RIMS Kôkyûroku Bessatsu
B37 (2013), 203–209

Nonlinear partial differential equations and logarithmic singularities

By

Hidetoshi Tahara* and Hideshi Yamane**

Abstract

We study logarithmic singularities of solutions to some nonlinear partial differential equations near noncharacteristic hypersurfaces. It is a logarithmic analogue of the Painlevé PDE test for integrable equations.

§ 1. Painlevé PDE test and WTC expansions

Weiss, Tabor and Carnevale ([7]) constructed a family of meromorphic solutions to some integrable equations. Let us review their calculation in the case of the KdV equation:

$$u_{ttt} - 6uu_t + u_x = 0 \quad (t, x \in \mathbb{R}). \tag{1.1}$$

For any real-analytic function $\psi(x)$, set $T = t - \psi(x)$. Then it has been proved in [7] that the equation (1.1) has a family of meromorphic solutions of the form

$$u = \frac{2}{T^2} - \frac{1}{6}\psi_x + gT^2 - \frac{1}{36}\psi_{xx}T^3 + hT^4 - \frac{1}{24}g_x T^5 + \dots, \tag{1.2}$$

Received June 25, 2008.
2000 Mathematics Subject Classification(s): Primary 35A20 ; Secondary 35L70
Key Words: singular solutions, Fuchsian equations, logarithmic singularities, nonlinear wave equations.
This research was partially supported by Grant-in-Aid for Scientific Research (No.16540169, No.17540182), Japan Society for the Promotion of Science. Parts of this work have been done during the authors' stay at Wuhan University. They thank Professor Chen Hua for hospitality and fruitful discussions.
*Department of Mathematics, Sophia University, Kioicho, Chiyoda-ku, Tokyo 102-8554, Japan.
Current address: Department of Information and Communication Sciences, Sophia University, Kioicho, Chiyoda-ku, Tokyo 102-8554, Japan.
**Department of Physics, Kwansei Gakuin University, Sanda 669-1337, Japan.
Current address: Department of Mathematical Sciences, Kwansei Gakuin University, Sanda 669-1337, Japan.

where $g = g(x)$ and $h = h(x)$ are arbitrary real-analytic functions. Since the function ψ is arbitrary, the noncharacteristic surface $T = t - \psi(x) = 0$ is a multi-dimensional analogue of moving singularities of the Painlevé equations.

Similar solutions exist for other integrable equations and their expansions are called WTC expansions. If an equation has a family of solutions which are expressed by WTC expansions, then it is said to pass the Painlevé PDE test. (See [1] and [7] for details).

When non-integrable equations are considered, different kinds of expansions have to be introduced. The solutions are generally multi-valued as is proved in [2] and [3]. We mainly follow the latter in the present paper.

§ 2. Kobayashi's theory ([3])

We introduce some notation following Kobayashi and explain his result. Let $(t, x) = (t, x_1, \ldots, x_n) \in \mathbb{C} \times \mathbb{C}^n$, fix $m \in \mathbb{N}^*$ and set $I_m = \{(j, \alpha) \in \mathbb{N} \times \mathbb{N}^n\,;\, j + |\alpha| \le m \text{ and } j < m\}$, $N =$ the cardinal of I_m, and $U = (U_{j,\alpha})_{(j,\alpha)\in I_m} \in \mathbb{C}^N$. Moreover, we set $\partial_t = \partial/\partial t$, $\partial_x^\alpha = (\partial/\partial x_1)^{\alpha_1} \ldots (\partial/\partial x_n)^{\alpha_n}$ for $\alpha = (\alpha_1, \ldots, \alpha_n)$.

We study nonlinear PDEs of the form

$$\partial_t^m u = f\Big(t, x, (\partial_t^j \partial_x^\alpha u)_{(j,\alpha)\in I_m}\Big). \tag{2.1}$$

Here $f(t, x, U)$ is holomorphic in $\{(t, x) \in \mathbb{C}_t \times \mathbb{C}_x^n; |t| < r_0, |x| < R_0\} \times \mathbb{C}_U^N$, where r_0 and R_0 are positive constants. (Although f is assumed to be a polynomial in [3], this condition can be relaxed).

