RIMS Kôkyûroku Bessatsu
B38 (2013), 1–13

Construction of generators for the module of liftable vector fields

By

Yusuke Mizota *

§ 1. Introduction

Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. In this paper, unless otherwise stated, all mappings are assumed to be smooth (that is, of class C^∞ if $\mathbb{K} = \mathbb{R}$ or holomorphic if $\mathbb{K} = \mathbb{C}$). Let S be a finite subset of $\mathbb{K}^n$ and $|S|$ the number of distinct points of S. A map germ $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ is called a *multigerm*. When $S = \{s_1, s_2, \ldots, s_r\}$, a multigerm $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ is identified with $(f_1, f_2, \ldots, f_r)$, where $f_i : (\mathbb{K}^n, s_i) \to (\mathbb{K}^p, 0)$ $(i = 1, 2, \ldots, r)$. By considering the coordinate change by parallel translation of the source of f_i, a multigerm $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ is identified with $(f_1, f_2 \ldots, f_r)$, where $f_i : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ $(i = 1, 2, \ldots, r)$. In this paper, for a multigerm $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ given by $f = (f_1, f_2 \ldots, f_r)$ with $f_i : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ $(i = 1, 2, \ldots, r)$, we consider S to be a set consisting of r distinct points. For example, $f : (\mathbb{K}, S) \to (\mathbb{K}^2, 0)$ given by $f_1(x) = (x^2, x^3)$, $f_2(x) = (-x^3, x^2)$, means that $S = \{s_1, s_2\}$ $(s_1 \neq s_2)$ and $f_1(x) = ((x - s_1)^2, (x - s_1)^3)$, $f_2(x) = (-(x - s_2)^3, (x - s_2)^2)$.

Let C_S (resp., C_0) be the set of function germs $(\mathbb{K}^n, S) \to \mathbb{K}$ (resp., $(\mathbb{K}^p, 0) \to \mathbb{K}$), and let m_S (resp., m_0) be the subset of C_S (resp., C_0) consisting of function germs $(\mathbb{K}^n, S) \to (\mathbb{K}, 0)$ (resp., $(\mathbb{K}^p, 0) \to (\mathbb{K}, 0)$). The sets C_S and C_0 have natural $\mathbb{K}$-algebra structures. For a multigerm $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$, let $f^* : C_0 \to C_S$ be the $\mathbb{K}$-algebra homomorphism defined by $f^*(\psi) = \psi \circ f$.

For a map germ $f : (\mathbb{K}^n, S) \to \mathbb{K}^p$, let $\theta_S(f)$ be the C_S-module consisting of germs of vector fields along f. Note that $\theta_S(f)$ is identified with the direct sum of p copies of C_S. Put $\theta_S(n) = \theta_S(\mathrm{id}_{(\mathbb{K}^n,S)})$ and $\theta_0(p) = \theta_{\{0\}}(\mathrm{id}_{(\mathbb{K}^p,0)})$, where $\mathrm{id}_{(\mathbb{K}^n,S)}$ (resp., $\mathrm{id}_{(\mathbb{K}^p,0)}$) is the germ of the identity mapping of $(\mathbb{K}^n, S)$ (resp., $(\mathbb{K}^p, 0)$). For a multigerm

Received February 23, 2012. Revised April 02, 2012.
2000 Mathematics Subject Classification(s):
*Graduate School of Mathematics, Kyushu University, 744, Motooka, Nishi-ku, Fukuoka 819-0395, Japan.

$f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$, following Mather [5], we define tf and ωf as

$$tf : \theta_S(n) \to \theta_S(f), \quad tf(\eta) = df \circ \eta,$$

$$\omega f : \theta_0(p) \to \theta_S(f), \quad \omega f(\xi) = \xi \circ f,$$

where df is the differential of f. Following Wall [8], we put

$$T\mathcal{R}_e(f) = tf(\theta_S(n)),$$

$$T\mathcal{L}_e(f) = \omega f(\theta_0(p)),$$

$$T\mathcal{A}_e(f) = T\mathcal{R}_e(f) + T\mathcal{L}_e(f).$$

For a multigerm $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$, a vector field $\xi \in \theta_0(p)$ is said to be *liftable over* f if $\xi \circ f \in T\mathcal{R}_e(f)$. The set of vector fields liftable over f is denoted by L_f. Note that L_f has a natural C_0-module structure. Let $(x_1, x_2, \ldots, x_n)$ (resp., $(X_1, \ldots, X_p)$) be the standard local coordinates of $\mathbb{K}^n$ (resp., $\mathbb{K}^p$) at the origin. Sometimes (x_1, x_2) (resp., (X_1, X_2)) is denoted by (x, y) (resp., (X, Y)). We see easily that

