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Wave Front Evolution and Pedal Evolution

Dedicated to Professors Satoshi Koike and Laurentiu Paunescu on the occasion of their sixtieth birthdays

By

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Abstract

The calculus correspondence has been known to exist between generic pedal evolutions and generic wave front evolutions. In this paper, we first extend the known results on the calculus correspondence to evolutions with multi-parameters, and then give applications of calculus correspondence. Moreover, we discuss the possibility of generalization of the calculus correspondence to degenerate pedal evolutions and degenerate wave front evolutions.

§ 1. Introduction

Throughout this paper, all maps, map-germs and vector fields are of class C^∞ unless otherwise stated.

A map-germ $\Phi : (\mathbb{R}^m, 0) \to (\mathbb{R}^{m+1}, 0)$ is called a *Legendrian map-germ* if there exists a germ of unit vector field ν_Φ along Φ such that the following 2 conditions hold, where the dot in the center stands for the scalar product of two vectors.

1. $\frac{\partial\Phi}{\partial x_1}(x_1,\ldots,x_m)\cdot\nu_\Phi(x_1,\ldots,x_m) = \cdots = \frac{\partial\Phi}{\partial x_m}(x_1,\ldots,x_m)\cdot\nu_\Phi(x_1,\ldots,x_m) = 0.$
2. The map-germ $L_\Phi : (\mathbb{R}^m, 0) \to T_1\mathbb{R}^{m+1}$ defined by

$$L_\Phi(x_1,\ldots,x_m) = (\Phi(x_1,\ldots,x_m), \nu_\Phi(x_1,\ldots,x_m))$$

   is non-singular, where $T_1\mathbb{R}^{m+1}$ is the unit tangent bundle of $\mathbb{R}^{m+1}$.

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The vector field ν_Φ, the map-germ L_Φ and the image of a Legendrian map-germ are called a *unit normal vector field of* Φ, a *Legendrian lift of* Φ and a *wave front* respectively. Singularities of Legendrian map-germs have been relatively well-studied (for instance, see [3, 10, 22, 23]).

Let $\mathbf{r} : (a,b) \to \mathbb{R}^2$ $(0 \in (a,b))$ be a non-singular plane curve without inflection point (namely, a non-degenerate curve), and let P be a point of $\mathbb{R}^2$. Then, the *pedal curve of* $\mathbf{r}$ *relative to the pedal point* P is defined as the trajectory of the foot of perpendicular to the tangent line $\{\mathbf{r}(s) + u\mathbf{r}'(s) \mid u \in \mathbb{R}\}$ at $\mathbf{r}(s)$ from P, and it is denoted by $ped_{\mathbf{r},P} : (a,b) \to \mathbb{R}^2$. The given point P is called the *pedal point*. Let $WF_{\mathbf{r},P} : (a,b) \to \mathbb{R}^2$ be the solution curve of

$$\frac{d}{ds} WF_{\mathbf{r},P}(s) = ped_{\mathbf{r},P}(s) - P, \quad WF_{\mathbf{r},P}(0) = (0,0),$$

where s is the arc-length parameter of $\mathbf{r}$. Then, by definition of pedal curve, $\mathbf{r}'(s)$ (which is the unit tangent vector to $\mathbf{r}$ at $\mathbf{r}(s)$) is a unit normal vector to $WF_{\mathbf{r},P}$ at $WF_{\mathbf{r},P}(s)$. Thus, the Legendrian lift $L_{WF_{\mathbf{r},P}} : (a,b) \to T_1\mathbb{R}^2$ given by

$$L_{WF_{\mathbf{r},P}}(s) = (WF_{\mathbf{r},P}(s), \mathbf{r}'(s))$$

is well-defined. Since the original curve $\mathbf{r}$ is without inflection point, by the Serret-Frenet formula (for the Serret-Frenet formula, see for instance [4]), $L_{WF_{\mathbf{r},P}}$ is non-singular. Thus, the image of $WF_{\mathbf{r},P}$ must be a wave front curve.

Next, we move the pedal point P. Let $P : U \to \mathbb{R}^2$ be a map, where U is an open neighborhood of the origin of $\mathbb{R}^n$, Then, we obtain two corank one maps $Un\text{-}ped_{\mathbf{r},P} : (a,b) \times U \to (\mathbb{R}^2 \times \mathbb{R}^n, 0)$ and $Un\text{-}WF_{\mathbf{r},P} : (a,b) \times U \to (\mathbb{R}^2 \times \mathbb{R}^n, 0)$ defined by

$$Un\text{-}ped_{\mathbf{r},P}(s,u) = \left(ped_{\mathbf{r},P(u)}(s), u\right) \text{ and } Un\text{-}WF_{\mathbf{r},P}(s,u) = \left(WF_{\mathbf{r},P(u)}(s), u\right)$$

respectively. The map $Un\text{-}ped_{\mathbf{r},P}$ (resp., $Un\text{-}WF_{\mathbf{r},P}$) is called the *pedal unfolding of* $ped_{\mathbf{r},P(0)}$ (resp., the *wave front unfolding of* $WF_{\mathbf{r},P(0)}$).

In [1], the evolution of generic wave fronts by time has been studied. We want to construct a different method from [1] to study generic wave front evolutions. In order to do so, we pay attention to the relation between $Un\text{-}ped_{\mathbf{r},P}$ and $Un\text{-}WF_{\mathbf{r},P}$ since we have the following Proposition 1.1 for $Un\text{-}ped_{\mathbf{r},P}$.

Proposition 1.1. *Let* $\mathbf{r} : (a,b) \to \mathbb{R}^2$ $(0 \in (a,b))$ *be a non-singular plane curve without inflection point such that* $\mathbf{r}(0) = 0$ *and let* U *be an open neighborhood of the origin of* $\mathbb{R}^n$. *Moreover, we let* $P : U \to \mathbb{R}^2$ *be a map such that* $\mathbf{r}(0) = P(0) = 0$. *Then, the map-germ* $Un\text{-}ped_{\mathbf{r},P} : ((a,b) \times U, (0,0)) \to (\mathbb{R}^2 \times \mathbb{R}^n, (0,0))$ *is* $\mathcal{A}$*-equivalent to the normal form of (Whitney umbrella)*$\times\mathbb{R}^{n-1}$ *if and only if the origin* $(0,0)$ *of* $(a,b)\times U$ *is a regular point of the map* $(\mathbf{r}, P) : (a,b) \times U \to \mathbb{R}^{n+2}$ *defined by* $(\mathbf{r}, P)(s,u) = (\mathbf{r}(s), P(u))$.