The function f can be expanded in the variable U:

$$f(t, x, U) = \sum_{\mu\in\mathcal{M}} f_\mu(t, x) U^\mu, \quad \mu = (\mu_{j,\alpha})_{(j,\alpha)\in I_m}, \quad U^\mu = \prod_{(j,\alpha)\in I_m} U_{j,\alpha}^{\mu_{j,\alpha}} \tag{2.2}$$

for some subset $\mathcal{M}$ of $\mathbb{N}^N$, the set of $\mathbb{N}$-valued functions on I_m. We assume that $f_\mu(t, x)$ does not vanish identically.

Next we expand $f_\mu(t, x)$ in t:

$$f_\mu(t, x) = t^{k_\mu} \sum_{k=0}^{\infty} f_{\mu,k}(x) t^k. \tag{2.3}$$

We assume that $f_{\mu,0}(x)$ does not vanish identically.

We set

$$|\mu| = \sum_{(j,\alpha)\in I_m} \mu_{j,\alpha}, \quad \gamma(\mu) = \sum_{(j,\alpha)\in I_m} j\mu_{j,\alpha}, \quad r(\mu) = \frac{\gamma(\mu) - m - k_\mu}{|\mu| - 1} \ \ (\mu \in \mathcal{M}, |\mu| \ge 2).$$

Kobayashi ([3]) defined his exponent σ_c by

$$\sigma_c = \sup_{\mu\in\mathcal{M}, |\mu|\ge 2} r(\mu).$$

He assumed that σ_c was a rational number and imposed a kind of nonresonance condition on it. In particular, it must avoid the values $0, 1, 2, \dots, m-2$. Under some additional assumptions, he constructed solutions of the form

$$u = t^{\sigma_c} \sum_{j=0}^{\infty} u_j(x) t^{j/p}, \tag{2.4}$$

where u_n's are holomorphic in a common neighborhood of the origin, $u_0 \not\equiv 0$ and p is the smallest positive integer such that $p\sigma_c \in \mathbb{Z}$. A general noncharacteristic surface $t = \psi(x)$ can be transformed to $t = 0$ by a change of coordinates and the WTC expansion (1.2) for the KdV equation is just an example of (2.4).

Kichenassamy-Srinivasan ([2]) chose a different formulation and introduced expansions involving logarithms in the higher order terms.

§ 3. Logarithmic singularities

We consider the resonant case where $\sigma_c \in \{0, 1, 2, \dots, m-2\}$. The expansion is radically different from the one in (2.4) in that the leading term involves a logarithm.

We assume the following:

(A0) $\sigma_c = l \in \{0, 1, 2, \dots, m-2\}$.

(A1) $\mathcal{M}_0 \underset{\text{def}}{=} \{\mu \in \mathcal{M}; |\mu| \geq 2, \, r(\mu) = l (= \sigma_c)\}$ is non-empty.

(A2) If $\mu \in \mathcal{M}_0$ and $\mu_{j\alpha} \neq 0$, then $j \geq l+1$ and $\alpha = 0$.

(A3) For a sufficiently small positive constant $C > 0$, we have

$$m - l + k_\mu - \gamma(\mu) + l|\mu| \geq C \sum_{\substack{(j,\alpha)\in I_m \\ j \leq l}} \mu_{j,\alpha}$$

for any $\mu \in \mathcal{M} \setminus \mathcal{M}_0$. (This is the case if f is a polynomial).