$$\xi = (\psi_1(X_1, X_2, \ldots, X_p), \cdots, \psi_p(X_1, X_2, \ldots, X_p)) \in L_f,$$

where $\psi_j : (\mathbb{K}^p, 0) \to \mathbb{K}$ $(j = 1, 2, \ldots, p)$, if and only if for every $i \in \{1, \ldots, r\}$ there exist a vector field

$$\eta_i = (\phi_{i,1}(x_1, x_2, \ldots, x_n), \ldots, \phi_{i,n}(x_1, x_2, \ldots, x_n)),$$

where $\phi_{i,k} : (\mathbb{K}^n, 0) \to \mathbb{K}$ $(k = 1, 2, \ldots, n)$, such that $\xi \circ f_i = df_i \circ \eta_i$ i.e.

$$\begin{pmatrix} \psi_1(X_1, X_2, \ldots, X_p) \\ \vdots \\ \psi_p(X_1, X_2, \ldots, X_p) \end{pmatrix} \circ f_i$$

$$= \begin{pmatrix} \frac{\partial(X_1 \circ f_i)}{\partial x_1} & \cdots & \frac{\partial(X_1 \circ f_i)}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial(X_p \circ f_i)}{\partial x_1} & \cdots & \frac{\partial(X_p \circ f_i)}{\partial x_n} \end{pmatrix} \begin{pmatrix} \phi_{i,1}(x_1, x_2, \ldots, x_n) \\ \vdots \\ \phi_{i,n}(x_1, x_2, \ldots, x_n) \end{pmatrix}.$$

Example 1.1. Let $f : (\mathbb{K}^2, 0) \to (\mathbb{K}^2, 0)$ be given by $f(x, y) = (x, y^3 + xy)$. Then, the following vector fields are liftable over f:

$$\begin{pmatrix} 2X \\ 3Y \end{pmatrix}, \begin{pmatrix} -9Y \\ 2X^2 \end{pmatrix}.$$

This is verified as follows:

$$\begin{pmatrix} 2X \\ 3Y \end{pmatrix} \circ f = \begin{pmatrix} 2x \\ 3y^3 + 3xy \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ y & 3y^2 + x \end{pmatrix} \begin{pmatrix} 2x \\ y \end{pmatrix},$$

$$\begin{pmatrix} -9Y \\ 2X^2 \end{pmatrix} \circ f = \begin{pmatrix} -9y^3 - 9xy \\ 2x^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ y & 3y^2 + x \end{pmatrix} \begin{pmatrix} -9y^3 - 9xy \\ 3y^2 + 2x \end{pmatrix}.$$

The notion of liftable vector fields was introduced by Arnol'd [1] for studying bifurcations of wave front singularities. By integrating vector fields liftable over f, we can obtain diffeomorphisms preserving the image of f. As applications of liftable vector fields, Bruce and West [2] classified functions on a cross cap, and Houston and Littlestone [4] classified linear functions on the image of the minimal cross cap mapping to find $\mathcal{A}_e$-codimension 1 maps. When $n < p$, to the best of author's knowledge, there seems to have been no general results except for [3, 4, 6, 7] and no general algorithms on constructing generators for L_f.

In this paper, following Nishimura [7], construction of explicit generators for L_f is carried out by some ways when $n < p$. Nishimura has obtained a kind of method to construct explicit generators for L_f when $n \leq p$. In Section 2, manual construction of generators for the module of liftable vector fields is carried out by his method. Then, we notice that it is difficult to obtain generators by hand. Thus, we need to develop an approach to obtain generators for the module of liftable vector fields using a computer. In Section 3, we give a method to find polynomial liftable vector fields and obtain explicit generators by a computer.

Acknowledgement

The author would like to thank Takashi Nishimura who gave me invaluable comments and warm encouragements. He is also indebted to anonymous referees for many pieces of helpful advice.

§ 2. Manual construction of generators

For a non-negative integer i, Let m_S^i (resp., m_0^i) be the subset of C_S (resp., C_0) consisting of function germs such that terms of their Taylor series up to $i-1$ are zero. Thus, $m_S^0 = C_S$ and $m_0^0 = C_0$. A multigerm $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ is said to be *finitely determined* if there exists a positive integer k such that the inclusion $m_S^k \theta_S(f) \subset T\mathcal{A}_e(f)$ holds. *Corank at most one* for a multigerm $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ $(n \leq p)$ implies that $\max\{n - \operatorname{rank} Jf(s_j) \mid 1 \leq j \leq |S|\} \leq 1$ holds, where $Jf(s_j)$ is the Jacobian matrix of f at $s_j \in S$.