Here, two map-germs $f, g : (\mathbb{R}^m, 0) \to (\mathbb{R}^{m+1}, 0)$ are said to be $\mathcal{A}$-*equivalent* if there exist germs of diffeomorphism $h_s : (\mathbb{R}^m, 0) \to (\mathbb{R}^m, 0)$ and $h_t : (\mathbb{R}^{m+1}, 0) \to (\mathbb{R}^{m+1}, 0)$ such that $f = h_t \circ g \circ h_s$, and the *normal form of (Whitney umbrella)*$\times\mathbb{R}^{n-1}$ is the map-germ defined by $(s, u) \mapsto (su_1, s^2, u)$ where $u = (u_1, \ldots, u_n)$. Proposition 1.1 in the case $n = 1$ is a special case of Theorem 1 in [17], Proof of Proposition 1.1 is given in §2.

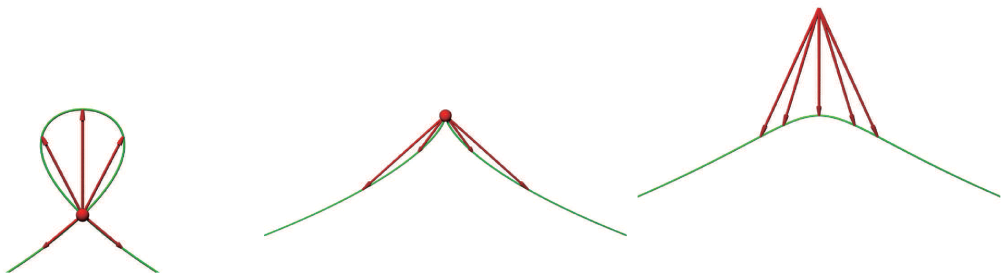

Figure 1. Pedal Evolution.

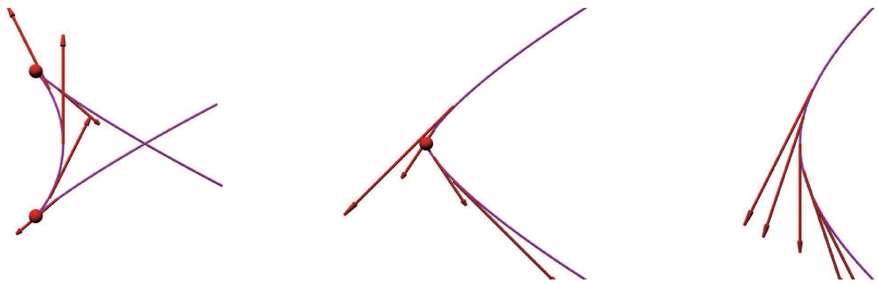

Figure 2. Wave Front Evolution.

By Figures 1 and 2, it is easily conjectured that the pedal evolution $Un\text{-}ped_{\mathbf{r},P}$ is $\mathcal{A}$-equivalent to the normal form of (Whitney umbrella)$\times\mathbb{R}^{n-1}$ if and only if the wave front evolution $Un\text{-}WF_{\mathbf{r},P}$ is a (swallowtail)$\times\mathbb{R}^{n-1}$, where a *(swallowtail)*$\times\mathbb{R}^{n-1}$ is a map-germ $\mathcal{A}$-equivalent to $(s, u) \mapsto (3s^4 + s^2u_1, -4s^3 - 2su_1, u)$ $(u = (u_1, \ldots, u_n))$; and in the case $n = 1$ this conjecture has been actually proved in [18] (such a correspondence is called the *calculus correspondence*. For more details on the known calculus correspondence, see §2).

In this paper, we first extend the known results on the calculus correspondence to evolutions with multi-parameters, and then give applications of calculus correspondence. Moreover, we discuss the possibility of generalization of calculus correspondence to degenerate pedal evolutions and degenerate wave front evolutions.

In Section 2, known calculus correspondences are extended to evolutions with multi-parameters. The proof of Proposition 1.1 is also given in Section 2. In Section 3, applications of calculus correspondence are given. Finally, the possibility of generalization of calculus correspondence is discussed in Section 4.

§ 2. Extension of known calculus correspondences to evolutions with multi-parameters

Definition 2.1. A map-germ $\varphi : (\mathbb{R}\times\mathbb{R}^n, (0,0)) \to (\mathbb{R}^2\times\mathbb{R}^n, (0,0))$ having the following form is said to be *of pedal unfolding type*.

$$\varphi(x,y) = (n(x,y)p(x,y), p(x,y), y)$$

where $n : (\mathbb{R}\times\mathbb{R}^n, (0,0)) \to (\mathbb{R}, 0)$ is a function-germ satisfying $\frac{\partial n}{\partial x}(0,0) \neq 0$ and $y = (y_1, \ldots, y_n)$.

In the case $n = 1$, Definition 2.1 has been given in [18].

Proposition 2.2. *Let $\mathbf{r} : (a,b) \to \mathbb{R}^2$ $(0 \in (a,b))$ be a non-singular plane curve without inflection point such that $\mathbf{r}(0) = 0$ and let $P : U \to \mathbb{R}^2$ be a map such that $P(0) = \mathbf{r}(0) = 0$, where U is an open neighborhood of the origin of $\mathbb{R}^n$. Then, $Un\text{-}ped_{\mathbf{r},P}$ is $\mathcal{A}$-equivalent to a map-germ of pedal unfolding type.*

Proof of Proposition 2.2. Since $\mathbf{r} : (a,b) \to \mathbb{R}^2$ $(0 \in (a,b))$ is non-singular, we may assume that $\mathbf{r}(x) = (-x, r_2(x))$, $r_2(0) = \frac{dr_2}{dx}(0) = 0$ near 0. Put $n(x) = \frac{dr_2}{dx}(x)$. Then, $(n(x), 1)$ is a normal vector to $\mathbf{r}$ at $\mathbf{r}(x)$. Since $\mathbf{r}$ is without inflection point, we have that $\frac{dn}{dx}(x) \neq 0$. Then, since $\mathbf{r}(0) = P(0) = 0$, $ped_{\mathbf{r},P(0)}(x) \in T_{P(0)}\mathbb{R}^2$ has the form:

$$ped_{\mathbf{r},P(0)}(x) = p(x)(n(x), 1) = (n(x)p(x), p(x)).$$

Therefore, we have:

$$\begin{aligned} Un\text{-}ped_{\mathbf{r},P}(x,y) &= \left(ped_{\mathbf{r},P(y)}(x), y\right) \\ &= (p(x,y)\,(n(x),1) + P(y), y) \\ &= ((n(x)p(x,y), p(x,y)) + P(y), y) \\ &\sim_{\mathcal{A}} ((n(x)p(x,y), p(x,y), y)\,. \end{aligned}$$

