RIMS Kôkyûroku Bessatsu
B37 (2013), 223–229

On generating functions for certain sums of multiple zeta values and a formula of S. Zlobin

By

Noriko Wakabayashi*

Abstract

Generating functions for certain sums of multiple zeta-star values are discussed and an alternative proof of a formula of S. Zlobin is given.

§ 1. Generating functions for certain sums of multiple zeta values

The multiple zeta values are real numbers first considered in Euler [5]. These numbers have been appeared in various fields in mathematics and physics. There are many linear relations among these values with rational coefficients. One of the main problems in this area is to clarify the all relations among multiple zeta values.

The multiple zeta values are natural generalization of Riemann zeta values. There are two types of definition for multiple zeta values. For a multi-index $\mathbf{k} = (k_1, k_2, \ldots, k_n)$ $(k_i \in \mathbf{N}, k_1 \geq 2)$, the multiple zeta values are defined by

$$\zeta(\mathbf{k}) = \zeta(k_1, k_2, \ldots, k_n) = \sum_{m_1 > m_2 > \ldots > m_n > 0} \frac{1}{{m_1}^{k_1} {m_2}^{k_2} \cdots {m_n}^{k_n}} \tag{1.1}$$

and

$$\zeta^{\star}(\mathbf{k}) = \zeta^{\star}(k_1, k_2, \ldots, k_n) = \sum_{m_1 \geq m_2 \geq \ldots \geq m_n \geq 1} \frac{1}{{m_1}^{k_1} {m_2}^{k_2} \cdots {m_n}^{k_n}}. \tag{1.2}$$

Received September 30, 2008. Revised December 19, 2008.
2000 Mathematics Subject Classification(s): Primary 11M06, 40B05; Secondary 33C05.
Key Words: multiple zeta values, generalized hypergeometric functions.
Supported by Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists Grant-in-Aid No. 06J01019.
*Department of Mathematics, Kinki University, Osaka 577-8502, Japan.
e-mail: noriko@math.kindai.ac.jp
Current address: Faculty of Engineering, Kyushu Sangyo University, Fukuoka 813-8503, Japan.
e-mail: noriko@ip.kyusan-u.ac.jp

The former is normally called a multiple zeta value and used mainly in the mathematical literatures in last two decades and the latter is called multiple zeta-star value (MZV and MZSV, respectively, for short). There exist natural $\mathbf{Q}$-linear relations between MZVs and MZSVs. Any MZSV can be expressed as a linear combination of MZVs and vice versa. Therefore two $\mathbf{Q}$-vector spaces generated by both values coincide with each other. On the other hand, there are many $\mathbf{Q}$-linear relations that hold among multiple zeta values. Some part of these relations can be described clearly by using MZSVs.

For any multi-index $\mathbf{k} = (k_1, k_2, \ldots, k_n)$ $(k_i \in \mathbf{N})$, the weight, depth and i-height of $\mathbf{k}$ are by definition the integers $\mathrm{wt}(\mathbf{k}) = k_1 + k_2 + \cdots + k_n = k$, $\mathrm{dep}(\mathbf{k}) = n$ and $i\text{-ht}(\mathbf{k}) = \sharp\{l | k_l \geq i+1\}$, respectively. The generalized height i-ht is defined in Li [6]. If $i = 1$, it is introduced by Ohno-Zagier [7] and usually called (1-)height. An index $\mathbf{k} = (k_1, k_2, \ldots, k_n)$ is called admissible, if $k_1 \geq 2$.

Let r be a positive integer. For any integers $k, n, h_1, h_2, \ldots, h_r (k, n, h_i \geq 0)$, we set

$$I(k, n, h_1, h_2, \ldots, h_r) = \{\mathbf{k} \mid \mathrm{wt}(\mathbf{k}) = k, \mathrm{dep}(\mathbf{k}) = n, 1\text{-ht}(\mathbf{k}) = h_1, \ldots, r\text{-ht}(\mathbf{k}) = h_r\},$$
$$I_j(k, n, h_1, h_2, \ldots, h_r) = \{\mathbf{k} = (k_1, k_2, \ldots, k_n) \mid \mathbf{k} \in I(k, n, h_1, h_2, \ldots, h_r), k_1 \geq j+2\},$$

where $j = 0, 1, 2, \ldots, r-1$.