For a multigerm $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$, Nishimura [7] introduced the following concept to investigate the minimal number of generators for L_f when $n \leq p$. For every non-negative integer i and a multigerm $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$, we define the $\mathbb{K}$-linear map

$$ {}_i\overline{\omega}f : \frac{m_0^i \theta_0(p)}{m_0^{i+1}\theta_0(p)} \to \frac{f^* m_0^i \theta_S(f)}{(T\mathcal{R}_e(f) \cap f^* m_0^i \theta_S(f)) + f^* m_0^{i+1}\theta_S(f)} $$

by ${}_i\overline{\omega}f([\xi]) = [\omega f(\xi)]$, where $[\xi] = \xi + m_0^{i+1}\theta_0(p)$, $[\omega f(\xi)] = \omega f(\xi) + (T\mathcal{R}_e(f) \cap f^* m_0^i \theta_S(f)) + f^* m_0^{i+1}\theta_S(f)$. We call ${}_i\overline{\omega}f$ a *higher version of the reduced Kodaira-Spencer-Mather map* [6]. Nishimura characterized the minimal number of generators for the module of vector fields liftable over a given finitely determined multigerm of corank at most one when $n \leq p$, as follows.

Theorem 2.1 ([7]). *Let $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ $(n \leq p)$ be a finitely determined multigerm of corank at most one. Suppose that there exists a non-negative integer i such that ${}_i\overline{\omega}f$ is bijective. Then, the minimal number of generators for the module of vector fields liftable over f is exactly equal to* $\dim_{\mathbb{K}} \ker {}_{i+1}\overline{\omega}f$.

We can find examples such that ${}_i\overline{\omega}f$ is bijective for some non-negative integer i in [6, 7]. For a multigerm $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$, when there exists some non-negative integer i such that ${}_i\overline{\omega}f$ is bijective, the proof of Theorem 2.1 given in [7] gives, in principle, a kind of method to construct explicit generators for L_f as follows.

1. Find a basis $\{[\xi_1], [\xi_2], \ldots, [\xi_m]\}$ of $\ker {}_{i+1}\overline{\omega}f$, where $m = \dim_{\mathbb{K}} \ker {}_{i+1}\overline{\omega}f$.
2. For each ξ_j $(j = 1, 2, \ldots, m)$, find $\tilde{\xi_j} \in m_0^{i+2}\theta_0(p)$ such that $\xi_j + \tilde{\xi_j}$ is liftable over f (the existence of such a $\tilde{\xi_j} \in m_0^{i+2}\theta_0(p)$ is proved in [7]).
3. Then, $\{\xi_j + \tilde{\xi_j}\}_{1 \leq j \leq m}$ generates L_f.

Example 2.2. Let $f : (\mathbb{K}, 0) \to (\mathbb{K}^2, 0)$ be defined by $f(x) = (x^4, x^5 + x^7)$. Then, it is known [7] that ${}_1\overline{\omega}f$ is bijective and the minimal number of generators for L_f is equal to 2. Explicit generators are obtained as follows.

First, we can verify that the following vector fields constitute a basis of $\ker {}_2\overline{\omega}f$:

$$ \left[(20X^2 - 8Y^2)\frac{\partial}{\partial X} + (25XY)\frac{\partial}{\partial Y}\right], \left[(4XY)\frac{\partial}{\partial X} + (5Y^2)\frac{\partial}{\partial Y}\right]. $$

Second, put

$$ \xi_1 = (20X^2 - 8Y^2)\frac{\partial}{\partial X} + (25XY)\frac{\partial}{\partial Y} $$

and

$$ \eta_1 = (5x^5 - 2x^7)\frac{\partial}{\partial x}. $$

Then, we have

$$\xi_1 \circ f - df \circ \eta_1 = (-16x^{12} - 8x^{14})\frac{\partial}{\partial X} + (14x^{13})\frac{\partial}{\partial Y}.$$

Here,

$$\begin{aligned}(X^2Y) \circ f - Y^3 \circ f &= x^{13} - 3x^{17} - 3x^{19} - x^{21} \\ &= x^{13}(1 - x^8) - 3x^{12}(x^5 + x^7) \\ &= x^{13}((1 - X^2) \circ f) - 3((X^3Y) \circ f),\end{aligned}$$

and hence we have

$$x^{13} = \left(\frac{X^2Y - Y^3 + 3X^3Y}{1 - X^2}\right) \circ f.$$

Furthermore, we have

$$\begin{aligned}(XY^2) \circ f &= x^{14} + 2x^{16} + x^{18} \\ &= x^{14}(1 + x^4) + 2x^{16} \\ &= x^{14}((1 + X) \circ f) + 2(X^4 \circ f),\end{aligned}$$

and hence we have

$$x^{14} = \left(\frac{XY^2 - 2X^4}{1 + X}\right) \circ f.$$

Put

$$\widetilde{\xi_1} = \left(16X^3 + 8\left(\frac{XY^2 - 2X^4}{1 + X}\right)\right)\frac{\partial}{\partial X} - \left(14\left(\frac{X^2Y - Y^3 + 3X^3Y}{1 - X^2}\right)\right)\frac{\partial}{\partial Y}.$$