$\square$

Proposition 2.3. *Let $\mathbf{r} : (a,b) \to \mathbb{R}^2$ $(0 \in (a,b))$ be a non-singular plane curve without inflection point such that $\mathbf{r}(0) = 0$ and let $P : U \to \mathbb{R}^2$ be a map such that $P(0) = \mathbf{r}(0) = 0$, where U is an open neighborhood of the origin of $\mathbb{R}^n$. Then, $Un\text{-}ped_{\mathbf{r},P}$ is $\mathcal{A}$-equivalent to a map-germ of the form: $(x,y) \mapsto \left(x(x^2+q(y)), x^2+q(y), y\right)$.*

Proof of Proposition 2.3. As same as the proof of Proposition 2.2, we may assume that $\mathbf{r}(x) = (-x, r_2(x))$, $r_2(0) = \frac{dr_2}{dx}(0) = 0$ near 0. By definition of pedal curve, the following holds:

$$n(x)\left(n(x)p(x) + x\right) + \left(p(x) - r_2(x)\right) = 0.$$

Here, $n(x)$ and $p(x)$ are functions defined in the proof of Proposition 2.2. Thus, we have the following locally:

$$p(x) = \frac{r_2(x) - xn(x)}{1 + n^2(x)}.$$

Since $\mathbf{r}$ is without inflection point and $n(x) = \frac{dr_2}{dx}(x)$, there exists a function $\xi(x)$ such that $r_2(x) - xn(x) = x^2\xi(x)$ and $\xi(0) \neq 0$ by Hadamard's lemma (for Hadamard's lemma, see [4]). Thus, $p : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ is a Morse function-germ. By the Morse lemma with parameters (see [4]), we have that

$$Un\text{-}ped_{\mathbf{r},P}(x, y) = \left(ped_{\mathbf{r},P(y)}(x), y\right)$$

is $\mathcal{A}$-equivalent to a map-germ of the form:

$$(x, y) \mapsto \left(n(x, y)\left(x^2 + q(y)\right), x^2 + q(y), y\right). \tag{2.1}$$

Since $\frac{\partial n}{\partial x}(0) \neq 0$, by using the Malgrange preparation theorem (for the Malgrange preparation theorem, for instance see [3]), $(x, y) \mapsto \left(n(x, y)\left(x^2 + q(y)\right), x^2 + q(y), y\right)$ is $\mathcal{A}$-equivalent to

$$(x, y) \mapsto \left(x\left(x^2 + q(y)\right), x^2 + q(y), y\right). \tag{2.2}$$

$\square$

Note that Proposition 2.3 may be proved without using the Malgrange preparation theorem. Alternatively, we may adopt a simple method used to prove the criterion of cuspidal crosscap given in [7]. Namely, by dividing $n(x, y)$ into the sum of an odd function and an even function with respect to the variable x, it is possible to show that the map-germ (2.1) is $\mathcal{A}$-equivalent to the map-germ (2.2).

Definition 2.4 ([15]). Let k be a non-negative integer. Then, a map-germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is said to be *of S_k type* if f is $\mathcal{A}$-equivalent to the map-germ $(x, y) \mapsto (x(x^2 \pm y^{k+1}), x^2 \pm y^{k+1}, y)$.

By Proposition 2.3, singularities of one-parameter pedal unfoldings $Un\text{-}ped_{\mathbf{r},P}$ must be of S_k type (k is a certain non-zero integer) provided that $\mathbf{r}$ is non-degenerate, $\mathbf{r}(0) = P(0)$ and $q(y)$ is not flat. This fact has been already proved in [17] by using a characterization of spherical pedal given in [16]. Thus, the proof given here is an alternative proof. Even if we moved the given non-degenerate curve $\mathbf{r}$ depending on the parameter y, the proof of Proposition 2.3 shows that new singularities never occur for the map-germ of the form $(x, y) \mapsto \left(ped_{\mathbf{r}_y, P(y)}(x), y\right)$ provided that $\mathbf{r}_0$ is non-degenerate, $\mathbf{r}_0(0) = P(0)$ and $q(y)$ is not flat.

Proof of Proposition 1.1.[1] By Proposition 2.3, $Un\text{-}ped_{\mathbf{r},P}$ is $\mathcal{A}$-equivalent to a map-germ $\psi(x, y) = \left(x(x^2 + q(y)), x^2 + q(y), y\right)$ under the assumption of Proposition 1.1. It is easily seen that the origin $(0, 0)$ of $(a, b)\times U$ is a regular point of the map $(\mathbf{r}, P)$: $(a, b)\times U \to \mathbb{R}^{n+2}$ if and only if there exists an integer i $(1 \le i \le n)$ such that $\frac{\partial q}{\partial y_i}(0) \neq 0$ for $q(y)$. Thus, it is sufficient to show that $\psi(x, y) = \left(x(x^2 + q(y)), x^2 + q(y), y\right)$ is a (Whitney umbrella)$\times\mathbb{R}^{n-1}$ if and only if there exists an integer i $(1 \le i \le n)$ such that $\frac{\partial q}{\partial y_i}(0) \neq 0$ for $q(y)$.

Suppose that ψ is a (Whitney umbrella)$\times\mathbb{R}^{n-1}$. Let $S_1 \subset J^2(\mathbb{R}^{n+1}, \mathbb{R}^{n+2})$ be the set of corank one 2-jets. Note that $j^2 f$ is transverse to S_1 where f denotes the normal form of (Whitney umbrella)$\times\mathbb{R}^{n-1}$. Since ψ is $\mathcal{A}$-equivalent to f, $j^2\psi$, too, is transverse to S_1. Since $S_1(\psi) = (j^2\psi)^{-1}(S_1) = \{3x^2 + q(y) = 0, x = 0\}$, $j^2\psi$ is transverse to S_1 if and only if

$$\operatorname{rank}\left(d(3x^2 + q(y)), dx\right) = \operatorname{rank}\begin{pmatrix} 6x & 1 \\ dq(y) & 0 \end{pmatrix} = 2.$$

Therefore, there exists an integer i $(1 \le i \le n)$ such that $\frac{\partial q}{\partial y_i}(0) \neq 0$.

