RIMS Kôkyûroku Bessatsu
B38 (2013), 31–43

On embedding lifts over a Morse function on a circle

Dedicated to Professor Shyuich Izumiya on the occasion of his 60th birthday

By

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Abstract

Let $f : S^1 \to \mathbb{R}$ be a Morse function and $\Pi : \mathbb{R}^2 \to \mathbb{R}$ an orthogonal projection. In [4], Saeki and Takase posed the following problem: "Determine those Morse functions $f : S^1 \to \mathbb{R}$ which have an embedding $\tilde{f} : S^1 \to \mathbb{R}^2$ such that $\Pi \circ \tilde{f} = f$". In this paper, we give a complete answer to this problem and give an application to the existence problem of embedding lifts for fold maps.

§ 1. Introduction

Throughout the paper, all manifolds and maps are differentiable of class C^∞. Let $f : S^1 \to \mathbb{R}$ be a function on an oriented circle. A point $q \in S^1$ is a *singular point* of f if the differential df_q at q vanishes. We denote by $S(f)$ the set of singular points of f. For $q \in S(f)$, we call $f(q)$ a *singular value* of f. A function $f : S^1 \to \mathbb{R}$ is a *Morse function* if for each $q \in S(f)$, there exist local coordinates x around $q \in S^1$ and y around $f(q) \in \mathbb{R}$ such that f has the form $y \circ f = \pm x^2$. If the singular values of a Morse function f are all distinct, then we call f a *stable Morse function*.

Let $f : S^1 \to \mathbb{R}$ be a Morse function and $\Pi : \mathbb{R}^2 \to \mathbb{R}$ the orthogonal projection defined by $\Pi(y_1, y_2) = y_1$. Saeki and Takase [4] pose the following problem: Determine those Morse functions $f : S^1 \to \mathbb{R}$ which have an embedding $\tilde{f} : S^1 \to \mathbb{R}^2$ such that $\Pi \circ \tilde{f} = f$. We call such a map $\tilde{f} : S^1 \to \mathbb{R}^2$ an *embedding lift* over f. In this paper, we give a complete answer to this problem.

This paper is organized as follows. In Section 2, we give a necessary and sufficient condition for the existence of an embedding lift $\tilde{f} : S^1 \to \mathbb{R}^2$ over a given Morse function

Received June 26, 2012. Revised February 18, 2013.
2000 Mathematics Subject Classification(s): 57R45, 57R40
Key Words: circle, Morse function, embedding lift, fold map
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$f : S^1 \to \mathbb{R}$. In Section 3, as an application, for each interger $n > 1$, we give an example of a stable fold map $f : S^n \to \mathbb{R}^n$ of the n-dimensional sphere which has an immersion lift $f' : S^n \to \mathbb{R}^{n+1}$, but which does not have an embedding lift into $\mathbb{R}^{n+1}$.

The author would like to express his sincere gratitude to Prof. Osamu Saeki and Prof. Masamichi Takase for invaluable comments and encouragement. Especially, Section 3 is due to their advice. The author expresses his gratitude to the referee for pointing out many mistakes in earlier versions of the paper. The author also thanks Prof. Kentaro Saji who is the editor of this volume.

§ 2. Embedding lift over a Morse function

In this section, we study the existence of an embedding lift $\tilde{f} : S^1 \to \mathbb{R}^2$ over a given Morse function $f : S^1 \to \mathbb{R}$.

Definition 2.1. Let $f : S^1 \to \mathbb{R}$ be a Morse function on an oriented circle. Let $\{p_1, p_2, \ldots, p_k\}$, $k \geq 1$ (resp. $\{q_1, q_2, \ldots, q_l\}$, $l \geq 1$) be the set of those points in S^1 where f takes its minimum (resp. maximum). If, by using the orientation of S^1, $p_1, p_2, \ldots, p_k, q_1, q_2, \ldots, q_l$ are arranged as $p_1 < p_2 < \cdots < p_k < q_1 < q_2 < \cdots < q_l$ in a cyclic sense on S^1 (this means that if we start p_1 and go along S^1 in the direction given by the orientation, we encounter $p_2, p_3, \ldots, p_k, q_1, q_2, \ldots, q_l$ in this order and then p_1), we say that $\{p_1, p_2, \ldots, p_k\}$ and $\{q_1, q_2, \ldots, q_l\}$ are *separated*.

Then, we have the following theorem which answers the problem posed in [4].

Theorem 2.2. *Let $f : S^1 \to \mathbb{R}$ be a Morse function on an oriented circle. Let $\{p_1, p_2, \ldots, p_k\}$, $k \geq 1$ (resp. $\{q_1, q_2, \ldots, q_l\}$, $l \geq 1$) be the set of those points in S^1 where f takes its minimum (resp. maximum). Then there exists an embedding lift $\tilde{f} : S^1 \to \mathbb{R}^2$ over f if and only if $\{p_1, p_2, \ldots, p_k\}$ and $\{q_1, q_2, \ldots, q_l\}$ are separated.*

Proof. Suppose that $\{p_1, p_2, \ldots, p_k\}$ and $\{q_1, q_2, \ldots, q_l\}$ are separated. We may assume that by using the orientation of S^1, $p_1, p_2, \ldots, p_k, q_1, q_2, \ldots, q_l$ are arranged as $p_1 < p_2 < \cdots < p_k < q_1 < q_2 < \cdots < q_l$ in a cyclic sense on S^1. We cut S^1 at p_1 and q_1. Let A_1 (resp. A_2) be the oriented arc in S^1 whose initial point is p_1 (resp. q_1) and whose terminal point is q_1 (resp. p_1). First, we can construct an embedding lift $\tilde{f}_1 : A_1 \to \mathbb{R}^2$ over $f|A_1$ which satisfies the following properties (see Figure 1(1)).