Then, we have

$$(\xi_1 + \widetilde{\xi_1}) \circ f - df \circ \eta_1 = 0.$$

Here, since $\xi_1 + \widetilde{\xi_1}$ is liftable over f, so is $(1 - X^2)(\xi_1 + \widetilde{\xi_1})$. Therefore, the following vector field becomes one of the two generators for L_f:

$$(-36X^4 + 16X^3 + 8XY^2 + 20X^2 - 8Y^2)\frac{\partial}{\partial X}$$

$$+(-67X^3Y - 14X^2Y + 14Y^3 + 25XY)\frac{\partial}{\partial Y}.$$

Similarly, the other generator is obtained as

$$(4X^2Y + 4XY)\frac{\partial}{\partial X} + (-2X^4 + 2X^3 + 7XY^2 + 5Y^2)\frac{\partial}{\partial Y}.$$

Example 2.3. Let $f : (\mathbb{K}, S) \to (\mathbb{K}^2, 0)$ be given by $f_1(x) = (x^2, x^3)$, $f_2(x) = (x^3, x^2)$. Then, it is known [7] that ${}_1\overline{\omega} f$ is bijective and the minimal number of generators for L_f is equal to 2 and explicit generators are obtained as follows:

$$(-6X^2Y^2 + 6XY)\frac{\partial}{\partial X} + (-9XY^3 + 5X^3 + 4Y^2)\frac{\partial}{\partial Y},$$

$$(-9X^3Y + 5Y^3 + 4X^2)\frac{\partial}{\partial X} + (-6X^2Y^2 + 6XY)\frac{\partial}{\partial Y}.$$

We review how to construct these generators here. First, we can verify that the following vector fields constitute a basis of $\ker {}_2\overline{\omega} f$:

$$\left[(6XY)\frac{\partial}{\partial X} + (4Y^2)\frac{\partial}{\partial Y}\right], \left[(4X^2)\frac{\partial}{\partial X} + (6XY)\frac{\partial}{\partial Y}\right].$$

Second, put

$$\xi_1 = (6XY)\frac{\partial}{\partial X} + (4Y^2)\frac{\partial}{\partial Y},$$

$$\eta_{1,1} = (3x^4)\frac{\partial}{\partial x}$$

and

$$\eta_{2,1} = (2x^3)\frac{\partial}{\partial x}.$$

Then, we have

$$\xi_1 \circ f_1 - df_1 \circ \eta_{1,1} = (-5x^6)\frac{\partial}{\partial Y},$$

$$\xi_1 \circ f_2 - df_2 \circ \eta_{2,1} = 0.$$

Put

$$\xi_{1,2} = (5X^3)\frac{\partial}{\partial Y}.$$

Then, we have

$$(\xi_1 + \xi_{1,2}) \circ f_1 - df_1 \circ \eta_{1,1} = 0, \tag{2.1}$$

$$(\xi_1 + \xi_{1,2}) \circ f_2 - df_2 \circ \eta_{2,1} = (5x^9)\frac{\partial}{\partial Y}. \tag{2.2}$$

Put

$$\xi_{1,3} = (-5XY^3)\frac{\partial}{\partial Y}$$

and

$$\eta_{1,2} = \left(-\frac{5}{3}x^9\right)\frac{\partial}{\partial x}.$$

Then, we have

$$(\xi_1+\xi_{1,2}+\xi_{1,3})\circ f_1 - df_1\circ(\eta_{1,1}+\eta_{1,2}) = \left(\frac{10}{3}x^{10}\right)\frac{\partial}{\partial X},$$

$$(\xi_1+\xi_{1,2}+\xi_{1,3})\circ f_2 - df_2\circ\eta_{2,1} = 0.$$

Put

$$\xi_{1,4} = \left(-\frac{10}{3}X^2Y^2\right)\frac{\partial}{\partial X}$$

and

$$\eta_{2,2} = \left(-\frac{10}{9}x^8\right)\frac{\partial}{\partial x}.$$

Then, we have

$$(\xi_1+\xi_{1,2}+\xi_{1,3}+\xi_{1,4})\circ f_1 - df_1\circ(\eta_{1,1}+\eta_{1,2}) = 0, \tag{2.3}$$

$$(\xi_1+\xi_{1,2}+\xi_{1,3}+\xi_{1,4})\circ f_2 - df_2\circ(\eta_{2,1}+\eta_{2,2}) = \left(\frac{20}{9}x^9\right)\frac{\partial}{\partial Y}. \tag{2.4}$$