Conversely, suppose that there exists an integer i $(1 \le i \le n)$ such that $\frac{\partial q}{\partial y_i}(0) \neq 0$. Set

$$\begin{aligned} h_s(x, y_1, \ldots, y_n) &= (x, y_1, \ldots, y_{i-1}, x^2 + q(y), y_{i+1}, \ldots, y_n), \\ H_t(X_1, X_2, Y_1, \ldots, Y_n) &= (X_1, Y_i, Y_1, \ldots, Y_{i-1}, -X_2 + Y_i, Y_{i+1}, \ldots, Y_n)). \end{aligned}$$

Then, h_s (resp., H_t) is a germ of diffeomorphism of $(\mathbb{R} \times \mathbb{R}^n, (0, 0))$ (resp., $(\mathbb{R}^2 \times \mathbb{R}^n, (0, 0))$). Set also $f_i(x, y_1, \ldots, y_n) = (xy_i, x^2, y_1, \ldots, y_n)$. Then, we have:

$$H_t \circ f_i \circ h_s\,(x, y_1, \ldots, y_n) = \left(x(x^2 + q(y)), x^2 + q(y), y_1, \ldots, y_{i-1}, q(y), y_{i+1}, \ldots, y_n\right).$$

Since $\frac{\partial q}{\partial y_i}(0) \neq 0$, the map-germ $(y_1, \ldots, y_n) \mapsto (y_1, \ldots, y_{i-1}, q(y), y_{i+1}, \ldots, y_n)$ is a germ of diffeomorphism. Thus, $H_t \circ f_i \circ h_s$ is $\mathcal{A}$-equivalent to ψ. Since the map-germ f_i is clearly a (Whitney umbrella)$\times\mathbb{R}^{n-1}$, ψ must be a (Whitney umbrella)$\times\mathbb{R}^{n-1}$. □

Definition 2.5. For a map-germ of pedal unfolding type

$$\varphi(x, y) = (n(x, y)p(x, y), p(x, y), y),$$

[1] The author's original proof of Proposition 1.1 used Mather's infinitesimal characterization of stable map-germs ([13])) and Mather's classification theorem ([14]). The proof given here, which is self-contained, was suggested by the referee.

set

$$\mathcal{I}(\varphi)(x,y) = \left(\int_0^x n(x,y)p(x,y)dx, \int_0^x p(x,y)dx, y\right).$$

The map-germ $\mathcal{I}(\varphi) : (\mathbb{R} \times \mathbb{R}^n, 0) \to (\mathbb{R}^2 \times \mathbb{R}^n, 0)$ is called the *integration of* φ.

In the case $n = 1$, Definition 2.5 has been given in [18].

Definition 2.6. A Legendrian map-germ $\Phi : (\mathbb{R} \times \mathbb{R}^n, (0,0)) \to (\mathbb{R}^2 \times \mathbb{R}^n, (0,0))$ is said to be *normalized* if Φ satisfies the following three conditions:

1. The map-germ Φ has the following form where $y = (y_1, \ldots, y_n)$.

$$\Phi(x,y) = (\Phi_1(x,y), \Phi_2(x,y), y).$$

2. The condition $\frac{\partial \Phi_2}{\partial x}(0,0) = 0$ holds.
3. The vector $\nu_\Phi(0,0)$ is $\frac{\partial}{\partial X_1}$ or $-\frac{\partial}{\partial X_1}$, where $(X_1, X_2, Y_1, \ldots, Y_n)$ denotes the standard coordinate system of $(\mathbb{R}^2 \times \mathbb{R}^n, (0,0))$.

In the case $n = 1$, Definition 2.6 has been given in [18].

Definition 2.7. For a normalized Legendrian map-germ

$$\Phi(x,y) = (\Phi_1(x,y), \Phi_2(x,y), y),$$

set

$$\mathcal{D}(\Phi)(x,y) = \left(\frac{\partial \Phi_1}{\partial x}(x,y), \frac{\partial \Phi_2}{\partial x}(x,y), y\right).$$

The map-germ $\mathcal{D}(\Phi) : (\mathbb{R} \times \mathbb{R}^n, (0,0)) \to (\mathbb{R}^2 \times \mathbb{R}^n, (0,0))$ is called the *differential of* Φ.

In the case $n = 1$, Definition 2.7 has been given in [18].

Proposition 2.8.

1. *For a map-germ of pedal unfolding type $\varphi : (\mathbb{R} \times \mathbb{R}^n, (0,0)) \to (\mathbb{R}^2 \times \mathbb{R}^n, (0,0))$, $\mathcal{I}(\varphi)$ is a normalized Legendrian map-germ.*
2. *For a normalized Legendrian map-germ $\Phi : (\mathbb{R} \times \mathbb{R}^n, (0,0)) \to (\mathbb{R}^2 \times \mathbb{R}^n, (0,0))$, $\mathcal{D}(\Phi)$ is a map-germ of pedal unfolding type.*

In the case $n = 1$, Proposition 2.8 with its proof can be found in [18]. The proof given in [18] works well even to the case $n \geq 2$.

The following set is denoted by $\mathcal{W}$.

$$\left\{\varphi : (\mathbb{R}\times\mathbb{R}^n,(0,0)) \to (\mathbb{R}^2\times\mathbb{R}^n,(0,0)) \ \text{(Whitney umbrella)}\times\mathbb{R}^{n-1}\text{, pedal unfolding type}\right\}.$$

And set also

$$\begin{aligned}
\mathcal{S} &= \left\{\Phi : (\mathbb{R}\times\mathbb{R}^n,(0,0)) \to (\mathbb{R}^2\times\mathbb{R}^n,(0,0)) \ \text{ normalized (swallowtail)}\times\mathbb{R}^{n-1}\right\},\\
\mathcal{N} &= \left\{\varphi : (\mathbb{R}\times\mathbb{R}^n,(0,0)) \to (\mathbb{R}^2\times\mathbb{R}^n,(0,0)) \ \text{ non-singular, pedal unfolding type}\right\},\\
\mathcal{C} &= \left\{\Phi : (\mathbb{R}\times\mathbb{R}^n,(0,0)) \to (\mathbb{R}^2\times\mathbb{R}^n,(0,0)) \ \text{ normalized (cusp)}\times\mathbb{R}^{n}\right\},
\end{aligned}$$

where a map-germ $\Phi : (\mathbb{R}\times\mathbb{R}^n,(0,0)) \to (\mathbb{R}^2\times\mathbb{R}^n,(0,0))$ is called a *(cusp)*$\times\mathbb{R}^n$ if it is $\mathcal{A}$-equivalent to $(x,y)\mapsto(2x^3,-3x^2,y)$ $(y=(y_1,\ldots,y_n))$. The following Theorems 2.9 and 2.10 are extensions of known calculus correspondences to multi-parameters.

Theorem 2.9.