1. $\tilde{f}_1(A_1) \subset \mathbb{R} \times [0, \infty)$.
2. $\tilde{f}_1(p_1) = (f(p_1), 0) \in \mathbb{R}^2, \tilde{f}_1(q_1) = (f(q_1), 0) \in \mathbb{R}^2$.
3. If we put the critical points of $f|A_1$ as $p_1 = \alpha_0 < \alpha_1 < \cdots < \alpha_{a-1} < \alpha_a = q_1$, then $\tilde{f}_1(\alpha_i) = (f(\alpha_i), i) \in \mathbb{R}^2$ holds $(i = 0, 1, \ldots, a-1)$.

Similarly, we can construct an embedding lift $\tilde{f}_2 : A_2 \to \mathbb{R}^2$ over $f|A_2$ which satisfies the following properties (see Figure 1(2)).

1. $\tilde{f}_2(A_2) \subset \mathbb{R} \times (-\infty, 0]$.

2. $\tilde{f}_2(p_1) = (f(p_1), 0) \in \mathbb{R}^2, \tilde{f}_2(q_1) = (f(q_1), 0) \in \mathbb{R}^2$.

3. If we put the critical points of $f|A_2$ as $q_1 = \beta_0 < \beta_1 < \cdots < \beta_{b-1} < \beta_b = p_1$, then $\tilde{f}_2(\beta_j) = (f(\beta_j), -j) \in \mathbb{R}^2$ holds $(j = 0, 1, \ldots, b-1)$.

Then, by attaching the two maps $\tilde{f}_1$ and $\tilde{f}_2$, we have an embedding $\tilde{f} = \tilde{f}_1 \cup \tilde{f}_2 : S^1 = A_1 \cup A_2 \to \mathbb{R}^2$. This $\tilde{f}$ is the desired embedding lift over f.

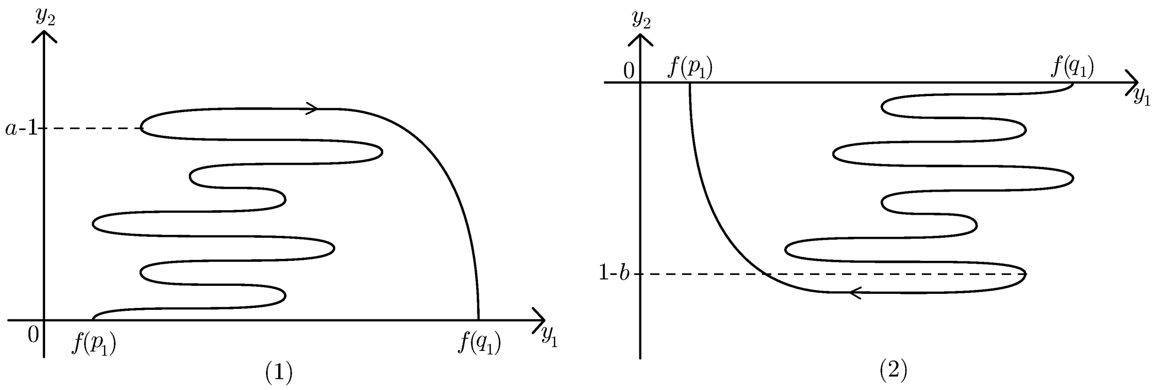

Figure 1. Embedding lifts (1) $\tilde{f}_1 : A_1 \to \mathbb{R}^2$ over $f|A_1$ and (2) $\tilde{f}_2 : A_2 \to \mathbb{R}^2$ over $f|A_2$.

Conversely, suppose that there exists an embedding lift $\tilde{f} : S^1 \to \mathbb{R}^2$ over the given Morse function f. By the Jordan curve theorem, $\mathbb{R}^2 \setminus \tilde{f}(S^1)$ is the union of two disjoint connected open sets U_1 and U_2. Let U_1 be the bounded set. We assume that the orientation of S^1 satisfies the following: when we walk along $\tilde{f}(S^1)$, we have U_1 to our right. Let $\{q_1, q_2, \ldots, q_l\}$, $l \geq 1$, be the set of those points in S^1 where f takes its maximum and K_i a sufficiently small oriented arc around $\tilde{f}(q_i)$ in $\tilde{f}(S^1)$ such that the initial point of K_i is s_i and the terminal point of K_i is t_i $(i = 1, 2, \ldots, l)$. Since $U_1 \subset \Pi^{-1}((-\infty, f(q_1)])$ must be satisfied, by renumbering the indices we may assume the following properties (see Figure 2).

1. $f(s_1) = f(t_1) = f(s_2) = f(t_2) = \cdots = f(s_l) = f(t_l)$.

2. (y_2-coordinate of s_1) $>$ (y_2-coordinate of $\tilde{f}(q_1)$) $>$ (y_2-coordinate of t_1) $> \cdots >$ (y_2-coordinate of s_l) $>$ (y_2-coordinate of $\tilde{f}(q_l)$) $>$ (y_2-coordinate of t_l).

Then, we have the following lemma.