We use the following notation:

- $\mathcal{R}(\mathbb{C} \setminus \{0\})$, the universal covering space of $\mathbb{C} \setminus \{0\}$,
- $S_\theta = \{t \in \mathcal{R}(\mathbb{C} \setminus \{0\}) \; ; \; |\arg t| < \theta\}$,
- $S(\varepsilon(y)) = \{t \in \mathcal{R}(\mathbb{C} \setminus \{0\}); 0 < |t| < \varepsilon(\arg t)\}$, where $\varepsilon(y)$ is a positive continuous function on $\mathbb{R}_y$,
- $D_r = \{x = (x_1, \dots, x_n) \in \mathbb{C}^n \; ; \; |x_i| < r \text{ for } i = 1, \cdots, n\}$.

Definition 3.1. $\widetilde{\mathcal{O}}_+$ denotes the set of all $v(t,x)$ satisfying the following two conditions:

i) $v(t,x)$ is a holomorphic function on $S(\varepsilon(y)) \times D_r$ for some positive continuous function $\varepsilon(y)$ on $\mathbb{R}_y$ and some constant $r > 0$.

ii) There exists a constant $a > 0$ such that for any $\tilde{r} \in]0, r[$ and $\theta > 0$ we have

$$\max_{x \in D_{\tilde{r}}} |v(t,x)| = O(|t|^a) \quad (\text{as } t \longrightarrow 0 \text{ in } S_\theta).$$

Our main result is the following:

Theorem 3.2. *Assume* (A0)–(A3) *and set* $\beta_{j,l} = (-1)^{j-l-1} l!(j-l-1)!$ *for* $j \geq l+1$. *Let* $A = a(x)$ *be a holomorphic solution to*

$$\sum_{\mu \in \mathcal{M}_0} f_{\mu,0}(x) \Big(\prod_{j=l+1}^{m-1} \beta_{j,l}^{\mu_{j,0}} \Big) A^{|\mu|-1} = \beta_{m,l} \tag{3.1}$$

in a neighborhood of $x = 0$. *Then, for any holomorphic function* $b(x)$ *in a neighborhood of* $x = 0$, *there exists a function* $v(t,x) \in \widetilde{\mathcal{O}}_+$ *such that*

$$\begin{aligned} u(t,x) &= a(x) t^l \log t + t^l b(x) + t^l v(t,x) \\ &= t^l \{a(x) \log t + b(x) + v(t,x)\} \end{aligned}$$

is a solution to (2.1).

Remark. The left hand side of (3.1) is entire in A. Picard's theorem in value distribution theory assures that there exists a solution to (3.1) in a generic case.

§ 4. Proof of Theorem 3.2

Set $u(t,x) = a(x)t^l \log t + t^l b(x) + t^l v(t,x)$. We shall derive an equation with a new unknown function $v(t,x)$. It is a nonlinear Fuchsian equation. Its coefficients are singular because they involve logarithms. We refer the reader to [6] for details.

§ 5. Moving singularities: nonlinear wave equation

In our main theorem, we claimed the existence of solutions with logarithmic singularities along $t = 0$. By a change of coordinates, we can construct solutions which are singular along other noncharacteristic hypersurfaces.

We consider

$$\Box u(s,y) = g(s,y; u, \partial_s u, \nabla_y u) \tag{5.1}$$

in an open set of $\mathbb{C}^{n+1} = \mathbb{C}_s \times \mathbb{C}^n_y$. Here $\Box = \partial^2/\partial s^2 - \sum_{i=1}^n \partial^2/\partial y_i^2$, $\nabla_y u = (\partial u/\partial y_1, \ldots, \partial u/\partial y_n)$. We assume that $g(s,y;z,\sigma,\eta)$ is a holomorphic function in all its arguments and is entire in (z,σ,η). Moreover we assume that it is a polynomial of degree 2 in (σ,η). Its homogeneous part of degree 2 is denoted by g_2.