1. *The map $\mathcal{I} : \mathcal{W}\to\mathcal{S}$ defined by $\mathcal{W}\ni\varphi\mapsto\mathcal{I}(\varphi)\in\mathcal{S}$ is well-defined and bijective.*
2. *The map $\mathcal{D} : \mathcal{S}\to\mathcal{W}$ defined by $\mathcal{S}\ni\Phi\mapsto\mathcal{D}(\Phi)\in\mathcal{W}$ is well-defined and bijective.*

Theorem 2.10.

1. *The map $\mathcal{I} : \mathcal{N}\to\mathcal{C}$ defined by $\mathcal{N}\ni\varphi\mapsto\mathcal{I}(\varphi)\in\mathcal{C}$ is well-defined and bijective.*
2. *The map $\mathcal{D} : \mathcal{C}\to\mathcal{N}$ defined by $\mathcal{C}\ni\Phi\mapsto\mathcal{D}(\Phi)\in\mathcal{N}$ is well-defined and bijective.*

In the case $n=1$, the proofs of Theorems 2.9 and 2.10 can be found in [18]. For the proof of Theorem 2.9 in the case $n=1$, two criteria (Theorems 2.13 and 2.15) have been used in [18]. Theorem 2.15 works well even in the case $n\geq 2$. Although it is uncertain that Theorem 2.13 works well even in the case $n\geq 2$, since a (Whitney umbrella)$\times\mathbb{R}^{n-1}$ is stable, by Mather's infinitesimal characterization of stable map-germs, Theorem 2.9 in general case can be proved. On the other hand, for the proof of Theorem 2.10 in the case $n=1$, Theorem 2.13 has not been used in [18] though Theorem 2.15 has been used. Hence the proof of Theorem 2.10 works well even in the case $n\geq 2$.

Besides Theorems 2.9 and 2.10, there is one more example of calculus correspondence (Proposition 2.11). Since Proposition 2.11 is almost trivial, its proof is omitted. Put

$$\begin{aligned}
&\mathcal{N}_{\text{non-zero}}\\
&= \left\{\varphi : (\mathbb{R}\times\mathbb{R}^n,(0,0)) \to \mathbb{R}^2\times\mathbb{R}^n\text{-}\{(0,0)\} \text{ non-singular, of pedal unfolding type}\right\},\\
&\widetilde{\mathcal{N}}\\
&= \left\{\Phi : (\mathbb{R}\times\mathbb{R}^n,(0,0)) \to (\mathbb{R}^2\times\mathbb{R}^n,(0,0)) \ \text{ normalized non-singular Legendrian}\right\}.
\end{aligned}$$

Proposition 2.11.

1. *The map* $\mathcal{I} : \mathcal{N}_{non\text{-}zero} \to \widetilde{\mathcal{N}}$ *defined by* $\mathcal{N}_{non\text{-}zero} \ni \varphi \mapsto \mathcal{I}(\varphi) \in \widetilde{\mathcal{N}}$ *is well-defined and bijective.*

2. *The map* $\mathcal{D} : \widetilde{\mathcal{N}} \to \mathcal{N}_{non\text{-}zero}$ *defined by* $\widetilde{\mathcal{N}} \ni \Phi \mapsto \mathcal{D}(\Phi) \in \mathcal{N}_{non\text{-}zero}$ *is well-defined and bijective.*

Definition 2.12 ([15])**.** Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation of the form $T(s, \lambda) = (-s, \lambda)$. Two function germs $p_1, p_2 : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ are said to be $\mathcal{K}^T$*-equivalent* if there exists a germ of diffeomorphism $h : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ having the form $h \circ T = T \circ h$ and a function-germ $M : (\mathbb{R}^2, (0, 0)) \to \mathbb{R}$ having the form $M \circ T = M$, $M(0, 0) \neq 0$ such that $p_1 \circ h(s, \lambda) = M(s, \lambda)p_2(s, \lambda)$.

Theorem 2.13 ([15])**.** *Two map-germs* $f_i : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ $(i = 1, 2)$ *of the following form*

$$f_i(x, y) = (n_i(x, y)p_i(x^2, y), x^2, y),$$

where $\frac{\partial n_i}{\partial x}(0, 0) \neq 0$ *and* $p_i(x, y)$ *is not flat for each* $i \in \{1, 2\}$*, are* $\mathcal{A}$*-equivalent if and only if the function-germs* $p_i(x^2, y)$ *are* $\mathcal{K}^T$*-equivalent.*

Definition 2.14. Let $\Phi : (\mathbb{R} \times \mathbb{R}^n, (0, 0)) \to (\mathbb{R}^2 \times \mathbb{R}^n, (0, 0))$ be a Legendrian map-germ and let ν_Φ be a unit normal vector field of Φ given in the definition of Legendrian map-germs. The function-germ $LJ_\Phi : (\mathbb{R} \times \mathbb{R}^n, 0) \to \mathbb{R}$ defined by the following is called the *Legendrian-Jacobian* of Φ where $(x, y) = (x, y_1, \ldots, y_n)$.

$$LJ_\Phi(x, y) = \det\left(\frac{\partial \Phi}{\partial x}(x, y), \frac{\partial \Phi}{\partial y_1}(x, y), \ldots, \frac{\partial \Phi}{\partial y_n}(x, y), \nu_\Phi(x, y)\right).$$

In the case $n = 1$, Definition 2.14 can be found in [18]. Note that if ν_Φ satisfies the conditions of unit normal vector field of Φ, then $-\nu_\Phi$ also satisfies them. Thus, the sign of $LJ_\Phi(x, y)$ depends on the particular choice of unit normal vector field ν_Φ. The Legendrian Jacobian of Φ is called also the *signed area density function* (for instance, see [20]). Although it seems reasonable to call LJ_Φ the area density function from the viewpoint of investigating the singular surface $\Phi(U)$ (U is a sufficiently small neighborhood of the origin of $\mathbb{R}^2$), it seems reasonable to call it the Legendrian Jacobian from the viewpoint of investigating the singular map-germ Φ.