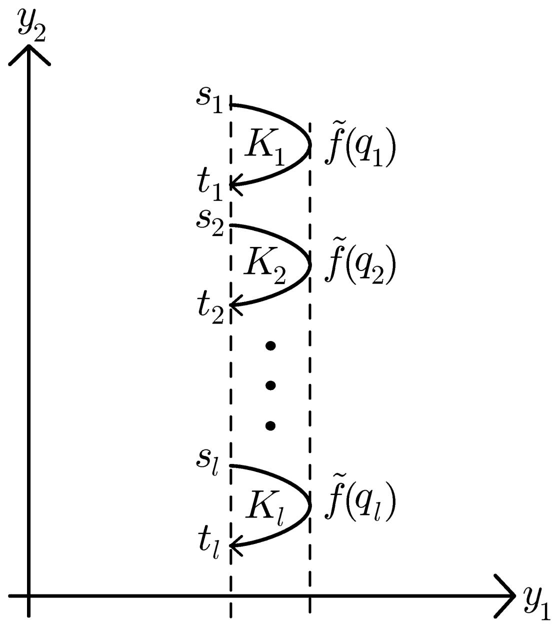

Figure 2. Arrangement of $\tilde{f}(q_i)$ and K_i $(i = 1, 2, \ldots, l)$ in $\mathbb{R}^2$.

Lemma 2.3. *The maximum points of f necessarily satisfy $q_1 < q_2 < \cdots < q_l$ in a cyclic sense on S^1.*

Proof. Let i be an integer such that $1 \leq i < l$. Suppose that q_j is the first maximum point that we encounter after leaving q_i in the direction given by the orientation of S^1 and suppose that $j > i + 1$.

Let A be the oriented arc in S^1 whose initial point is q_i and whose terminal point is q_j and L a line segment in $\mathbb{R}^2$ which connects $\tilde{f}(q_i)$ and $\tilde{f}(q_j)$. Note that $\tilde{f}(A) \cup L$ is a simple closed curve, and let V be the open region bounded by $\tilde{f}(A) \cup L$. Then U_1 and V must satisfy $U_1 \cap V = \emptyset$: however, V contains the small oriented arc K_m $(i < m < j)$ (see Figure 3). This is a contradiction. Therefore, $j \leq i + 1$ holds.

Let us consider the case $i = 1$. Since $j \leq 2$ and $q_1 \neq q_j$, we have $j = 2$. Then, for $i = 2$, since $j \leq 3$ and $q_2 \neq q_j$, we have $j = 1$ or $j = 3$. If $j = 1$, then we have exactly two maximum points on S^1: therefore, $l = 2$, but in the first paragraph of the proof, we assumed that $i = 2 < l$. Then we have $j = 3$. We can repeat this argument until $i = l - 1$. This completes the proof of the lemma. □

Now we go back to the proof of Theorem 2.2. Suppose that between two maximum points q_i and q_{i+1}, there exists a point p in S^1 where f takes its minimum and that q_i, q_{i+1}, p satisfy $q_i < p < q_{i+1}$ in a cyclic sense on S^1 for some $i = 1, 2, \ldots, l - 1$. Suppose that p is the first minimum point that we encounter after leaving q_i in the

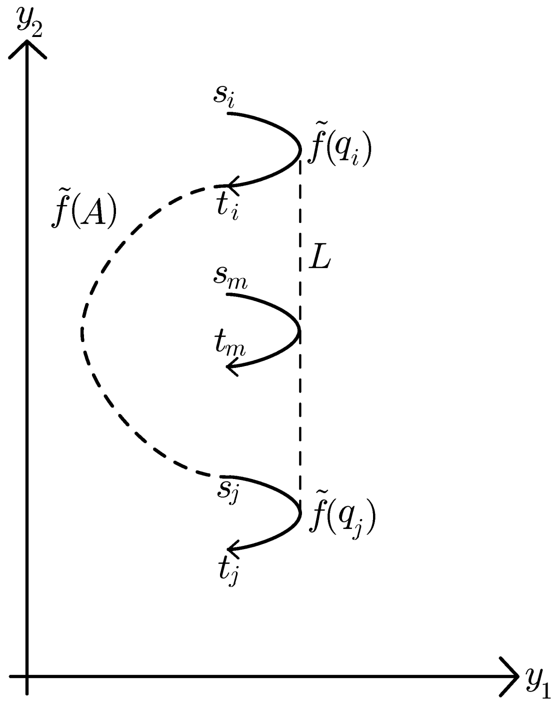

Figure 3. Arrangements of $\tilde{f}(A)$ and L.

direction given by the orientation. Let K be a sufficiently small oriented arc around $\tilde{f}(p)$ in $\tilde{f}(S^1)$ such that the initial point of K is s and the terminal point of K is t. Since $\Pi^{-1}\left([f(p), \infty)\right) \supset U_1$ must be satisfied, we may assume the following properties (see Figure 4).

1. $f(s) = f(t)$.

2. (y_2-coordinate of s) $<$ (y_2-coordinate of $\tilde{f}(p)$) $<$ (y_2-coordinate of t).

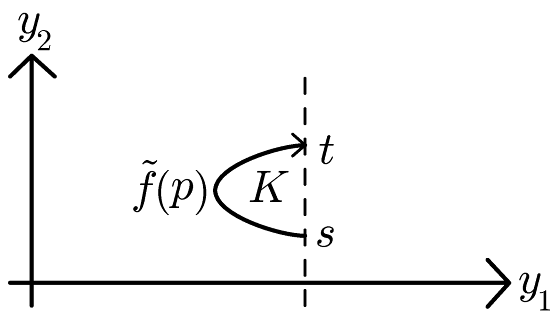

Figure 4. Arrangement of $\tilde{f}(p)$ and K in $\mathbb{R}^2$.