Here, note that the right-hand side of (2.3) (resp., the right-hand side of (2.4)) coincides with the right-hand side of (2.1) (resp., the right-hand side of (2.2)) multiplied by 4/9. Therefore, we have

$$\frac{9}{5}\left(\xi_1+\xi_{1,2}+\xi_{1,3}+\xi_{1,4}-\frac{4}{9}(\xi_1+\xi_{1,2})\right)\circ f_1 - df_1\circ\frac{9}{5}\left(\eta_{1,1}+\eta_{1,2}-\frac{4}{9}\eta_{1,1}\right) = 0,$$

$$\frac{9}{5}\left(\xi_1+\xi_{1,2}+\xi_{1,3}+\xi_{1,4}-\frac{4}{9}(\xi_1+\xi_{1,2})\right)\circ f_2 - df_2\circ\left(\eta_{2,1}+\eta_{2,2}-\frac{4}{9}\eta_{2,1}\right) = 0.$$

Here, we have

$$\begin{aligned}
&\frac{9}{5}\left(\xi_1+\xi_{1,2}+\xi_{1,3}+\xi_{1,4}-\frac{4}{9}(\xi_1+\xi_{1,2})\right)\\
&=\xi_1+\xi_{1,2}+\frac{5}{9}(\xi_{1,3}+\xi_{1,4})\\
&=(6XY-6X^2Y^2)\frac{\partial}{\partial X}+(4Y^2+5X^3-9XY^3)\frac{\partial}{\partial Y}
\end{aligned}$$

and hence

$$(6XY-6X^2Y^2)\frac{\partial}{\partial X}+(4Y^2+5X^3-9XY^3)\frac{\partial}{\partial Y}$$

is one of the two generators for L_f. The other generator is constructed similarly.

§ 3. Computer-based construction of generators

In Section 2, we found that it was very complicated to calculate generators by hand. The purpose of Section 3 is to develop an approach to obtain generators for the module of liftable vector fields using a computer. The proof of the following theorem gives a method by which we can find a non-zero liftable vector field for a multigerm.

Theorem 3.1. *Let $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ $(n < p)$ be a polynomial multigerm. Then, there exists a non-zero polynomial vector field in L_f such that the highest degree is at most*

$$\left[\sqrt[p-n]{(\alpha+1)\cdot(p-1)!} \right] + 1,$$

where

$$\alpha = r\binom{\frac{p\cdot 2^n - n}{n!}}{} \cdot (D+n-1)^n, \quad r = |S|,$$

$$D = \max\{D_i \,|\, i \in \{1,2,\ldots,r\}\}, \quad D_i = \max\{\deg(X_j \circ f_i) \,|\, j \in \{1,2,\ldots,p\}\},$$

and $[x]$ denotes the greatest integer not exceeding x.

Proof. We put

$$N = \left[\sqrt[p-n]{(\alpha+1)\cdot(p-1)!} \right] + 1,\ N' = (D+n-1)N.$$

Then, we show that there exists a coefficient vector

$$(a_1^{(0,0,\ldots,0)}, a_1^{(1,0,\ldots,0)}, \ldots, a_p^{(0,0,\ldots,N)}, a_{1,1}^{(0,0,\ldots,0)}, a_{1,1}^{(1,0,\ldots,0)}, \ldots, a_{r,n}^{(0,0,\ldots,N')}) \neq 0$$

such that for every $i \in \{1,2,\ldots,r\}$, the following polynomial equation with respect to the variables $x_1, x_2, \ldots, x_n$ holds:

$$\begin{pmatrix} \displaystyle\sum_{d=0}^{N} \left(\sum_{i_1+\cdots+i_p=d} a_1^{(i_1,i_2,\ldots,i_p)} \prod_{h=1}^{p} X_h^{i_h} \right) \\ \vdots \\ \displaystyle\sum_{d=0}^{N} \left(\sum_{i_1+\cdots+i_p=d} a_p^{(i_1,i_2,\ldots,i_p)} \prod_{h=1}^{p} X_h^{i_h} \right) \end{pmatrix} \circ f_i$$