Let $\psi(y)$ be a holomorphic function with

$$1 - \{\nabla_y \psi(y)\}^2 \neq 0, \tag{5.2}$$

where $\nabla_y \psi(y) = \big(\psi_1(y), \ldots, \psi_n(y)\big)$, $\psi_i(y) = \partial\psi(y)/\partial y_i \, (i = 1,2,\ldots,n)$, $\{\nabla_y \psi(y)\}^2 = \sum_{i=1}^n \psi_i(y)^2$. Moreover we assume that

$$g_2\big(\psi(y), y; 0, 1, -\nabla_y \psi(y)\big) \neq 0. \tag{5.3}$$

Theorem 5.1. *Assume* (5.2) *and* (5.3)*. Then, in a neighborhood of the hypersurface* $\Sigma = \{s = \psi(y)\}$*, there exists a family of solutions* $u(s,y)$ *to* (5.1) *with the asymptotic behavior*

$$u(s,y) \sim -\frac{1 - \{\nabla_y \psi(y)\}^2}{g_2\big(\psi(y), y; 0, 1, -\nabla_y \psi(y)\big)} \log\big(s - \psi(y)\big) \quad \textit{as} \quad s \to \psi(y).$$

Proof. Set $t = s - \psi(y)$, $x = y$, $\Psi = 1 - \{\nabla_y \psi(y)\}^2 (\neq 0)$. Then, we have $\partial_s = \partial_t$, $\partial_{y_i} = -\psi_i \partial_t + \partial_{x_i}$. We can apply Theorem 3.2 near $t = 0$. The assumption (5.3) corresponds to $k_\mu = 0$. Details are explained in [6]. ☐

§ 6. Moving singularities: third order case

Next we consider a third order equation. Let $P(s,y;\partial_s,\partial_y)$ be a linear partial differential operator of third order. Its principal symbol is denoted by $\sigma(P)$. Its coefficients are assumed to be holomorphic near $s = \psi(y)$. We consider

$$Pu = g\left(s,y;(\partial_s^j \partial_y^\alpha u)_{j+|\alpha|\leq 2}\right). \tag{6.1}$$

We assume that $g(s,y;(Y_{j,\alpha})_{j+|\alpha|\leq 2})$ is entire in $(Y_{j,\alpha})_{j+|\alpha|\leq 2}$. Moreover, we assume that g is a polynomial of degree 2 in $(Y_{j,\alpha})_{j+|\alpha|=1,2}$ and we denote its homogeneous part of degree 2 by g_2.

Set $Y^{(0)}_{j,\alpha} = (-\nabla_y \psi(y))^\alpha$ if $j + |\alpha| = 2$ and $Y^{(0)}_{j,\alpha} = 0$ otherwise. We assume

$$g_2\left(\psi(y), y; \big(Y^{(0)}_{j,\alpha}\big)_{j+|\alpha|\leq 2}\right) \neq 0 \tag{6.2}$$

$$\sigma(P)\big(s,y;1,-\nabla_y \psi(y)\big) \neq 0 \tag{6.3}$$

Theorem 6.1. *Assume* (6.2) *and* (6.3)*. Then, in a neighborhood of the hypersurface* $\Sigma = \{s = \psi(y)\}$*, there exists a family of solutions* $u(s, y)$ *to* (6.1) *with the asymptotic behavior*

$$u(s,y) \sim a(y) t \log t, \quad t = s - \psi(y), \quad a(y) = \frac{-\sigma(P)\big(s, y; 1, -\nabla_y \psi(y)\big)}{g_2\left(\psi(y), y; \big(Y^{(0)}_{j,\alpha}\big)\right)}$$

as $s \to \psi(y)$*.*

Proof. We set $t = s - \psi(y), x = y$ again. By using (6.3), we can rewrite (6.1) as

$$\partial_t^3 u = [\text{linear part}] + \sigma(P)^{-1} g\Big(t + \psi(x), x; \Big(\partial_t^j \big(-(\nabla_x \psi)\partial_t + \nabla_x\big)^\alpha u\Big)_{j+|\alpha|\le 2}\Big).$$

