{\partial t} = \lambda(x)w \tag{2.1}$$

as a candidate of the canonical form of (1.1) in a generic case. In order to justify this assertion, we need to discuss the following coupling equation

$$(\Phi) \qquad t\frac{\partial\phi}{\partial t}+\sum_{m\geq 0}(D^m F)(t,x,u_0,\ldots,u_{m+1})\frac{\partial\phi}{\partial u_m}=\lambda(x)\phi, \quad \text{or}$$

$$(\Psi) \qquad t\frac{\partial\psi}{\partial t}+\sum_{m\geq 0}\Big(\sum_{0\leq i\leq m}\lambda_{m,i}(x)w_i\Big)\frac{\partial\psi}{\partial w_m}=F\big(t,x,\psi,D\psi\big),$$

where

$$\lambda_{m,i}(x)=\frac{m!}{i!(m-i)!}\Big(\frac{\partial}{\partial x}\Big)^{m-i}\lambda(x), \quad 0\leq i\leq m,$$

and D is the totally derivative operator defined by

$$D=\frac{\partial}{\partial x}+\sum_{i\geq 0}u_{i+1}\frac{\partial}{\partial u_i}, \quad \text{or} \quad D=\frac{\partial}{\partial x}+\sum_{i\geq 0}w_{i+1}\frac{\partial}{\partial w_i}.$$

In the equation (Φ), $\phi=\phi(t,x,u_0,u_1,\ldots)$ is the unknown function with infinitely many variables $(t,x,u_0,u_1,\ldots)$; in the equation (Ψ), $\psi=\psi(t,x,w_0,w_1,\ldots)$ is the unknown function with infinitely many variables $(t,x,w_0,w_1,\ldots)$.

The formal meaning of the coupling equations is as follows:

Proposition 2.1. (1) *If $\phi(t,x,u_0,u_1,\ldots)$ is a solution of (Φ) and if $u(t,x)$ is a solution of* (1.1)*, then the function $w(t,x)=\phi(t,x,u,\partial u/\partial x,\ldots)$ is a solution of* (2.1)*.*

(2) *If $\psi(t,x,w_0,w_1,\ldots)$ is a solution of (Ψ) and if $w(t,x)$ is a solution of* (2.1)*, then the function $u(t,x)=\psi(t,x,w,\partial w/\partial x,\ldots)$ is a solution of* (1.1)*.*

Proof. We will show only (1). Let $\phi(t,x,u_0,\,u_1,\ldots)$ be a solution (Φ) and let $u(t,x)$ be a solution of (1.1). Set $u_i(t,x)=(\partial/\partial x)^i u(t,x)$ $(i=0,1,2,\ldots)$: we have $w(t,x)=\phi(t,x,u,\partial u/\partial x,\ldots)=\phi(t,x,u_0,u_1,\ldots)$ and $\partial w/\partial x=D[\phi](t,x,\,u_0,u_1,\ldots)$. Therefore we have

$$\begin{aligned}
t\frac{\partial w}{\partial t}&=t\frac{\partial\phi}{\partial t}+\sum_{i\geq 0}\frac{\partial\phi}{\partial u_i}\times t\frac{\partial u_i}{\partial t}=t\frac{\partial\phi}{\partial t}+\sum_{i\geq 0}\frac{\partial\phi}{\partial u_i}\Big(\frac{\partial}{\partial x}\Big)^i\Big[t\frac{\partial u}{\partial t}\Big]\\
&=t\frac{\partial\phi}{\partial t}+\sum_{i\geq 0}\frac{\partial\phi}{\partial u_i}\Big(\frac{\partial}{\partial x}\Big)^i\Big[F\Big(t,x,u,\frac{\partial u}{\partial x}\Big)\Big]\\
&=t\frac{\partial\phi}{\partial t}+\sum_{i\geq 0}\frac{\partial\phi}{\partial u_i}D^i[F](t,x,u_0,\ldots,u_{i+1})\\
&=G\Big(t,x,\phi,D[\phi]\Big)=G\Big(t,x,w,\frac{\partial w}{\partial x}\Big).
\end{aligned}$$

This shows that $w(t,x)$ is a solution of (2.1). □

Denote by $\mathcal{S}_1$ the set of all solutions of (1.1), and by $\mathcal{S}_2$ the set of all solutions of (2.1): if we have a solution $\phi(t,x,u_0,u_1,\ldots)$ of (Φ) (resp. a solution $\psi(t,x,w_0,w_1,\ldots)$ of (Ψ)) we can define a mapping $\Phi : \mathcal{S}_1 \ni u \longmapsto w = \phi(t,x,u,\partial u/\partial x,\ldots) \in \mathcal{S}_2$ (resp. $\Psi : \mathcal{S}_2 \ni w \longmapsto u = \psi(t,x,w,\partial w/\partial x,\ldots) \in \mathcal{S}_1$). Thus, to define the mappings Φ and Ψ we must solve the equations (Φ) and (Ψ).