$$= \begin{pmatrix} \frac{\partial(X_1\circ f_i)}{\partial x_1} & \cdots & \frac{\partial(X_1\circ f_i)}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial(X_p\circ f_i)}{\partial x_1} & \cdots & \frac{\partial(X_p\circ f_i)}{\partial x_n} \end{pmatrix} \begin{pmatrix} \displaystyle\sum_{d=0}^{N'} \left(\sum_{j_1+\cdots+j_n=d} a_{i,1}^{(j_1,j_2,\ldots,j_n)} \prod_{h=1}^{n} x_h^{j_h} \right) \\ \vdots \\ \displaystyle\sum_{d=0}^{N'} \left(\sum_{j_1+\cdots+j_n=d} a_{i,n}^{(j_1,j_2,\ldots,j_n)} \prod_{h=1}^{n} x_h^{j_h} \right) \end{pmatrix},$$

where $i_1, i_2, \ldots, i_p, j_1, j_2, \ldots, j_n$ are non-negative integers. Note that for every $i \in \{1, 2, \ldots, r\}$, the highest degree of the left-hand side (resp., right-hand side) is at most $N \cdot D_i$ (resp., $N' + D_i - 1$). By comparing the coefficients of the terms on the both sides, a system of linear equations with respect to

$$a_1^{(0,0,\ldots,0)}, a_1^{(1,0,\ldots,0)}, \ldots, a_p^{(0,0,\ldots,N)}, a_{1,1}^{(0,0,\ldots,0)}, a_{1,1}^{(1,0,\ldots,0)}, \ldots, a_{r,n}^{(0,0,\ldots,N')}$$

is obtained. Let the number of unknowns of the system of linear equations be denoted by U and the number of equations by E. The number of combinations of non-negative integers $j_1, j_2, \ldots, j_n$ such that $j_1 + \cdots + j_n = d$ is given by

$$\binom{d+n-1}{d} = \frac{(d+n-1)!}{d!(n-1)!} = \begin{cases} 1 & (n=1) \\ \dfrac{(d+n-1)\cdots(d+1)}{(n-1)!} & (n \geq 2). \end{cases}$$

Therefore, when $n \geq 2$ and $p \geq 2$, we have

$$U = p \sum_{d=0}^{N} \frac{(d+p-1)\cdots(d+1)}{(p-1)!} + nr \sum_{d=0}^{N'} \frac{(d+n-1)\cdots(d+1)}{(n-1)!}.$$

Here, since it is verified that

$$N \cdot D \leq N' + D - 1, \quad N \geq 1, \quad N' \geq 1,$$

we have for every $i \in \{1, \cdots, r\}$,

$$N \cdot D_i \leq N' + D_i - 1.$$

Therefore, we obtain

$$E \leq pr \sum_{d=0}^{N'+D-1} \frac{(d+n-1)\cdots(d+1)}{(n-1)!}.$$

Here, for a non-negative integer k, the following formula is known:

$$\sum_{d=1}^{n} d(d+1)\cdots(d+k) = \frac{n(n+1)\cdots(n+k+1)}{k+2}.$$

Therefore, we have

$$\begin{aligned} U &= p\left(\frac{(N+1)\cdots(N+p)}{p!}\right) + nr\left(\frac{(N'+1)\cdots(N'+n)}{n!}\right) \\ &> \frac{1}{(p-1)!}N^p + \frac{nr}{n!}(N')^n \end{aligned}$$

and

$$\begin{aligned} E &\leq pr\frac{(N'+D)\cdots(N'+D+n-1)}{n!} \\ &\leq pr\frac{(N'+D+n-1)^n}{n!} \\ &\leq pr\frac{(N'+N')^n}{n!} \\ &= \left(\frac{pr\cdot 2^n}{n!}\right)(N')^n. \end{aligned}$$

Thus, the following holds even in the case where $n = 1$ or $p = 1$:

$$U - E \geq \frac{1}{(p-1)!}N^p - r\left(\frac{p\cdot 2^n - n}{n!}\right)(N')^n.$$

Therefore, we get

$$\begin{aligned} U - E &\geq \frac{1}{(p-1)!}N^p - r\left(\frac{p\cdot 2^n - n}{n!}\right)(D+n-1)^n N^n \\ &= \frac{1}{(p-1)!}N^p - \alpha N^n \\ &= N^n\left(\frac{1}{(p-1)!}N^{p-n} - \alpha\right) \\ &\geq 1. \end{aligned}$$

Thus, there exists a non-zero element of L_f such that the highest degree is at most $\left[\sqrt[p-n]{(\alpha+1)\cdot(p-1)!}\right] + 1$. □

Note that we can usually take values for N and N' much smaller than those calculated in Theorem 3.1. This gives clues for finding generators of L_f for a multigerm f satisfying the assumption of Theorem 2.1. Some examples of calculating generators using a computer are given below.