Ohno-Zagier [7] gives a concrete expression of the generating function for sums of multiple zeta values of fixed weight, depth and 1-height;

$$\begin{aligned}(1.3)\quad \sum_{k,n,h_1\geq 0} &\left(\sum_{\mathbf{k}\in I_0(k,n,h_1)} \zeta(\mathbf{k})\right) x^{k-n-h_1} y^{n-h_1} z^{h_1-1} \\ &= \frac{1}{xy-z}\left\{1 - \exp\left(\sum_{n=2}^{\infty} \frac{\zeta(n)}{n}(x^n + y^n - a_1^n - a_2^n)\right)\right\} \\ &= \frac{1}{xy-z}\left(1 - {}_2F_1\left(\begin{matrix} a_1 - x, a_2 - x \\ 1-x \end{matrix}; 1\right)\right),\end{aligned}$$

where a_1 and a_2 are defined by $a_1 + a_2 = x + y$ and $a_1 a_2 = z$, ${}_2F_1\left(\begin{smallmatrix} a,b \\ c \end{smallmatrix}; t\right)$ denotes the Gauss hypergeometric function ${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; t\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!}$, and $(a)_n$ means the Pochhammer symbol $(a)_n = a(a+1)(a+2)\cdots(a+n-1)$. In particular, all of the coefficients of left-hand side of the above generating function can be expressed as polynomials in Riemann zeta values $\zeta(2), \zeta(3), \ldots$ with rational coefficients.

On the other hand, Aoki-Kombu-Ohno [1] (see also [2]) considers a counterpart of the result of Ohno-Zagier for multiple zeta-star values and obtains the following expression of generating function:

$$\sum_{k,n,h_1\geq 0}\left(\sum_{\mathbf{k}\in I_0(k,n,h_1)}\zeta^\star(\mathbf{k})\right)x^{k-n-h_1}y^{n-h_1}z^{2h_1-2} = \frac{1}{(1-x)(1-y)-z^2}\,{}_3F_2\left(\begin{matrix}1-x,1,1\\ b_1+2,b_2+2\end{matrix};1\right), \tag{1.4}$$

where b_1 and b_2 are defined by $b_1+b_2=-(x+y)$ and $b_1b_2=-(z^2-xy)$, and ${}_{r+1}F_r\left(\begin{smallmatrix}a_1,a_2,\ldots,a_{r+1}\\ b_1,b_2,\ldots,b_r\end{smallmatrix};t\right)$ denotes the generalized hypergeometric function

$${}_{r+1}F_r\left(\begin{matrix}a_1,a_2,\ldots,a_{r+1}\\ b_1,b_2,\ldots,b_r\end{matrix};t\right)=\sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n\cdots(a_{r+1})_n}{(b_1)_n(b_2)_n\ldots(b_r)_n}\frac{t^n}{n!} \tag{1.5}$$

for any positive integer r.

Li [6] generalizes the generating function of Ohno-Zagier and gives an expression of it in terms of generalized hypergeometric functions ${}_{r+1}F_r\left(\begin{smallmatrix}a_1,a_2,\ldots,a_{r+1}\\ b_1,b_2,\ldots,b_r\end{smallmatrix};t\right)$.

In this paper, we consider generating functions of multiple zeta values of fixed weight, depth and i-heights and introduce expressions of them by using generalized hypergeometric functions. This part is an announcement of the author's joint work [3] with Aoki and Ohno. Moreover we give an alternative proof of a formula of Zlobin by specializing the expression.

We define a sum X_0 for any integers $k,n,h_1,h_2,\ldots,h_r$ $(k,n,h_i\geq 0)$, and its generating function $\Phi_0^\star$ as follows:

$$X_0(k,n,h_1,h_2,\ldots,h_r)=\sum_{\mathbf{k}\in I_0(k,n,h_1,h_2,\ldots,h_r)}\zeta^\star(\mathbf{k}), \tag{1.6}$$

$$\begin{aligned}\Phi_0^\star &= \Phi_0^\star(x_1,x_2,\ldots,x_{r+2})\\ &= \sum_{\substack{k\geq n+\sum_{i=1}^r h_i\\ n\geq h_1\geq h_2\geq\ldots\geq h_r\geq 0,\ h_1\geq 1}} X_0(k,n,h_1,h_2,\ldots,h_r)\\ &\qquad\times {x_1}^{k-n-\sum_{i=1}^r h_i}{x_2}^{n-h_1}{x_3}^{h_1-h_2}\cdots {x_{r+1}}^{h_{r-1}-h_r}{x_{r+2}}^{h_r}.\end{aligned} \tag{1.7}$$

In the above definitions, the sum is treated as 0 whenever the index set is empty. Our main result of [3] will then be