For $k \in \mathbb{N}^*$ and $R > 0$ we denote by $\mathcal{H}_{k,R}[t,u_0,\ldots,u_{k-1}]$ the set of all homogeneous polynomials of degree k in $(t,u_0,\ldots,u_{k-1})$ with holomorphic coefficients in a neighborhood of $D_R = \{x \in \mathbb{C}\,;\, |x| \le R\}$.

For $r > 0$, $c > 0$, $s > 0$ and $\varepsilon > 0$ we write

$$\begin{aligned} U_k(r,c,s,\varepsilon) = \big\{&(t,x,u_0,\ldots,u_{k-1}) \in \mathbb{C}\times\mathbb{C}\times\mathbb{C}^k\,;\, |t| \le r\varepsilon, |x| \le s,\\ &|u_0| \le 0!\varepsilon, |u_1| \le 1!\varepsilon/c,\ldots,|u_{k-1}| \le (k-1)!\varepsilon/c^{k-1}\big\},\\ W_k(c,s,\varepsilon) = \big\{&(t,x,w_0,\ldots,w_{k-1}) \in \mathbb{C}\times\mathbb{C}\times\mathbb{C}^k\,;\, |t| \le \varepsilon, |x| \le s,\\ &|w_0| \le 0!\varepsilon, |w_1| \le 1!\varepsilon/c,\ldots,|w_{k-1}| \le (k-1)!\varepsilon/c^{k-1}\big\} \end{aligned}$$

$(k = 1,2,\ldots)$. For a holomorphic function $f(t,x,u_0,\ldots,u_{k-1})$ on $U_k = U_k(r,c,s,\varepsilon)$ we define the norm $\|f\|_{U_k}$ by

$$\|f\|_{U_k} = \max_{U_k} |f(t,x,u_0,\ldots,u_{k-1})|.$$

The norm $\|g\|_{W_k}$ is defined in the same way. We have the following result.

Theorem 2.2. *Let $R > 0$ be sufficiently small. Suppose the conditions* A_1), A_2), A_3) *and*

$$|\,i + \lambda(x)(j-1)| \ge \sigma(i+j) \quad \text{on } D_R \qquad \text{for any } (i,j) \in \mathbb{N}\times\mathbb{N}\setminus\{(0,0),(0,1)\} \tag{2.2}$$

for some $\sigma > 0$. Then, we can find constants $r_0 > 0$ and $0 < c_0 < R$ so that the following results hold.

(1) *The coupling equation* (Φ) *has a unique formal solution of the form*

$$\phi = \frac{-a(x)}{1-\lambda(x)}\,t + u_0 + \sum_{k\ge 2} \phi_k(t,x,u_0,\ldots,u_{k-1}) \tag{2.3}$$

$$\text{with } \phi_k(t,x,u_0,\ldots,u_{k-1}) \in \mathcal{H}_{k,R}[t,u_0,\ldots,u_{k-1}] \ (k = 2,3,\ldots);$$

moreover, for any $0 < r \le r_0$, $0 < c \le c_0$ and $s = R - c$ there is an $\varepsilon > 0$ such that

$$\sum_{k\ge 1} \|\phi_k\|_{U_k} \quad \text{with } U_k = U_k(r,c,s,\varepsilon)$$

is convergent, where $\phi_1 = -a(x)/(1-\lambda(x))t + u_0$.

(2) *The coupling equation* (Ψ) *has a unique formal solution of the form*

$$\psi = \frac{a(x)}{1-\lambda(x)}\, t + w_0 + \sum_{k\geq 2} \psi_k(t,x,w_0,\ldots,w_{k-1}) \tag{2.4}$$

with $\psi_k(t,x,w_0,\ldots,w_{k-1}) \in \mathcal{H}_{k,R}[t,w_0,\ldots,w_{k-1}]$ $(k=2,3,\ldots)$;

moreover, for any $r>0$, $0<c\leq c_0$ *and* $0<s<R$ *there is an* $\varepsilon>0$ *such that*

$$\sum_{k\geq 1} \|\psi_k\|_{W_k} \quad \text{with } W_k = W_k(c,s,\varepsilon)$$

is convergent, where $\psi_1 = a(x)/(1-\lambda(x))t + w_0$.