Theorem 2.15 ([19])**.** *Let* $\Phi : (\mathbb{R} \times \mathbb{R}^n, (0, 0)) \to (\mathbb{R}^2 \times \mathbb{R}^n, (0, 0))$ *be a normalized Legendrian map-germ,*

1. *The given* Φ *is a (swallowtail)*$\times\mathbb{R}^{n-1}$ *if and only if the following two hold where* $y = (y_1, \ldots y_n)$*:*

$$Q\left(LJ_\Phi, \frac{\partial LJ_\Phi}{\partial x}\right) \cong Q(x, y_1), \;\; \frac{\partial^2 LJ_\Phi}{\partial x^2}(0, 0) \neq 0.$$

2. *The given Φ is a (cusp)$\times\mathbb{R}^n$ if and only if the following two hold:*

$$Q\left(LJ_\Phi\right) \cong Q(x),\ \frac{\partial LJ_\Phi}{\partial x}(0,0) \neq 0.$$

Here, $Q(f_1,\ldots,f_\ell)$ stands for Mather's local algebra for function-germs $f_1,\ldots,f_\ell$. For Mather's local algebra, see [14, 21]. Theorems 2.13 (resp., Theorem 2.15) is used as a criterion of Whitney umbrella (resp., (swallowtail)$\times\mathbb{R}^{n-1}$). Theorems 2.13 and 2.15 are connected by the following simple lemma.

Lemma 2.16. *For a normalized Legendrian map-germ $\Phi : (\mathbb{R}\times\mathbb{R}^n,(0,0)) \to (\mathbb{R}^2\times\mathbb{R}^n,(0,0))$,*

$$LJ_\Phi(x,y) = (-1)^{n+1}\frac{\frac{\partial\Phi_2}{\partial x}(x,y)}{\nu_1(x,y)}.$$

Here $\nu_\Phi(x,y) = \nu_1(x,y)\frac{\partial}{\partial X_1} + \nu_2(x,y)\frac{\partial}{\partial X_2} + \cdots + \nu_{n+2}(x,y)\frac{\partial}{\partial X_{n+2}}$.

In the case $n=1$, Lemma 2.16 with its proof can be found in [18]. The proof given in [18] works well in general case.

§ 3. Applications of calculus correspondence

In order to show that the calculus correspondence is significant and useful, we give two applications of Theorem 2.9.

Proposition 3.1. *Let $\Phi : (\mathbb{R}\times\mathbb{R}^n,(0,0)) \to (\mathbb{R}^2\times\mathbb{R}^n,(0,0))$ be given by*

$$\Phi(x,y) = \left(ax^4 + x^2\sum_{i=1}^n b_iy_i + \Phi_1(x,y), cx^3 + x\sum_{i=1}^n d_iy_i + \Phi_2(x,y), y\right)$$

where $y = (y_1,\ldots,y_n) \in \mathbb{R}^n$, $\{a, b_i, c, d_i\} \subset \mathbb{R}$ and $\Phi_i : (\mathbb{R}^2,0) \to (\mathbb{R},0)$ is a C^∞ function-germ such that $j^{5-i}\Phi_i(0,0) = 0$ $(i=1,2)$. Then, the following two are equivalent.

1. *The given Φ is a (swallowtail)$\times\mathbb{R}^{n-1}$, that is, it is $\mathcal{A}$-equivalent to the normal form of (swallowtail)$\times\mathbb{R}^{n-1}$ which is the following*

$$(x,y) \mapsto \left(3x^4 + x^2y_1, -4x^3 - 2xy_1, y\right).$$

2. *The following three hold:*
   (a) *There exists an i $(1 \le i \le n)$ such that $b_ic \neq 0$ is satisfied.*
   (b) *The equality $2ad_i = 3b_ic$ holds for any i $(1 \le i \le n)$*

(c) *The function-germ* $\frac{4ax^3+2x\sum_{i=1}^n b_iy_i+\frac{\partial\Phi_1}{\partial x}(x,y)}{3cx^2+\sum_{i=1}^n d_iy_i+\frac{\partial\Phi_2}{\partial x}(x,y)}$ *is well defined and of class* C^∞.

Proof of Proposition 3.1. Suppose that Φ is a (swallowtail)$\times\mathbb{R}^{n-1}$. Then, since Φ is Legendrian, there exists a unit normal vector field

$$\nu_\Phi(x,y) = (\nu_1(x,y), \nu_2(x,y), \ldots, \nu_{n+2}(x,y))$$

such that the following two hold:

(3.1)
$$\nu_1(x,y)\left(4ax^3+2x\sum_{i=1}^n b_iy_i+\frac{\partial\Phi_1}{\partial x}(x,y)\right)+\nu_2(x,y)\left(3cx^2+\sum_{i=1}^n d_iy_i+\frac{\partial\Phi_2}{\partial x}(x,y)\right)=0,$$

(3.2)
$$\nu_1(x,y)\left(b_ix^2+\frac{\partial\Phi_1}{\partial y}(x,y)\right)+\nu_2(x,y)\left(d_ix+\frac{\partial\Phi_2}{\partial y}(x,y)\right)+\nu_{2+i}(x,y)=0 \quad (1\le i\le n).$$

Since Φ is a (swallowtail)$\times\mathbb{R}^{n-1}$, $\nu_2(0,0)$ (resp., $\nu_{2+i}(0,0)$) must be zero by the equality (3.1) (resp., (3.2)). Since $\nu_\Phi(0,0)$ is a unit vector, $\nu_1(0,0)$ must be ±1. It is clear that the given Φ satisfies the first and the second conditions of Definition 2.6. Thus, Φ is a normalized (swallowtail)$\times\mathbb{R}^{n-1}$. By Theorem 2.9,

$$\mathcal{D}(\Phi)(x,y) = \left(4ax^3+2x\sum_{i=1}^n b_iy_i+\frac{\partial\Phi_1}{\partial x}(x,y), 3cx^2+\sum_{i=1}^n d_iy_i+\frac{\partial\Phi_2}{\partial x}(x,y), y\right)$$

is a (Whitney umbrella)$\times\mathbb{R}^{n-1}$ of pedal unfolding type. Since $\mathcal{D}(\Phi)$ is a (Whitney umbrella)$\times\mathbb{R}^{n-1}$, there must exist an i $(1\le i\le n)$ such that $b_ic\neq 0$. Since $\mathcal{D}(\Phi)$ is of pedal unfolding type, we have that $2ad_i = 3b_ic$ for any i $(1\le i\le n)$ and the function-germ $\frac{4ax^3+2x\sum_{i=1}^n b_iy_i+\frac{\partial\Phi_1}{\partial x}(x,y)}{3cx^2+\sum_{i=1}^n d_iy_i+\frac{\partial\Phi_2}{\partial x}(x,y)}$ is well-defined and of class C^∞.