Let $L_1 = \{(f(p), y_2) \in \mathbb{R}^2 \mid y_2 \geq (y_2\text{-coordinate of } \tilde{f}(p))\}$ and $L_2 = \{(f(q_i), y_2) \in \mathbb{R}^2 \mid y_2 \geq (y_2\text{-coordinate of } \tilde{f}(q_i))\}$ be half lines and A the oriented arc in S^1 whose initial point is q_i and whose terminal point is p. The proper arc $L_1 \cup \tilde{f}(A) \cup L_2$ separates $\mathbb{R}^2$ into two open regions. Since the point t is in one open region and the point s_{i+1} is in the other open region, the arc connecting t and s_{i+1} must cross the proper arc $L_1 \cup \tilde{f}(A) \cup L_2$ (see Figure 5). This is a contradiction.

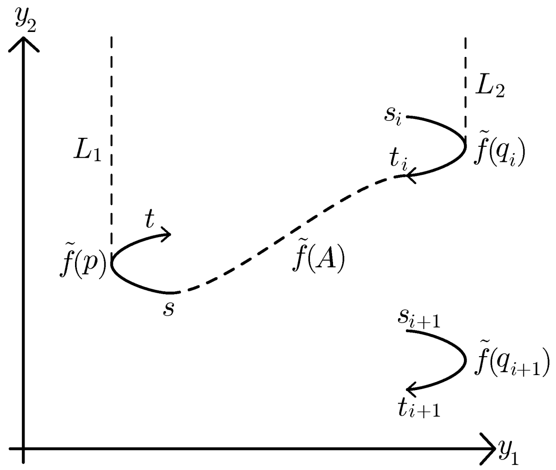

Figure 5. Arrangements of L_1, L_2 and $\tilde{f}(A)$.

Therefore, in the oriented arc in S^1 whose initial point is q_1 and whose terminal point is q_l, there does not exist a point where f takes its minimum. This means that the set of points in S^1 where f takes its minimum and the set of points in S^1 where f takes its maximum are separated. This completes the proof of Theorem 2.2. □

Remark 2.4. If a Morse function $f : S^1 \to \mathbb{R}$ has exactly one minimum or exactly one maximum, then f has an embedding lift $\tilde{f} : S^1 \to \mathbb{R}^2$. In particular, every stable Morse function has an embedding lift.

§ 3. Application

In this section, as an application of Theorem 2.2, we study the existence of an embedding lift over a stable fold map of a closed n-dimensional manifold into $\mathbb{R}^n$ $(n > 1)$.

Let M be a closed n-dimensional manifold and $f : M \to \mathbb{R}^n$ a smooth map. A point $q \in M$ is a *singular point* of f if the rank of the differential df_q at q is strictly smaller than n. We denote by $S(f)$ the set of singular points of f. A smooth map $f : M \to \mathbb{R}^n$ is a *fold map* if for each $q \in S(f)$, there exist local coordinates $(x_1, x_2, \ldots, x_n)$ around

$q \in M$ and $(y_1, y_2, \ldots, y_n)$ around $f(q) \in \mathbb{R}^n$ such that f has the form

$$\begin{cases} y_i \circ f = x_i \qquad (i = 1, 2, \ldots, n-1), \\ y_n \circ f = x_n^2. \end{cases}$$

We remark that for a fold map $f : M \to \mathbb{R}^n$, $S(f)$ is an $(n-1)$-dimensional closed submanifold of M and $f|S(f)$ is an immersion. We mention that the fold maps that appear here are exactly the same as the special generic maps as defined, for example, in [4]. If a fold map $f : M \to \mathbb{R}^n$ satisfies that $f|S(f)$ is an immersion with normal crossings, then we call f a *stable fold map* (see [1]).

Let $\Pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be an orthogonal projection and $f : M \to \mathbb{R}^n$ a stable fold map. If there exists an embedding $\tilde{f} : M \to \mathbb{R}^{n+1}$ such that $\Pi \circ \tilde{f} = f$, then we say that f has an *embedding lift* $\tilde{f}$. Note that if M is orientable, any fold map $f : M \to \mathbb{R}^n$, has an immersion $f' : M \to \mathbb{R}^{n+1}$ such that $\Pi \circ f' = f$ (see [5]).

Let $f : M \to \mathbb{R}^n$ be a stable fold map and $l \subset \mathbb{R}^n$ an embedded arc such that $f|S(f)$ is transverse to l, $l \cap f(M) \neq \emptyset$ and $f^{-1}(\partial l) = \emptyset$. We call such an arc l a *t-arc* for f. Let $\tilde{\Pi} : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be a submersion such that $\tilde{\Pi}(l)$ is a point in $\mathbb{R}^{n-1}$. Since $f|S(f)$ is transverse to l, if we restrict $\tilde{\Pi} \circ f$ to the interior of a sufficiently small tubular neighborhood $N(f^{-1}(l))$ of $f^{-1}(l)$, $\tilde{\Pi}(l)$ is a regular value of the restricted map $\tilde{\Pi} \circ f|\mathrm{Int}\big(N(f^{-1}(l))\big) : \mathrm{Int}\big(N(f^{-1}(l))\big) \to \mathbb{R}^{n-1}$. Thus, $f^{-1}(l) \subset (\tilde{\Pi} \circ f)^{-1}(\tilde{\Pi}(l))$ is a finite disjoint union of circles and $f|f^{-1}(l) : f^{-1}(l) \to l$ is a Morse function.