(3) *Moreover, we have the following equalities:* $u_0 = \psi(t,x,\phi,D\phi,D^2\phi,\ldots)$ *as a function with respect to the variables* $(t,x,u_0,u_1,u_2,\ldots)$, *and also* $w_0 = \phi(t,x,\psi,D\psi, D^2\psi,\ldots)$ *as a function with respect to the variables* $(t,x,w_0,w_1,w_2,\ldots)$.

The proof will be published in [5]. We will give here only a construction of the formal solution. The proof of the convergence is done by the majorant method; but the details are very much complicated.

Construction of the formal solution of (Φ). By the conditions A_1), A_2) and A_3) we have the expession

$$F(t,x,u_0,u_1) = a(x)t + \lambda(x)u_0 + \sum_{i+j+\alpha\geq 2} c_{i,j,\alpha}(x) t^i {u_0}^j {u_1}^\alpha \tag{2.5}$$

where $a(x)$, $\lambda(x)$ and $c_{i,j,\alpha}(x)$ $(i+j+\alpha\geq 2)$ are all holomorphic functions in a neighborhood of D_R. We set

$$R_p(t,x,u_0,u_1) = \sum_{i+j+\alpha=p} c_{i,j,\alpha}(x) t^i {u_0}^j {u_1}^\alpha \in \mathcal{H}_{p,R}[t,u_0,u_1], \quad p\geq 2.$$

Then, we have $F(t,x,u_0,u_1) = a(x)t + \lambda(x)u_0 + \sum_{p\geq 2} R_p(t,x,u_0,u_1)$ and so

$$\begin{aligned} &D^m[F](t,x,u_0,\ldots,u_{m+1}) \\ &\quad = a^{(m)}(x)t + \sum_{0\leq i\leq m} \lambda_{m,i}(x)u_i + \sum_{p\geq 2} D^m[R_p](t,x,u_0,\ldots,u_{m+1}) \end{aligned} \tag{2.6}$$

for any $m\in\mathbb{N}$, where $a^{(m)}(x) = (\partial/\partial x)^m a(x)$. Thus, by substituting the unknown function $\phi(t,x,u_0,u_1,\ldots)$ into the coupling equation (Φ) we see that our coupling equation (Φ) is written in the form

$$\big(\tau - \lambda(x)\big)\phi = -\sum_{m\geq 0}\sum_{p\geq 2} D^m[R_p](t,x,u_0,\ldots,u_{m+1}) \frac{\partial\phi}{\partial u_m} \tag{2.7}$$

where

$$\tau = t\frac{\partial}{\partial t} + \sum_{m\geq 0}\Big(a^{(m)}(x)t + \sum_{0\leq i\leq m}\lambda_{m,i}(x)u_i\Big)\frac{\partial}{\partial u_m}$$

a vector field with infinitely many variables $(t, u_0, u_1, \ldots)$.

Now, let us solve the equation (2.7). Let

$$\phi = \sum_{k\geq 1}\phi_k(t,x,u_0,\ldots,u_{k-1}) \in \sum_{k\geq 1}\mathcal{H}_{k,R}[t,u_0,\ldots,u_{k-1}] \tag{2.8}$$

be the unknown function. Since $D^m[R_p](t,x,u_0,\ldots,u_{m+1})$ belongs in the class $\mathcal{H}_{p,R}[t,u_0,\ldots,u_{m+1}]$, by substituting (2.8) into (2.7) and by comparing the homogeneous part of degree k with respect to $(t,u_0,\ldots,u_{k-1})$ we see that (2.7) is decomposed into the following recurrent formulas:

$$\big(\tau_1 - \lambda(x)\big)\phi_1 = 0 \quad \text{in } \mathcal{H}_{1,R}[t,u_0] \tag{2.9}$$

and for $k \geq 2$

$$\begin{aligned}(2.10)\qquad &\big(\tau_k - \lambda(x)\big)\phi_k \\ &= -\sum_{1\leq q\leq k-1}\ \sum_{0\leq m\leq q-1} D^m\big[R_{k-q+1}\big](t,x,u_0,\ldots,u_{m+1})\frac{\partial\phi_q}{\partial u_m} \\ &\qquad\qquad\qquad\qquad \text{in } \mathcal{H}_{k,R}[t,u_0,\ldots,u_{k-1}],\end{aligned}$$

where

$$\tau_k = t\frac{\partial}{\partial t} + \sum_{0\leq m\leq k-1}\Big(a^{(m)}(x)t + \sum_{0\leq i\leq m}\lambda_{m,i}(x)u_i\Big)\frac{\partial}{\partial u_m}, \quad k = 1,2,\ldots.$$

Thus, if we note the following lemma, we can get a formal solution (2.3) of (Φ).