Conversely, suppose that there exists an i $(1\le i\le n)$ such that $b_ic\neq 0$ is satisfied, the equality $2ad_i = 3b_ic$ holds for any i $(1\le i\le n)$ and the function-germ $\frac{4ax^3+2x\sum_{i=1}^n b_iy_i+\frac{\partial\Phi_1}{\partial x}(x,y)}{3cx^2+\sum_{i=1}^n d_iy_i+\frac{\partial\Phi_2}{\partial x}(x,y)}$ is well-defined and of class C^∞. Then, $\mathcal{D}(\Phi)$ is a (Whitney umbrella)$\times\mathbb{R}^{n-1}$ of pedal unfolding. Therefore, by Theorem 2.9, $\Phi = \mathcal{I}(\mathcal{D}(\Phi))$ is a normalized (swallowtail)$\times\mathbb{R}^{n-1}$. □

As a special case of Proposition 3.1, we have the following:

Corollary 3.2. *Let* $\Phi : (\mathbb{R}\times\mathbb{R}^n, (0,0)) \to (\mathbb{R}^2\times\mathbb{R}^n, (0,0))$ *be given by*

$$\Phi(x,y) = \left(ax^4+x^2\sum_{i=1}^n b_iy_i, cx^3+x\sum_{i=1}^n d_iy_i, y\right)$$

where $y=(y_1,\ldots,y_n)\in\mathbb{R}^n$ *and* $\{a,b_i,c,d_i\}\subset\mathbb{R}$. *Then, the following two are equivalent.*

1. *The given Φ is a (swallowtail)$\times\mathbb{R}^{n-1}$.*

2. *The following two hold:*

   (a) *There exists an i $(1 \leq i \leq n)$ such that $b_i c \neq 0$ is satisfied.*

   (b) *The equality $2ad_i = 3b_i c$ holds for any i $(1 \leq i \leq n)$.*

It is interesting to compare Proposition 3.1 or Corollary 3.2 with the following fact which has been used in the proof of Proposition 3.1.

Fact 3.3. *Let $\Phi : (\mathbb{R} \times \mathbb{R}^n, (0,0)) \to (\mathbb{R}^2 \times \mathbb{R}^n, (0,0))$ be given by*

$$\Phi(x,y) = \left(ax^4 + x^2 \sum_{i=1}^{n} b_i y_i + \Phi_1(x,y), cx^3 + x \sum_{i=1}^{n} d_i y_i + \Phi_2(x,y), y \right)$$

where $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, $\{a, b_i, c, d_i\} \subset \mathbb{R}$ and $\Phi_i : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ is a C^∞ function-germ such that $j^{5-i}\Phi_i(0,0) = 0 \quad (i = 1,2)$. Then, the following two are equivalent.

1. *The map-germ $\mathcal{D}(\Phi)$ is a (Whitney umbrella)$\times\mathbb{R}^{n-1}$, that is, it is $\mathcal{A}$-equivalent to the normal form of (Whitney umbrella)$\times\mathbb{R}^{n-1}$ which is the following*

$$(x,y) \mapsto \left(xy_1, x^2, y\right).$$

2. *There exists an i $(1 \leq i \leq n)$ such that $b_i c \neq 0$ is satisfied.*

As another application of Theorem 2.9, we give an alternative proof of Arnol'd's observation given in [2] (for Arnol'd's observation, see also [8]), Namely, we show the following:

Observation 3.4. *Let $\gamma : (\mathbb{R}, 0) \to (\mathbb{R}^3, 0)$ be the space curve given by $\gamma(x) = (x^4, x^3, x^2)$. Then, the tangent developable of γ, which is the following, is a swallowtail.*

$$\Phi(x,y) = \left(x^4, x^3, x^2\right) + y\left(4x^2, 3x, 2\right).$$

Proof of Observation 3.4. Put $\widetilde{y} = x^2 + 2y$. Then, Φ is $\mathcal{R}$-equivalent to $\widetilde{\Phi}(x, \widetilde{y}) = \left(-x^4 + 2x^2\widetilde{y}, -\frac{1}{2}x^3 + \frac{3}{2}x\widetilde{y}, \widetilde{y}\right)$. It is easily seen that $\mathcal{D}(\widetilde{\Phi})$ is a Whitney umbrella of pedal unfolding type. Thus, by Theorem 2.9, $\widetilde{\Phi} = \mathcal{I}(\mathcal{D}(\widetilde{\Phi}))$ is a normalized swallowtail. $\square$

§ 4. Questions around calculus correspondences

The following question is a multi-parameter version of the question posed in [18].

Question 4.1.

1. *Let $\varphi_1, \varphi_2 : (\mathbb{R}\times\mathbb{R}^n, (0,0)) \to (\mathbb{R}^2\times\mathbb{R}^n, (0,0))$ be two map-germs of pedal unfolding type. Suppose that φ_1 is $\mathcal{A}$-equivalent to φ_2. Is $\mathcal{I}(\varphi_1)$ necessarily $\mathcal{A}$-equivalent to $\mathcal{I}(\varphi_2)$?*

2. *Let $\Phi_1, \Phi_2 : (\mathbb{R}\times\mathbb{R}^n, (0,0)) \to (\mathbb{R}^2\times\mathbb{R}^n, (0,0))$ be two normalized Legendrian map-germs. Suppose that Φ_1 is $\mathcal{A}$-equivalent to Φ_2. Is $\mathcal{D}(\Phi_1)$ necessarily $\mathcal{A}$-equivalent to $\mathcal{D}(\Phi_2)$?*

Question 4.1 seems to be difficult to solve completely in general. In the following two subsections, we discuss special cases of Question 4.1.

§ 4.1. S_k type singularities and Legendrian S_k type singularities

Recall that a map-germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is of S_k type if f is $\mathcal{A}$-equivalent to the map-germ $f_{k,\pm}(x, y) = \left(x\left(x^2 \pm y^{k+1}\right), x^2 \pm y^{k+1}, y\right)$ (Definition 2.4). Since the map-germ $f_{k,\pm}$ is of pedal unfolding type, the following map-germ (which is $\mathcal{I}(f_{k,\pm})$) is normalized Legendrian map-germ by Proposition 2.8.

$$F_{k,\pm}(x, y) = \left(\frac{1}{4}x^4 \pm \frac{1}{2}x^2y^{k+1}, \frac{1}{3}x^3 \pm xy^{k+1}, y\right).$$

The Legendrian map-germ $\mathcal{I}(f_{k,\pm})$ is called the *normal form of Legendrian S_k type* and any Legendrian map-germ $\mathcal{A}$-equivalent to $\mathcal{I}(f_{k,\pm})$ is said to be *of Legendrian S_k type.*

Question 4.2.