Then we have the following observation.

Observation 3.1. *Let M be a closed n-dimensional manifold and $f : M \to \mathbb{R}^n$ a stable fold map $(n > 1)$. If f has an embedding lift $\tilde{f} : M \to \mathbb{R}^{n+1}$, then for any t-arc l for f, $\tilde{f}|f^{-1}(l) : f^{-1}(l) \to \Pi^{-1}(l)$ is an embedding lift over the Morse function $f|f^{-1}(l) : f^{-1}(l) \to l$.*

Using Observation 3.1, we construct stable fold maps $f : M \to \mathbb{R}^n$ which do not have an embedding lift.

Example 3.2. Let f_1 and $f_2 : S^1 \times D^1 \to \mathbb{R}^2$ be orientation preserving immersions such that $f_1|S^1 \times \partial D^1 = f_2|S^1 \times \partial D^1$ and that their images are as shown in Figure 6.

By attaching f_1 and f_2 along $S^1 \times \partial D^1$ and changing the orientation of the source $S^1 \times D^1$ of f_2, we have a stable fold map of a torus T^2 into $\mathbb{R}^2$. We denote by $f : T^2 \to \mathbb{R}^2$ this stable fold map. Let l be a t-arc, as shown in Figure 7, which passes through the two normal crossings of $f(S(f))$. Then, $f^{-1}(l)$ is a circle and the Morse function $f|f^{-1}(l) : f^{-1}(l) \to l$ has two minimum p_1, p_2 and two maximum q_1, q_2 which are not separated in the sense of Definition 2.1. Therefore, f does not have an embedding lift by Theorem 2.2 and Observation 3.1. See Figure 7.

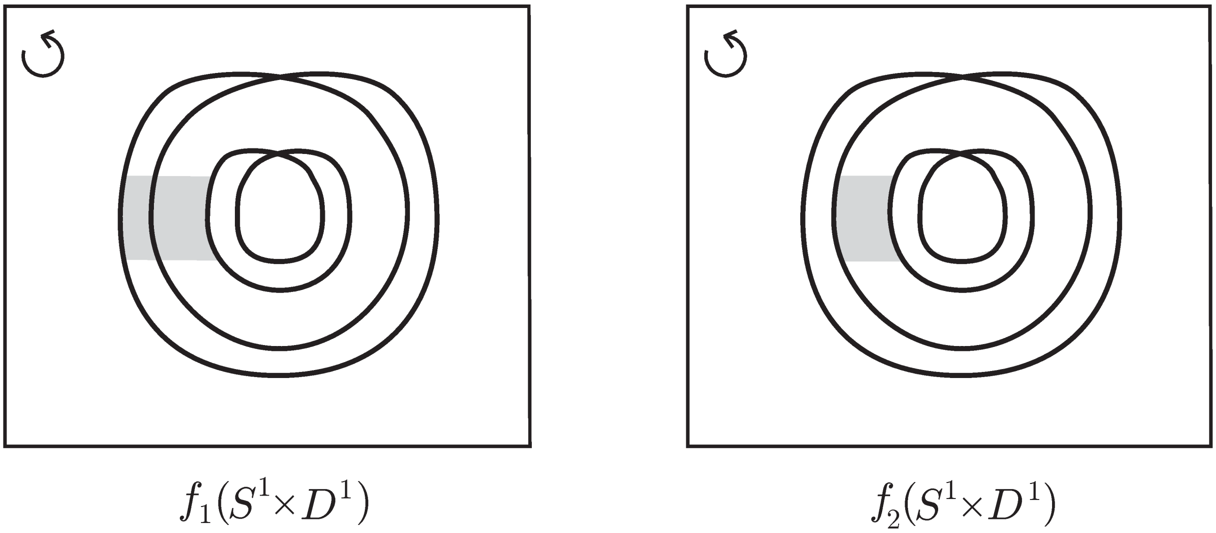

Figure 6. Orientation preserving immersions of $S^1 \times D^1$ into $\mathbb{R}^2$. The black curves represent the images $f_i(S^1 \times \partial D^1)$ and the gray strips represent parts of $f_i(S^1 \times D^1)$ $(i = 1, 2)$.

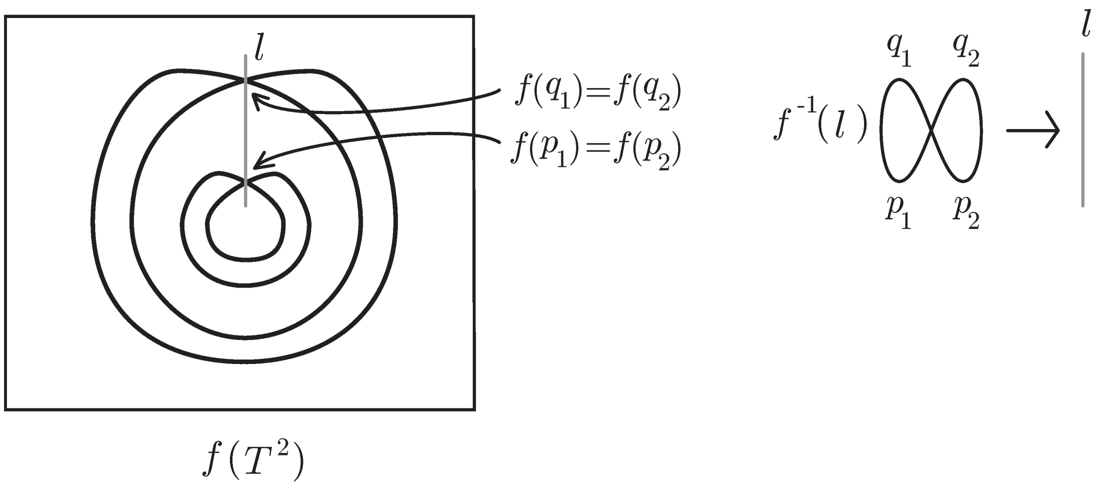

Figure 7. The t-arc l for $f : T^2 \to \mathbb{R}^2$ such that $f|f^{-1}(l)$ does not have an embedding lift.