Lemma 2.3. (1) *If $\lambda(x) \neq 1$ on D_R, the equation $(\tau_1 - \lambda(x))\phi_1 = 0$ has a solution $\phi_1 \in \mathcal{H}_{1,R}[t,u_0]$ of the form*

$$\phi_1 = \frac{-a(x)\beta(x)}{1-\lambda(x)}\,t + \beta(x)u_0$$

and $\beta(x)$ can be chosen arbitrarily.

(2) *Let $k \geq 2$. If $|\,i + \lambda(x)(j-1)| \neq 0$ on D_R for any $(i,j) \in \mathbb{N}\times\mathbb{N}$ with $i+j=k$, then for any $f_k \in \mathcal{H}_{k,R}[t,u_0,\ldots,u_{k-1}]$ the equation $(\tau_k - \lambda(x))\phi_k = f_k$ has a unique solution $\phi_k \in \mathcal{H}_{k,R}[t,u_0,\ldots,u_{k-1}]$.*

Construction of the formal solution of (Ψ). As in (2.5), we have the expression

$$F(t,x,w_0,w_1) = a(x)t + \lambda(x)w_0 + \sum_{i+j+\alpha\geq 2} c_{i,j,\alpha}(x)t^i {w_0}^j {w_1}^\alpha$$

where $a(x)$, $\lambda(x)$ and $c_{i,j,\alpha}(x)$ $(i+j+\alpha \geq 2)$ are all holomorphic functions in a neighborhood of D_R. Therefore, the coupling equation (Ψ) is expressed in the form

$$\big(\tau^* - \lambda(x)\big)\psi = a(x)t + \sum_{i+j+\alpha \geq 2} c_{i,j,\alpha}(x) t^i \psi^j \big(D[\psi]\big)^\alpha \tag{2.11}$$

where

$$\tau^* = t\frac{\partial}{\partial t} + \sum_{m \geq 0}\Big(\sum_{0 \leq i \leq m} \lambda_{m,i}(x) w_i\Big)\frac{\partial}{\partial w_m}$$

a vector field of infinitely many variables $(t, w_0, w_1, \ldots)$, and $\lambda_{m,i}(x) = m!/(i!(m-i)!)(\partial/\partial x)^{m-i}\lambda(x)$ $(0 \leq i \leq m)$. We note that $\lambda_{m,m}(x) = \lambda(x)$ holds for all $m = 0, 1, 2, \ldots$.

Let

$$\psi = \sum_{k \geq 1} \psi_k(t, x, w_0, \ldots, w_{k-1}) \in \sum_{k \geq 1} \mathcal{H}_{k,R}[t, w_0, \ldots, w_{k-1}] \tag{2.12}$$

be the unknown function. Then, by substituting (2.12) into (2.11) and by comparing the homogeneous parts of degree k with respect to $(t, w_0, \ldots, w_{k-1})$ we see that (2.11) is decomposed into the following recurrent formulas:

$$(\tau_1^* - \lambda(x))\psi_1 = a(x)t \quad \text{in } \mathcal{H}_{1,R}[t, w_0] \tag{2.13}$$

and for $k \geq 2$

$$\begin{aligned}(\tau_k^* - \lambda(x))\psi_k = \sum_{2 \leq i+j+\alpha \leq k} c_{i,j,\alpha}(x)\, t^i \Bigg[&\sum_{|p(j)|+|q(\alpha)|=k-i} \psi_{p_1} \times \\ &\times \cdots \times \psi_{p_j} \times D\big[\psi_{q_1}\big] \times \cdots \times D\big[\psi_{q_\alpha}\big]\Bigg] \\ \text{in } \mathcal{H}_{k,R}[t, w_0, \cdots, w_{k-1}],&\end{aligned} \tag{2.14}$$

where

$$\tau_k^* = t\frac{\partial}{\partial t} + \sum_{0 \leq m \leq k-1}\Big(\sum_{0 \leq i \leq m} \lambda_{m,i}(x) w_i\Big)\frac{\partial}{\partial w_m}, \quad k = 1, 2, \ldots,$$

$|p(j)| = p_1 + \cdots + p_j$ and $|q(\alpha)| = q_1 + \cdots + q_\alpha$. Thus, if we note the following lemma, we can get a formal solution (2.4) of (Ψ).