Example 3.2. For f in Example 2.2, we obtain generators using Mathematica as follows. Although we get $N = 23$ and $N' = 161$ by using the argument in the proof of Theorem 3.1, we can put $N = 4$ and $N' = 25$. The calculation is carried out as follows.

```
SolveAlways[{Sum[
    Sum[a[1, i, d] (x^4)^i (x^5 + x^7)^(d - i), {i, 0, d}], {d, 0, 4}],
   Sum[Sum[a[2, i, d] (x^4)^i (x^5 + x^7)^(d - i), {i, 0, d}], {d, 0,
     4}]}
  ==
```

```
{{4 x^3}, {5 x^4 + 7 x^6}}.{Sum[b[i] x^i, {i, 0, 25}]}, x]

{{a[1, 0, 0] -> 0, a[2, 0, 0] -> 0, b[0] -> 0, a[2, 1, 1] -> 0,
a[1, 1, 1] -> 0, b[1] -> 0, a[2, 0, 1] -> 0, a[1, 0, 1] -> 0,
b[2] -> 0, b[3] -> 0, b[4] -> 0, a[2, 2, 2] -> 0,
a[1, 2, 2] -> -(20/49) a[2, 3, 4] - (180 b[15])/49,
b[5] -> -(5/49) a[2, 3, 4] - (45 b[15])/49,
a[2, 1, 2] -> -(25/49) a[2, 3, 4] - (225 b[15])/49,
a[1, 1, 2] -> -2 a[2, 4, 4], b[6] -> -(1/2) a[2, 4, 4],
a[2, 0, 2] -> -(5/2) a[2, 4, 4],
a[1, 0, 2] -> 8/49 a[2, 3, 4] + (72 b[15])/49,
b[7] -> 2/49 a[2, 3, 4] + (18 b[15])/49, b[8] -> -(1/2) a[2, 4, 4],
a[2, 3, 3] -> -a[2, 4, 4],
a[1, 3, 3] -> -(8/49) a[2, 3, 4] + (124 b[15])/49,
b[9] -> 2/49 a[2, 3, 4] + (67 b[15])/49,
a[2, 2, 3] -> 24/49 a[2, 3, 4] + (461 b[15])/49,
a[1, 2, 3] -> -2 a[2, 4, 4], b[10] -> -(1/2) a[2, 4, 4],
a[2, 1, 3] -> -(7/2) a[2, 4, 4], a[1, 1, 3] -> 4 b[15],
b[11] -> 2/49 a[2, 3, 4] + (67 b[15])/49, a[2, 0, 3] -> 7 b[15],
a[1, 0, 3] -> 0, b[12] -> -(1/2) a[2, 4, 4],
a[1, 4, 4] -> 4/7 a[2, 3, 4] + (8 b[15])/7,
b[13] -> 1/7 a[2, 3, 4] + (16 b[15])/7, a[1, 3, 4] -> 0, b[14] -> 0,
 a[2, 2, 4] -> 0, a[1, 2, 4] -> 0, a[2, 1, 4] -> 0, a[1, 1, 4] -> 0,
 b[16] -> 0, a[2, 0, 4] -> 0, a[1, 0, 4] -> 0, b[17] -> 0,
b[18] -> 0, b[19] -> 0, b[20] -> 0, b[21] -> 0, b[22] -> 0,
b[23] -> 0, b[24] -> 0, b[25] -> 0}}
```

Vector fields liftable over f are expressed by three parameters as follows.

```
{Sum[Sum[a[1, i, d] (X)^i (Y)^(d - i), {i, 0, d}], {d, 0, 4}],
 Sum[Sum[a[2, i, d] (X)^i (Y)^(d - i), {i, 0, d}], {d, 0, 4}]} /. %

{{-2 X Y a[2, 4, 4] - 2 X^2 Y a[2, 4, 4] +
  X^2 (-(20/49) a[2, 3, 4] - (180 b[15])/49) + 4 X Y^2 b[15] +
  X^4 (4/7 a[2, 3, 4] + (8 b[15])/7) +
  Y^2 (8/49 a[2, 3, 4] + (72 b[15])/49) +
  X^3 (-(8/49) a[2, 3, 4] + (124 b[15])/49),
 X^3 Y a[2, 3, 4] - X^3 a[2, 4, 4] + X^4 a[2, 4, 4] -
  5/2 Y^2 a[2, 4, 4] - 7/2 X Y^2 a[2, 4, 4] +
```

```
X Y (-(25/49) a[2, 3, 4] - (225 b[15])/49) + 7 Y^3 b[15] +
X^2 Y (24/49 a[2, 3, 4] + (461 b[15])/49)}}
```