Example 3.3. Let g_1 and $g_2 : D^2 \to \mathbb{R}^2$ be orientation preserving immersions such that $g_1|\partial D^2 = g_2|\partial D^2$ and that their images are as shown in Figure 8. These immersions are known as Milnor's example (see [3]).

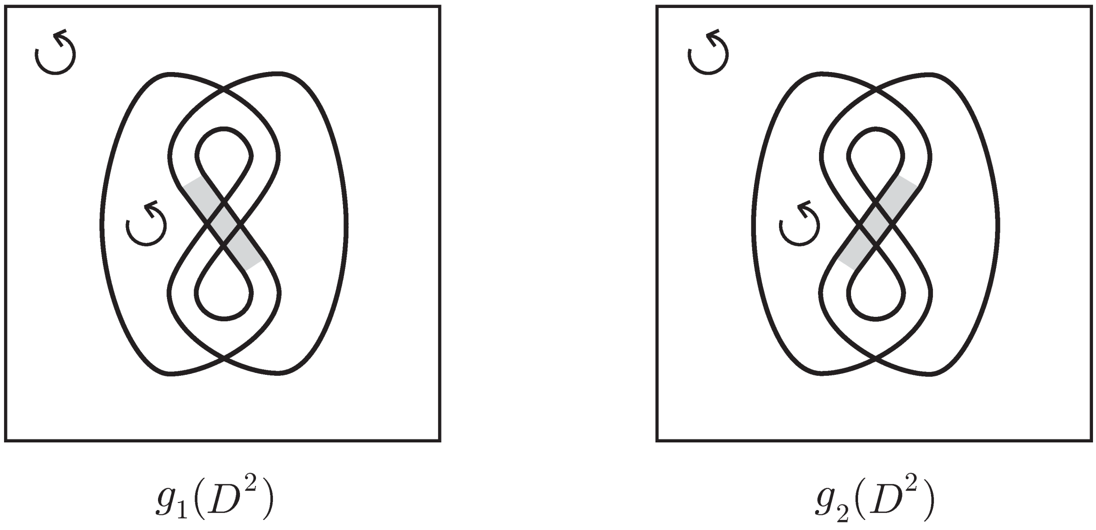

Figure 8. Orientation preserving immersions of D^2 into $\mathbb{R}^2$ which are known as Milnor's example. The black curves represent the images $g_i(\partial D^2)$ and the gray strips represent parts of $g_i(D^2)$ $(i = 1, 2)$.

By attaching g_1 and g_2 along ∂D^2 and changing the orientation of the source D^2 of g_2, we have a stable fold map $g : S^2 \to \mathbb{R}^2$. Let l be a t-arc, as shown in Figure 9, which passes through the two outermost normal crossings of $g(S(g))$. Then, $g^{-1}(l)$ is a circle and the Morse function $g|g^{-1}(l) : g^{-1}(l) \to l$ has two minimum p_1, p_2 and two maximum q_1, q_2 which are not separated in the sense of Definition 2.1. Therefore, g does not have an embedding lift by Theorem 2.2 and Observation 3.1. See Figure 9.

Example 3.4. Using the stable fold map $g : S^2 \to \mathbb{R}^2$ which is constructed in Example 3.3, we can construct a family of stable fold maps $h_n : S^n \to \mathbb{R}^n, n \geq 2$, which have an immersion lift but which do not have an embedding lift into $\mathbb{R}^{n+1}$ as follows.

Let $\Pi' : \mathbb{R}^n \to \mathbb{R}$ be the orthogonal projection defined by $\Pi'(y_1, y_2, \ldots, y_n) = y_1$. Suppose that a stable fold map $h_n : S^n \to \mathbb{R}^n$ satisfies the following conditions.

(1_n) $h_n(S^n) \subset [1, 3] \times \mathbb{R}^{n-1}$.

(2_n) $\{(2, y_2, 0, \ldots, 0) \mid y_2 \in \mathbb{R}\} \subset \mathbb{R}^n$ contains a t-arc l_n for h_n such that

$$h_n|h_n^{-1}(l_n) : h_n^{-1}(l_n) \to l_n$$

does not have an embedding lift.