Theorem 1.1. *Let r be a positive integer. Then we obtain*

$$\Phi_0^\star = \left\{ 1 - (x_1 + x_2) - \sum_{q=0}^{r-1} (x_{r+2-q} - x_1 x_{r+1-q}) \right\}^{-1} \times \sum_{m=0}^{r-1} A_m \left(\frac{d}{dt} \right)^m t \; {}_{r+2}F_{r+1} \left(\left. \begin{matrix} 1 - x_1, \overbrace{1, \dots, 1}^{r+1} \\ b_1 + 2, b_2 + 2, \dots, b_{r+1} + 2 \end{matrix} ; t \right) \right|_{t=1}, \tag{1.8}$$

where b_i $(i = 1, \dots, r+1)$ are defined by

$$\begin{cases} b_1 + b_2 + \cdots + b_{r+1} = -(x_1 + x_2), \\ \displaystyle\sum_{1 \le i_1 < i_2 < \dots < i_j \le r+1} b_{i_1} b_{i_2} \cdots b_{i_j} = -(x_{j+1} - x_1 x_j), \;\; j = 2, 3, \dots, r+1, \end{cases} \tag{1.9}$$

A_m is expressed by using the Stirling numbers of the second kind as

$$A_m = \sum_{j=m}^{r-1} (x_{r+2-j} - x_1 x_{r+1-j}) \left\{ \begin{matrix} j \\ m \end{matrix} \right\} + x_1 x_2 \left\{ \begin{matrix} r-1 \\ m \end{matrix} \right\}, \tag{1.10}$$

and the Stirling number of the second kind $\left\{ \begin{smallmatrix} n \\ m \end{smallmatrix} \right\}$ is the number of ways of partitioning a set of n elements into m non-empty subsets.

§ 2. A formula of S. Zlobin

In some special cases, we can explicitly evaluate $\Phi_0^\star$ by using Theorem 1.1. For example, we get one of the results of Zlobin [9]. By setting $r = l-1$, $x_1 = x_2 = \cdots = x_l = 0$, $x_{l+1} = x$, we have

$$\begin{cases} \mathrm{wt}(\mathbf{k}) = ln, \\ \mathrm{dep}(\mathbf{k}) = n, \\ 1\text{-}\mathrm{ht}(\mathbf{k}) = 2\text{-}\mathrm{ht}(\mathbf{k}) = \cdots = (l-1)\text{-}\mathrm{ht}(\mathbf{k}) = n, \\ l\text{-}\mathrm{ht}(\mathbf{k}) = \cdots = 0. \end{cases}$$

From the definition of $\Phi_0^\star$, we have

$$\begin{aligned} \Phi_0^\star &= \sum_{n=1}^{\infty} X_0(ln, \underbrace{n, n, \dots, n}_{l-1}) x^n \\ &= \sum_{n=1}^{\infty} \zeta^\star(\underbrace{l, l, \dots, l}_{n}) x^n. \end{aligned}$$

On the other hand, substituting $l-1$ for r of Theorem 1.1 ($x_1 = x_2 = \cdots = x_l = 0,\ x_{l+1} = x$), we get the following:

$$\begin{aligned}
\Phi_0^\star &= \frac{x}{1-x} t \left[{}_{l+1}F_l \left(\begin{matrix} \overbrace{1,1,\ldots,1}^{l+1} \\ b_1+2, b_2+2, \ldots, b_l+2 \end{matrix} ; t \right) \right] \Bigg|_{t=1} \\
&= \frac{x}{1-x} \sum_{m=0}^{\infty} \frac{(m!)^{l+1}}{(b_1+2)_m (b_2+2)_m \cdots (b_l+2)_m m!} \\
&= \frac{x}{1-x} \sum_{m=0}^{\infty} \frac{(m!)^{l}}{(b_1+2)_m (b_2+2)_m \cdots (b_l+2)_m}, \qquad (2.1)
\end{aligned}$$

where b_i $(i = 1, \ldots, l)$ are defined by

$$\begin{cases} b_1 + b_2 + \cdots + b_l = 0, \\ \displaystyle\sum_{1 \le i_1 < i_2 < \ldots < i_j \le l} b_{i_1} b_{i_2} \cdots b_{i_j} = 0, j = 2, 3, \ldots, l-1, \\ b_1 b_2 \cdots b_l = -x. \end{cases} \qquad (2.2)$$

Putting $f(z) = (z+b_1)(z+b_2)(z+b_3)\cdots(z+b_l) = z^l - x$, we obtain

$$(b_1+2)_m (b_2+2)_m \cdots (b_l+2)_m = \prod_{i=1}^{l} \prod_{j=2}^{m+1} (b_i + j).$$

Interchanging two products, we have

$$\prod_{j=2}^{m+1} \prod_{i=1}^{l} (b_i + j) = \prod_{j=2}^{m+1} f(j) = (2^l - x)(3^l - x) \cdots ((m+1)^l - x).$$