Here, we fix the parameters for obtaining two liftable vector fields as follows.

```
% /. {a[2, 4, 4] -> 2, a[2, 3, 4] -> 0, b[15] -> 0}

{{-4 X Y - 4 X^2 Y, -2 X^3 + 2 X^4 - 5 Y^2 - 7 X Y^2}}


% /. {a[2, 4, 4] -> 0, a[2, 3, 4] -> 49, b[15] -> 0}

{{-20 X^2 - 8 X^3 + 28 X^4 + 8 Y^2, -25 X Y + 24 X^2 Y + 49 X^3 Y}}
```

It is seen that both these vector fields belong to $\ker {}_2\overline{\omega}f$ and constitute a basis of $\ker {}_2\overline{\omega}f$. Therefore, the following vector fields constitute a set of generators for L_f:

$$(28X^4 - 8X^3 - 20X^2 + 8Y^2)\frac{\partial}{\partial X} + (49X^3Y + 24X^2Y - 25XY)\frac{\partial}{\partial Y},$$
$$(-4X^2Y - 4XY)\frac{\partial}{\partial X} + (2X^4 - 2X^3 - 7XY^2 - 5Y^2)\frac{\partial}{\partial Y}.$$

Note that the generators obtained above are different from those in Example 2.2.

Example 3.3. Let $f : (\mathbb{K}, S) \to (\mathbb{K}^2, 0)$ be the multigerm given by $f_1(x) = (x^2, x^3)$, $f_2(x) = (-x^3, x^2)$ and $f_3(x) = (x^2 - x^3, x^2 + x^3)$. Then, ${}_2\overline{\omega}f$ is bijective, which is shown by the same method as [6]. Put $Q(f) = C_S/f^*m_0C_S$, and let $\delta(f)$ be the dimension of the $\mathbb{K}$-vector space $Q(f)$. Then, $\delta(f) = 6$ holds. By Proposition 4 and Proposition 5 in [7], it is seen that

$$\dim_{\mathbb{K}} \ker {}_3\overline{\omega}f = 2.$$

Therefore, the minimal number of generators for L_f is equal to 2 by Theorem 2.1. Although $N = 29$ and $N' = 87$ by Theorem 3.1, we can put $N = 6$ and $N' = 16$. In the same way as before we obtain the following liftable vector fields:

$$\begin{gathered}(-15X^6 - 45X^5Y - 45X^4Y^2 + 19X^3Y^3 + 4X^2Y^4 - 4X^5 \\ -64X^4Y + 45X^3Y^2 + 41X^2Y^3 + 57XY^4 - 7Y^5 + 4X^4 \\ -12X^3Y - 8X^2Y^2 + 52XY^3 - 14Y^4 + 8X^3 - 16X^2Y)\frac{\partial}{\partial X} \\ +(-10X^5Y - 30X^4Y^2 - 38X^3Y^3 + 18X^2Y^4 + 6XY^5 + 8X^5 \\ -8X^4Y - 46X^3Y^2 + 34X^2Y^3 + 24XY^4 + 56Y^5 - 4X^4 \\ +6X^3Y - 26X^2Y^2 - 10XY^3 + 28Y^4 + 12X^2Y - 20XY^2)\frac{\partial}{\partial Y},\end{gathered}$$

$$
\begin{gathered}
(-6X^5Y - 18X^4Y^2 + 38X^3Y^3 + 30X^2Y^4 + 10XY^5 - 56X^5 \\
-24X^4Y - 34X^3Y^2 + 46X^2Y^3 + 8XY^4 - 8Y^5 - 28X^4 \\
+10X^3Y + 26X^2Y^2 - 6XY^3 + 4Y^4 + 20X^2Y - 12XY^2)\frac{\partial}{\partial X} \\
+(-4X^4Y^2 - 19X^3Y^3 + 45X^2Y^4 + 45XY^5 + 15Y^6 + 7X^5 \\
-57X^4Y - 41X^3Y^2 - 45X^2Y^3 + 64XY^4 + 4Y^5 + 14X^4 \\
-52X^3Y + 8X^2Y^2 + 12XY^3 - 4Y^4 + 16XY^2 - 8Y^3)\frac{\partial}{\partial Y}.
\end{gathered}
$$

It is seen that both these vector fields belong to $\ker {}_3\overline{\omega} f$ and constitute a basis of $\ker {}_3\overline{\omega} f$. Therefore, the above vector fields constitute a set of generators for L_f.

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