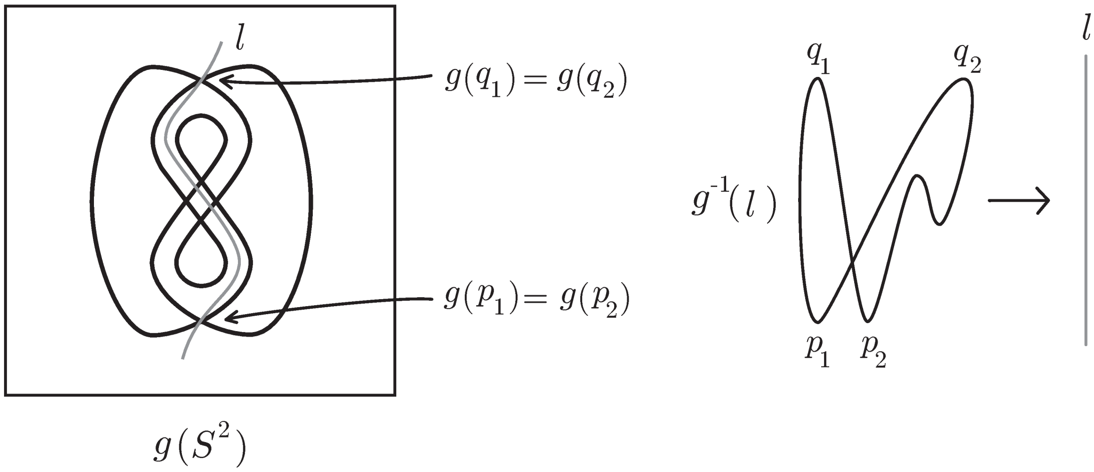

Figure 9. The t-arc l for $g : S^2 \to \mathbb{R}^2$ such that $g|g^{-1}(l)$ does not have an embedding lift.

(3_n) The set $(\Pi' \circ h_n)^{-1}(\{1\})$ (resp. $(\Pi' \circ h_n)^{-1}(\{3\})$) consists exactly of one point in S^n and we denote this point by p^n (resp. q^n). Furthermore, $\Pi' \circ h_n$ takes its minimum (resp. maximum) at p^n (resp. q^n). Both p^n and q^n are non-degenerate critical points of $\Pi' \circ h_n$.

(4_n) For a sufficiently small real number ε with $0 < \varepsilon < 1$, $h_n|S(h_n)$ is transverse to $\Pi'^{-1}(\{1+\varepsilon\})$ and $B_n = (\Pi' \circ h_n)^{-1}([1, 1+\varepsilon])$ is an n-ball around p^n such that $\partial B_n = (\Pi' \circ h_n)^{-1}(\{1+\varepsilon\})$ and $\partial B_n \cap S(h_n)$ consists of an $(n-2)$-dimensional sphere.

(5_n) $h_n|\partial B_n : \partial B_n \to \Pi'^{-1}(\{1+\varepsilon\})$ is a stable fold map such that $\partial B_n \cap S(h_n) = S(h_n|\partial B_n)$.

(6_n) $B_n \cap S(h_n)$ is an $(n-1)$-ball around p^n and $h_n|(B_n \cap S(h_n)) : B_n \cap S(h_n) \to h_n(S(h_n))$ is an embedding.

By composing $g : S^2 \to \mathbb{R}^2$ in Example 3.3 with a suitable diffeomorphism of $\mathbb{R}^2$, we get a stable fold map $h_2 : S^2 \to \mathbb{R}^2$ which satisfies the conditions (1_2)–(6_2). See Figure 10.

Let us now construct h_{n+1}, $n \geq 2$, inductively as follows. Consider the stable fold map $h^1_{n+1} : S^n \times S^1 \to \mathbb{R}^{n+1}$ defined by

$$h^1_{n+1}(x, (\cos\theta, \sin\theta)) = \big((y_1 \circ h_n(x))\cos\theta, y_2 \circ h_n(x), \ldots, y_n \circ h_n(x), (y_1 \circ h_n(x))\sin\theta\big).$$

Here, S^1 is identified with the boundary of the unit disk $D^2 = \{(r\cos\theta, r\sin\theta) \in \mathbb{R}^2 \mid 0 \leq r \leq 1, 0 \leq \theta < 2\pi\}$, $(\cos\theta, \sin\theta) \in S^1 = \partial D^2$ and $x \in S^n$. Let h^2_{n+1} :

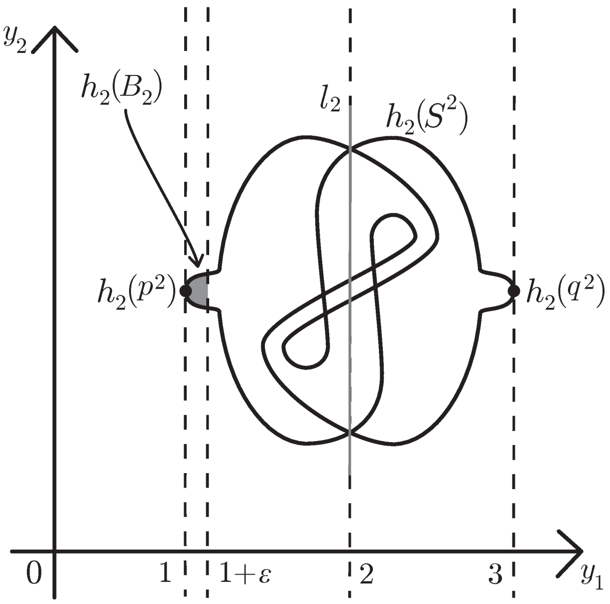

Figure 10. Positions of the t-arc l_2, $h_2(p^2)$, $h_2(q^2)$ and $h_2(B_2)$ for $h_2 : S^2 \to \mathbb{R}^2$.