Lemma 2.4. (1) *If $\lambda(x) \neq 1$ on D_R, the equation $(\tau_1^* - \lambda(x))\psi_1 = a(x)t$ has a solution $\psi_1 \in \mathcal{H}_{1,R}[t, w_0]$ of the form*

$$\psi_1 = \frac{a(x)}{1 - \lambda(x)}\, t + \beta(x) w_0$$

and $\beta(x)$ can be chosen arbitrarily.

(2) *Let $k \geq 2$. If $|\, i+\lambda(x)(j-1)| \neq 0$ on D_R for any $(i,j) \in \mathbb{N}\times\mathbb{N}$ with $i+j=k$, then for any $f_k \in \mathcal{H}_{k,R}[t, w_0, \dots, w_{k-1}]$ the equation $(\tau_k^* - \lambda(x))\psi_k = f_k$ has a unique solution $\psi_k \in \mathcal{H}_{k,R}[t, w_0, \dots, w_{k-1}]$.*

§ 3. Equivalence of two PDEs

Let $\mathcal{F}$ and $\mathcal{G}$ be function spaces in which we can consider the following two partial differential equations:

$$\text{(A)} \qquad t\frac{\partial u}{\partial t} = F\Big(t, x, u, \frac{\partial u}{\partial x}\Big) \quad \text{in } \mathcal{F},$$

$$\text{(B)} \qquad t\frac{\partial w}{\partial t} = \lambda(x) w \quad \text{in } \mathcal{G}.$$

Set

$$\mathcal{S}_A = \text{ the set of all solutions of (A) in } \mathcal{F},$$
$$\mathcal{S}_B = \text{ the set of all solutions of (B) in } \mathcal{G}.$$

Then, if we can find function-spaces $\mathcal{F}$ and $\mathcal{G}$ so that the two mappings

$$\Phi : \mathcal{F} \ni u(t,x) \longmapsto w(t,x) = \phi(t, x, u, \partial u/\partial x, \dots) \in \mathcal{G},$$
$$\Psi : \mathcal{G} \ni w(t,x) \longmapsto u(t,x) = \psi(t, x, w, \partial w/\partial x, \dots) \in \mathcal{F}$$

are well defined, by Proposition 2.1 and Theorem 2.2 we see that the two mappings

$$\Phi : \mathcal{S}_A \ni u(t,x) \longmapsto w(t,x) = \phi(t, x, u, \partial u/\partial x, \dots) \in \mathcal{S}_B,$$
$$\Psi : \mathcal{S}_B \ni w(t,x) \longmapsto u(t,x) = \psi(t, x, w, \partial w/\partial x, \dots) \in \mathcal{S}_A$$

are well defined and that one is the inverse of the other. In this case, we say that two equations (A) and (B) are equivalent.

Let us intrduce such function-spaces. We denote by $\mathcal{R}(\mathbb{C}\setminus\{0\})$ the universal covering space of $\mathbb{C}\setminus\{0\}$, and we write: $S_\theta(r) = \{t \in \mathcal{R}(\mathbb{C}\setminus\{0\})\ ;\ |\arg t| < \theta, 0 < |t| < r\}$ and $D_R = \{x \in \mathbb{C}\ ;\ |x| \leq R\}$.

Definition 3.1. (1) We denote by $\mathscr{S}_0$ the set of all $u(t,x)$ satisfying the following i) and ii): i) $u(t,x)$ is a holomorphic function on $S_\theta(r) \times D_R$ for some $\theta > 0$, $r > 0$ and $R > 0$; and ii) we have

$$\max_{x \in D_R} |u(t,x)| = o(1) \quad (\text{as } t \longrightarrow 0 \text{ in } S_\theta(r)).$$

(2) We denote by $\mathscr{S}_+$ the set of all $u(t,x)$ satisfying the following i) and ii): i) $u(t,x)$ is a holomorphic function on $S_\theta(r)\times D_R$ for some $\theta>0$, $r>0$ and $R>0$; and ii) there is an $a>0$ such that

$$\max_{x\in D_R}|u(t,x)|=O(|t|^a)\quad (\text{as } t\longrightarrow 0 \text{ in } S_\theta(r)).$$