Here $\sigma(P) = \sigma(P)\big(t+\psi(x), x; 1, -\nabla_x\psi(x)\big)$ and $-(\nabla_x\psi)\partial_t + \nabla_x$ is the n-tuple of vector fields whose i-th component is $-(\partial\psi/\partial x_i)\partial_t + \partial_{x_i}$. Expanding the right hand side as in (2.2) and (2.3), we find the term

$$\sigma(P)^{-1} g_2\left(\psi(x), x; (Y^{(0)}_{j,\alpha})_{j+|\alpha|\le 2}\right) U_{2,0}^2, \quad U_{2,0} = \partial_t^2 u.$$

This term is $f_{\bar\mu,0}(x) U^{\bar\mu}$ ($\mu = \bar\mu, k = 0$) in the notation of (2.2) and (2.3), where we define $\bar\mu$ by

$$\bar\mu_{2,0} = 2, \quad \bar\mu_{j,\alpha} = 0 \text{ (otherwise)}.$$

The assumption (6.2) implies $k_{\bar\mu} = 0$, and we have $\gamma(\bar\mu) = 4, |\bar\mu| = 2, r(\bar\mu) = 1$.

We claim that $\mathcal{M}_0 = \{\bar\mu\}$. First, the assumption on g means $\sum_{j+|\alpha|=1,2} \mu_{j,\alpha} \le 2$ for each $\mu \in \mathcal{M}$. If $\mu_{2,0} = 2$, then $\mu_{j,\alpha} = 0\,(j + |\alpha| = 1, 2)$. It follows that $\gamma(\mu) = \gamma(\bar\mu)$. Since $|\mu| \ge |\bar\mu|$ and $k_\mu \ge k_{\bar\mu} = 0$, we obtain the estimate $r(\mu) \le r(\bar\mu) = 1$, the equality being true if and only if $\mu = \bar\mu$. On the other hand, if $\mu_{2,0} \le 1$, then we have $\gamma(\mu) \le |\mu|$ and $r(\mu) < 1$.

We have $f_{\bar\mu,0}(x) = \sigma(P)^{-1} g_2\left(\psi(x), x, (Y^{(0)}_{j,\alpha})\right)$. So (3.1) becomes

$$\sigma(P)^{-1} g_2(\psi, x, (Y^{(0)}_{j,\alpha}))A = -1.$$

Hence we have $A = a(x) = -\sigma(P)/g_2\big(\psi(x), x, (Y^{(0)}_{j,\alpha})\big), \quad u \sim a(x) t \log t.$ □

References

[1] Ablowitz M. J. and Clarkson P. A., *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, *London Mathematical Society Lecture Note Series* 149, Cambridge University Press, 1996.

[2] Kichenassamy S. and Srinivasan G. K., The structure of WTC expansions and applications, *J. Phys. A*, **28**(1995), 1977-2004.

[3] Kobayashi T., Singular solutions and prolongation of holomorphic solutions to nonlinear differential equations, *Publ. Res. Inst. Math. Sci.*, **34**(1998), 43-63.

[4] Tahara H., On the singularities of solutions of nonlinear partial differential equations in the complex domain, *Microlocal Analysis and Complex Fourier Analysis*, World Sci. Publ., Hackensack, NJ (2002), 273-283.

[5] Tahara H., On the singularities of solutions of nonlinear partial differential equations in the complex domain, II, *Differential equations & asymptotic theory in mathematical physics*, Ser. Anal., World Sci. Publ., Hackensack, NJ (2004), 343-354.

[6] Tahara H. and Yamane H., Logarithmic singularities of solutions to nonlinear partial differential equations , *J. Math. Soc. Japan*, **60** (2) (2008), 603-630.

[7] Weiss J., Tabor, M. and Carnevale, G., The Painlevé property for partial differential equations, *J. Math. Phys.* **24** (1983), 522-526.

[8] Yamane H., Nonlinear wave equations and singular solutions, *Proc. Amer. Math. Soc.*, **135**(11) (2007), 3659-3667.