$\partial B_n \times D^2 \to \mathbb{R}^{n+1}$ be the smooth map defined by

$$\begin{aligned} &h^2_{n+1}(x, (r\cos\theta, r\sin\theta)) \\ &\qquad = \big((y_1 \circ h_n(x))r\cos\theta, y_2 \circ h_n(x), \dots, y_n \circ h_n(x), (y_1 \circ h_n(x))r\sin\theta\big). \end{aligned}$$

From these constructions, we get the $(n+1)$-dimensional manifold

$$\big((S^n \times S^1) \setminus (\mathrm{Int}(B_n) \times S^1)\big) \cup (\partial B_n \times D^2) \tag{3.1}$$

which is diffeomorphic to S^{n+1}, and the map $\big(h^1_{n+1} \,|\, \big((S^n \times S^1) \setminus (\mathrm{Int}(B_n) \times S^1)\big)\big) \cup h^2_{n+1}$. By slightly deforming $h^1_{n+1} \,|\, \big((S^n \times S^1) \setminus (\mathrm{Int}(B_n) \times S^1)\big)$ and h^2_{n+1} around the boundary $\partial B_n \times S^1$, we get a stable fold map h'_{n+1} of S^{n+1} into $\mathbb{R}^{n+1}$. In the following, we will identify the manifold (3.1) with S^{n+1}. By construction, the stable fold map $h'_{n+1} : S^{n+1} \to \mathbb{R}^{n+1}$ satisfies the following conditions.

$(1'_{n+1})$ $h_{n+1}(S^{n+1}) \subset [-3, 3] \times \mathbb{R}^n$.

$(2'_{n+1})$ $\{(2, y_2, 0, \dots, 0) \mid y_2 \in \mathbb{R}\} \subset \mathbb{R}^{n+1}$ contains a t-arc l'_{n+1} for h'_{n+1} such that

$$h'_{n+1}|(h'_{n+1})^{-1}(l'_{n+1}) : (h'_{n+1})^{-1}(l'_{n+1}) \to l'_{n+1}$$

does not have an embedding lift.

$(3'_{n+1})$ For $(-1, 0) \in S^1$ and $(1, 0) \in S^1$, we define $(p^{n+1})' \in \big((S^n \times S^1) \setminus (\mathrm{Int}(B_n) \times S^1)\big) \subset S^{n+1}$ by $(p^{n+1})' = q^n \times \{(-1, 0)\}$ and define $(q^{n+1})' \in \big((S^n \times S^1) \setminus (\mathrm{Int}(B_n) \times S^1)\big) \subset$

S^{n+1} by $(q^{n+1})' = q^n \times \{(1,0)\}$. By the definition, we have $(\Pi' \circ h'_{n+1})^{-1}(\{-3\}) = \{(p^{n+1})'\}$ and $(\Pi' \circ h'_{n+1})^{-1}(\{3\}) = \{(q^{n+1})'\}$, and $\Pi' \circ h'_{n+1}$ takes its minimum (resp. maximum) at $(p^{n+1})'$ (resp. $(q^{n+1})'$). Both $(p^{n+1})'$ and $(q^{n+1})'$ are non-degenerate critical points of $\Pi' \circ h'_{n+1}$.

By composing $h'_{n+1} : S^{n+1} \to \mathbb{R}^{n+1}$ with a suitable diffeomorphism of $\mathbb{R}^{n+1}$, we get a stable fold map $h_{n+1} : S^{n+1} \to \mathbb{R}^{n+1}$ which satisfies the conditions (1_{n+1})–(6_{n+1}). By [5], h_{n+1} has an immersion lift into $\mathbb{R}^{n+2}$. However, by Observation 3.1, h_{n+1} does not have an embedding lift into $\mathbb{R}^{n+2}$.

We end this paper by stating a future problem.

Problem 3.5. If M is a closed orientable n-dimensional manifold, then does the converse of Observation 3.1 hold? That is, let M be a closed orientable n-dimensional manifold and $f : M \to \mathbb{R}^n$ a stable fold map ($n > 1$). Suppose that for any t-arc l for f, the Morse function $f|f^{-1}(l) : f^{-1}(l) \to l$ has an embedding lift. Then, does the stable fold map f has an embedding lift?

When M is nonorientable, we have a counterexample as follows. Let $f : M \to \mathbb{R}^2$ be a stable fold map of the Klein bottle such that $f(S(f))$ consists of two concentric circles and $f(M)$ is an embedded annulus whose boundary is $f(S(f))$. See Figure 11. Since $f(S(f))$ does not have normal crossings, for any t-arc l, $f|f^{-1}(l) : f^{-1}(l) \to l$ is a stable Morse function and it has an embedding lift (see Remark 2.4). But it is known that the Klein bottle cannot be embedded in $\mathbb{R}^3$. Therefore, f does not have an embedding lift. By Haefliger's theorem [2], f does not have an immersion lift either.

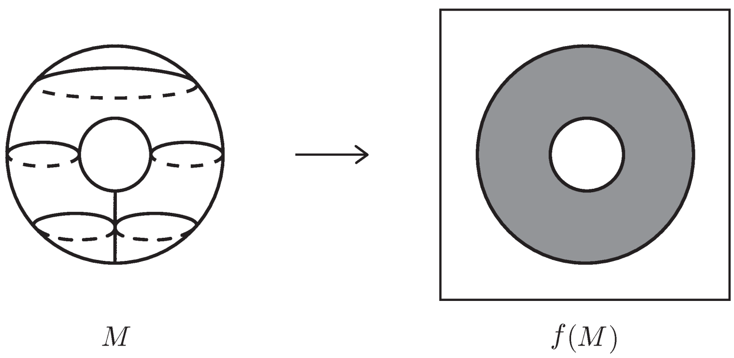

Figure 11. The stable fold map $f : M \to \mathbb{R}^2$ of the Klein bottle M.

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