1. *Let $\varphi : (\mathbb{R}\times\mathbb{R}, (0,0)) \to (\mathbb{R}^2\times\mathbb{R}, (0,0))$ be a map-germ of pedal unfolding type. Suppose that φ is of S_k type. Is $\mathcal{I}(\varphi)$ necessarily of Legendrian S_k type ?*

2. *Let $\Phi : (\mathbb{R}\times\mathbb{R}, (0,0)) \to (\mathbb{R}^2\times\mathbb{R}, (0,0))$ be a normalized Legendrian map-germ. Suppose that Φ is of Legendrian S_k type. Is $\mathcal{D}(\Phi)$ necessarily of S_k type ?*

In the case $k = 0$, both $f_{0,+}, f_{0,-}$ are $\mathcal{A}$-equivalent to the normal form of Whitney umbrella, and both $F_{0,+}, F_{0,-}$ are $\mathcal{A}$-equivalent to the normal form of swallowtail (namely, the map-germ $(x, y) \mapsto (3x^4 + x^2y, -4x^3 - 2xy, y)$). In this case, we have the calculus correspondence by Theorem 2.9.

In the case $k = 1$, $f_{1,+}$ (resp., $F_{1,+}$) is not $\mathcal{A}$-equivalent to $f_{1,-}$ (resp., $F_{1,-}$). It is known that only the map-germs of S_1 type are $\mathcal{A}_e$-codimension one singularities of mono-germs from the plane to the 3-space (for $\mathcal{A}_e$-codimension, see [21] and for the classification of $\mathcal{A}_e$-codimension one singularities $(\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$, see [5, 6, 15]). Theorem 2.13 can be applied as a criterion of S_1 singularities. On the other hand, criteria of Legendrian S_1 singularities have been obtained by Izumiya-Saji-Takahashi ([11]).

Thus, by replacing Saji-Umehara-Yamada criterion (Theorem 2.15) with Izumiya-Saji-Takahashi criteria given in [11], the proof of Theorem 2.9 is expected to work well to show calculus correspondence between S_1 singularities of pedal unfolding type and normalized Legendrian S_1 singularities.

Next, we discuss the case $k \geq 2$. Even in this case, Theorem 2.13 can be applied as a criterion of S_k singularities. However, there seems to be no criteria for Legendrian S_k singularities in the case $k \geq 2$. Hence, it seems that we cannot expect an analogy of the proof of Theorem 2.9.

§ 4.2. Legendrian A_k type singularities

Definition 4.3 ([19]). Let k, n be non-negative integers such that $k \leq n+1$.

1. The map-germ $G_k : (\mathbb{R} \times \mathbb{R}^n, (0,0)) \to (\mathbb{R}^2 \times \mathbb{R}^n, (0,0))$ given by

$$G_k(x,y) = \left((k+1)x^{k+2} + \sum_{j=1}^{k-1} j x^{j+1} y_j, -(k+2)x^{k+1} - \sum_{j=1}^{k-1} (j+1) x^j y_j, y \right)$$

   is called the *normal form of Legendrian A_{k+1} type*, where $(x,y) = (x, y_1, \ldots, y_n)$.

2. A map-germ $\Phi : (\mathbb{R} \times \mathbb{R}^n, (0,0)) \to (\mathbb{R}^2 \times \mathbb{R}^n, (0,0))$ is said to be *of Legendrian A_{k+1} type* if Φ is $\mathcal{A}$-equivalent to G_k.

Note that the image of G_k is the envelope of the following one parameter family of hyperplanes. By this reason, G_k is called the normal form of A_{k+1} type.

$$\left\{ (X_1, X_2, Y_1, \ldots, Y_n) \mid x^{k+2} + Y_{k-1}x^k + \cdots + Y_1 x^2 + X_2 x + X_1 = 0. \right\}$$

For the normal form of Legendrian A_{k+1} type, we have

$$\mathcal{D}(G_k)(x,y) = (n(x,y)p(x,y), p(x,y), y),$$

where $n(x,y) = -x$ and $p(x,y) = -(k+2)(k+1)x^k - \sum_{j=1}^{k-1} j(j+1)x^{j-1}y_j$. Since, $p(0,0) = 0$ and $\frac{\partial n}{\partial x}(0,0) \neq 0$, $\mathcal{D}(G_k)$ is of pedal unfolding type. Therefore, $G_k = \mathcal{I}(\mathcal{D}(G_k))$ is normalized Legendrian by Proposition 2.8.

Question 4.4.

1. *Let $\varphi : (\mathbb{R} \times \mathbb{R}^n, (0,0)) \to (\mathbb{R}^2 \times \mathbb{R}^n, (0,0))$ be a map-germ of pedal unfolding type. Suppose that φ is $\mathcal{A}$-equivalent to $\mathcal{D}(G_k)$. Is $\mathcal{I}(\varphi)$ necessarily of Legendrian A_{k+1} type ?*

2. *Let $\Phi : (\mathbb{R} \times \mathbb{R}^n, (0,0)) \to (\mathbb{R}^2 \times \mathbb{R}^n, (0,0))$ be a normalized Legendrian map-germ. Suppose that Φ is of Legendrian A_{k+1} type. Is $\mathcal{D}(\Phi)$ necessarily $\mathcal{A}$-equivalent to $\mathcal{D}(G_k)$?*

Question 4.4 was asked by G. Ishikawa ([9]), and independently by T. Gaffney during AMS Spring Western Section Meeting at the University of Hawaii (2012). It is easily seen that G_1 is non-singular, G_2 is the normal form of (cusp)$\times\mathbb{R}^n$ and G_3 is the normal form of (swallowtail)$\times\mathbb{R}^{n-1}$. Thus, in the case $k = 0, 1, 2$, Proposition 2.11, Theorem 2.10 and Theorem 2.9 are the affirmative answers to Question 4.4 respectively.

Therefore, Question 4.4 asks essentially the case $k \geq 3$. Even in this case, there is a criterion of Legendrian A_{k+1} singularities (Theorem 4.5). However, there seems to be no criteria for the $\mathcal{A}$-equivalence class of $\mathcal{D}(G_k)$ ($k \geq 3$). Hence, it seems that we cannot expect an analogy of the proof of Theorem 2.9.

Theorem 4.5 ([19]). *For a normalized Legendrian map-germ $\Phi : (\mathbb{R}\times\mathbb{R}^n, (0,0)) \to (\mathbb{R}^2 \times \mathbb{R}^n, (0,0))$, Φ is of Legendrian A_{k+1} type if and only if the following two hold:*

$$Q\left(LJ_\Phi, \frac{\partial LJ_\Phi}{\partial x}, \ldots, \frac{\partial^{k-1} LJ_\Phi}{\partial x^{k-1}}\right) \cong Q(x, y_1, \ldots, y_{k-1}), \ \frac{\partial^k LJ_\Phi}{\partial x^k}(0,0) \neq 0.$$

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