Applying the above relation to the denominator of (2.1), we get

$$\begin{aligned}
\Phi_0^\star &= \frac{x}{1-x} \sum_{m=0}^{\infty} \frac{(m!)^l}{(2^l - x)(3^l - x)\cdots((m+1)^l - x)} \\
&= \sum_{m=0}^{\infty} \frac{1^l 2^l \cdots m^l x}{(1^l - x)(2^l - x)\cdots((m+1)^l - x)}.
\end{aligned}$$

Multiplying $(m+1)^l$ to the numerator and the denominator, the above equality becomes

$$\begin{aligned}
\sum_{m=0}^{\infty} \left(\prod_{j=1}^{m+1} \frac{j^l}{j^l - x} \right) \frac{x}{(m+1)^l} &= \sum_{m=0}^{\infty} \left(\prod_{j=1}^{m+1} \left(1 - \frac{x}{j^l}\right)^{-1} \right) \frac{x}{(m+1)^l} \\
&= \sum_{m=1}^{\infty} \left(\prod_{j=1}^{m} \left(1 - \frac{x}{j^l}\right)^{-1} \right) \frac{x}{m^l}. \qquad (2.3)
\end{aligned}$$

To complete our computation, we need the following formula obtained by Euler [4]:

$$1+\sum_{m=1}^{\infty}\frac{a_m}{(1-a_1)(1-a_2)\cdots(1-a_m)}=\prod_{m=1}^{\infty}(1-a_m)^{-1}, \tag{2.4}$$

where we assume that $\sum|a_m|$ is convergent.

The following equation is immediately obtained when we apply (2.4) to equation (2.3).

$$\Phi_0^{\star}=\prod_{j=1}^{\infty}\left(1-\frac{x}{j^l}\right)^{-1}-1.$$

Thus we obtain

$$1+\sum_{n=1}^{\infty}\zeta^{\star}(\underbrace{l,l,\dots,l}_{n})x^n=\prod_{j=1}^{\infty}\left(1-\frac{x}{j^l}\right)^{-1}. \tag{2.5}$$

Zlobin [9] (see also Vasil'ev [8]) obtains (2.5) in the course of the proof of his formula;

$$\zeta^{\star}(\{2,\{1\}_{l-2}\}_m,1)=l\zeta(lm+1), \tag{2.6}$$

where we denote $(\underbrace{k,\dots,k}_{n})$ by $\{k\}_n$. Therefore our specialization of Theorem 1.1 provides another proof of (2.5).

Acknowledgements

The author expresses her sincere thanks to Professor Kentaro Ihara for his advice on Euler's results. Thanks are also due to Professor Wadim Zudilin for his helpful advice on notation and encouragement. The author is grateful to collaborators Professors Takashi Aoki and Yasuo Ohno. This work was supported by Grant-in-Aid for Japan Society for the Promotion of Science Fellows.

References

[1] Aoki, T., Kombu, Y. and Ohno, Y., A generating function for sum of multiple zeta values and its applications, *Proc. Amer. Math. Soc.*, **136** (2008), 387–395.

[2] Aoki, T. and Ohno, Y., Sum relations for multiple zeta values and connection formulas for the Gauss hypergeometric functions, *Publ. Res. Inst. Math. Sci.*, **41** (2005), 329–337.

[3] Aoki, T., Ohno, Y. and Wakabayashi, N., On generating functions of multiple zeta values and generalized hypergeometric functions, *Manuscripta Math.*, **134** (2011), 139–155.

[4] Euler, L., Demonstratio theorematis circa ordinem in summis divisorum obserbatum, *Novi Comm. Acad. Sci. Petropol*, **5** (1760), 75–83. Reprinted in Opera Omnia ser. I, vol. 2, B. G. Teubner, Berlin (1915), 390–398.

[5] Euler, L., Meditationes circa singulare serierum genus, *Novi Comm. Acad. Sci. Petropol*, **20** (1775), 140–186. Reprinted in Opera Omnia ser. I, vol. 15, B. G. Teubner, Berlin (1927), 217–267.

[6] Li, Z. h., Sum of multiple zeta values of fixed weight, depth and i-height, *Math. Z.*, **258** (2008), 133–142.

[7] Ohno, Y. and Zagier, D., Multiple zeta values of fixed weight, depth, and height, *Indag. Math.*, **12** (2001), 483–487.

[8] Vasil'ev, D. V., Some formulas for Riemann zeta function at integer points, *Vestnik Moskov. Univ. Ser. I Mat. Mekh.*, (1996), 81–84; English transl., *Moscow Univ. Math. Bull.* **51** (1996), 41–43.

[9] Zlobin, S. A., Generating functions for values of a multiple zeta function, *Vestnik Moskov. Univ. Ser. I Mat. Mekh.*, (2005), 55–59; English transl., *Moscow Univ. Math. Bull.* **60** (2005), 44–48.