Then we have

Proposition 3.2. *Let $\phi(t,x,u_0,u_1,\ldots)$ and $\psi(t,x,w_0,w_1,\ldots)$ be the solutions in Theorem 2.2. Then the following two mappings are well defined:*

$$\Phi:\mathscr{S}_0\ (\text{resp.}\,\mathscr{S}_+)\ni u(t,x)\longmapsto w(t,x)=\phi(t,x,u,\partial u/\partial x,\ldots)\in\mathscr{S}_0\ (\text{resp.}\,\mathscr{S}_+),$$
$$\Psi:\mathscr{S}_0\ (\text{resp.}\,\mathscr{S}_+)\ni w(t,x)\longmapsto u(t,x)=\psi(t,x,w,\partial w/\partial x,\ldots)\in\mathscr{S}_0\ (\text{resp.}\,\mathscr{S}_+).$$

Thus, we have the following result.

Theorem 3.3 (Equivalence). *Suppose the conditions* A$_1$), A$_2$), A$_3$) *and*

$$|\,i+\lambda(x)(j-1)|\ge\sigma(i+j)\quad \text{on } D_R \quad \text{for any } (i,j)\in\mathbb{N}\times\mathbb{N}\setminus\{(0,0),(0,1)\} \tag{3.1}$$

for some $\sigma>0$and $R>0$. Then, the following two equations are equivalent:

$$t\,\frac{\partial u}{\partial t}=F\Big(t,x,u,\frac{\partial u}{\partial x}\Big)\quad \text{in } \mathscr{S}_0\ (\text{resp. } \mathscr{S}_+), \tag{3.2}$$

$$t\,\frac{\partial w}{\partial t}=\lambda(x)w\quad \text{in } \mathscr{S}_0\ (\text{resp. } \mathscr{S}_+). \tag{3.3}$$

In other words, if we denote by S_A the set of all solutions of (3.2) *and by S_B the set of all solutions of* (3.3)*, the following two mappings are bijective and one is the inverse of the other:*

$$\mathcal{S}_A \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \mathcal{S}_B$$

§ 4. Application

Let us consider the following Briot-Bouquet type partial differential equation

$$t\frac{\partial u}{\partial t}=F\Big(t,x,u,\frac{\partial u}{\partial x}\Big). \tag{4.1}$$

We denote by $\mathscr{S}_0((4.1))$ (resp. $\mathscr{S}_+((4.1))$ the set of all solutions of (4.1) belonging in the class $\mathscr{S}_0$ (resp. $\mathscr{S}_+$). Then we have

Theorem 4.1 (Structure of solutions). *Suppose the conditions* A_1), A_2), A_3) *and* (3.1). *Then we have*

$$\begin{aligned}\mathscr{S}_0((4.1)) &= \mathscr{S}_+((4.1)) \\ &= \begin{cases} \{\Psi[h(x)t^{\lambda(x)}]\ ;\ h(x) \in \mathbb{C}\{x\}\}, & \textit{when } Re\lambda(0) > 0, \\ \{\Psi[0]\}, & \textit{when } Re\lambda(0) \le 0. \end{cases}\end{aligned}$$

Proof. This follows from Theorem 3.3 and the following fact:

$$S_B = \begin{cases} \{h(x)t^{\lambda(x)}\ ;\ h(x) \in \mathbb{C}\{x\}\}, & \text{when } \mathrm{Re}\lambda(0) > 0, \\ \{0\}, & \text{when } \mathrm{Re}\lambda(0) \le 0. \end{cases}$$

□

Note that $\Psi[0] = \psi(t, x, 0, 0, \ldots)$ is nothing but the unique holomorphic solution $u_0(t, x)$ obtained in Theorem 1.1, and so we have

Corollary 4.2 (Analytic continuation). *Suppose the conditions* A_1), A_2), A_3), $\mathrm{Re}\lambda(0) \le 0$ *and* $\lambda(0) \notin (-\infty, 0]$. *If* $u(t, x)$ *is a solution of* (4.1) *on* $S_\theta(r) \times D_R$ *satisfying* $u(t, x) \longrightarrow 0$ *uniformly on* D_R *(as* $t \longrightarrow 0$ *in* $S_\theta(r)$*), then* $u(t, x)$ *has an analytic continuation up to some neighborhood of* $(0, 0) \in \mathbb{C}^2$.

The details will be published in [5].

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