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Triangulating Stein factorizations of generic maps and Euler characteristic formulas

Dedicated to Professor Masahiko Suzuki on the occasion of his sixtieth birthday

By

Jorge T. Hiratuka* and Osamu Saeki**

Abstract

For a map between manifolds, the space of the connected components of its fibers is called the Stein factorization. In this paper, we show that for a large class of C^∞ maps between manifolds, their Stein factorizations are triangulable. As applications, we obtain several new Euler characteristic formulas concerning singularities of C^∞ stable maps in certain dimensions.

§ 1. Introduction

Let $g\colon M \to N$ be a generic C^∞ map between smooth manifolds. The space of the connected components of fibers of g is denoted by W_g. Then, we have the canonical quotient map $q_g\colon M \to W_g$ and the natural map $\bar{g}\colon W_g \to N$ such that $g = \bar{g} \circ q_g$. Such a decomposition of g into the composition of q_g and $\bar{g}$ is called the *Stein factorization* of g. Sometimes the quotient space W_g is also called the Stein factorization of g.

It is known that when $\dim M > \dim N$, the Stein factorization of $g\colon M \to N$, or the quotient space W_g, is a very important tool in studying the topological properties of the map g. Refer to [3, 4, 10, 11, 12, 14, 17, 20, 21], for example. In all the known cases, the quotient spaces of generic C^∞ maps are polyhedrons and their local structures have been determined.

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*Departamento de Matemática, Instituto de Matemática e Estatística, Universidade de São Paulo, Caixa Postal 66.281, CEP: 05389-970, São Paulo, SP, Brazil.
e-mail: `jotahira@ime.usp.br`
**Institute of Mathematics for Industry, Kyushu University, Motooka 744, Nishi-ku, Fukuoka 819-0395, Japan.
e-mail: `saeki@imi.kyushu-u.ac.jp`

In this paper, we first show that if g is a so-called Thom map, then its Stein factorization is triangulable. In particular, topologically stable C^∞ maps have triangulable Stein factorizations, and their quotient spaces are polyhedrons. This is a fundamental result in the topological study of generic C^∞ maps in the sense that it enables us to study given C^∞ maps by using the polyhedral structures of their quotient spaces.

As examples, we give several Euler characteristic formulas for generic C^∞ maps, by means of triangulations of Stein factorizations. We pay special attention to C^∞ stable maps of 3-dimensional manifolds into $\mathbb{R}^2$ and those of 4-dimensional manifolds into $\mathbb{R}^3$, and obtain several new formulas.

The paper is organized as follows. In §2 we give a precise definition of the Stein factorization of a continuous map between topological spaces and its triangulation. Then, we state the triangulation theorem of Stein factorizations of proper triangulable maps. As a corollary, we see that the Stein factorization of a proper Thom map is always triangulable, since by Shiota [22] proper Thom maps are triangulable. In §3 we prove that the Stein factorization of a proper simplicial map between locally finite simplicial complexes is triangulable. The idea is to subdivide the given simplicial map and to define an abstract simplicial complex whose underlying space is homeomorphic to the quotient space. In §4, we recall Euler characteristic formulas obtained in [16] for simplicial maps and show that local indices defined in [16] also decompose according to the Stein factorization. In §5 we consider C^∞ stable maps of 3-dimensional manifolds into the plane and get a formula relating the Euler characteristics of the quotient space and the image together with the number of singular fibers of specific types. In §6, we consider C^∞ stable maps of 4-dimensional manifolds into $\mathbb{R}^3$ and get a similar formula.

Throughout the paper, we will often abuse the terminology "simplicial complex" (or "simplicial map") to indicate the corresponding polyhedron (resp. PL map), although we will try to use correct notations until §3 in order to avoid confusion. For a space X, id_X denotes the identity map of X.

§ 2. Preliminaries

In this section, we define the notion of a triangulation of the Stein factorization of a map and state our triangulation theorem of Stein factorizations.

Definition 2.1. Let $g\colon M \to N$ be a continuous map between topological spaces M and N. Two points $x, x' \in M$ are *g-equivalent* if $g(x) = g(x')$ and the points x and x' are in the same connected component of $g^{-1}(g(x)) = g^{-1}(g(x'))$. We denote by W_g the quotient space with respect to the g-equivalence, endowed with the quotient topology. The quotient map is denoted by $q_g\colon M \to W_g$. Then there exists a unique continuous map $\bar{g}\colon W_g \to N$ such that $g = \bar{g} \circ q_g$. The quotient space W_g or the commutative diagram

$$\begin{tikzcd} M \arrow[rr, "g"] \arrow[dr, "q_g"'] & \circlearrowleft & N \\ & W_g \arrow[ur, "\bar{g}"'] & \end{tikzcd}$$

is called the *Stein factorization* of g.

There is a one-to-one correspondence between the quotient space and the space of the connected components of the fibers of g. Note that each fiber of the quotient map q_g is connected.

Remark 2.2. Even if $g\colon M \to N$ is a C^∞ map between smooth manifolds, the Stein factorization W_g may not necessarily be a manifold. For example, consider the Morse function $g\colon T^2 \to \mathbb{R}$ on the 2-dimensional torus as in Fig. 1. The Stein factorization W_g is clearly not a manifold.

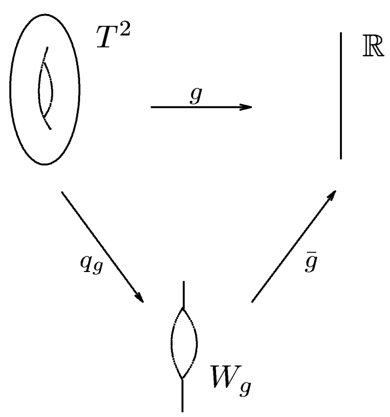

Figure 1. Stein factorization may not be a manifold

In the following, for a simplicial complex K, its associated polyhedron is denoted by $|K|$, and for a simplicial map $f\colon K \to L$ between simplicial complexes, $|f|\colon |K| \to |L|$ denotes the associated PL map.

Definition 2.3. Let $g\colon M \to N$ be a continuous map, where M and N are topological spaces. We say that g is *triangulable* if there exist simplicial complexes K and L, a simplicial map $f\colon K \to L$ and homeomorphisms $\lambda\colon |K| \to M$ and $\mu\colon |L| \to N$

such that the following diagram is commutative:

$$\begin{tikzcd} M \arrow[r, "g"] & N \\ {|K|} \arrow[u, "\lambda"] \arrow[r, "|f|"] & {|L|}. \arrow[u, "\mu"'] \end{tikzcd}$$

We call f, K and L *triangulations* of g, M and N, respectively.

Definition 2.4. Let $g \colon M \to N$ be a continuous map, where M and N are topological spaces. We say that the Stein factorization of g is *triangulable* if there exist simplicial complexes K, L and V, simplicial maps $f \colon K \to L$, $\varphi \colon K \to V$ and $\psi \colon V \to L$, homeomorphisms $\lambda \colon |K| \to M, \mu \colon |L| \to N$ and $\Theta \colon |V| \to W_g$ such that the following diagram is commutative:

$$\begin{tikzcd} M \arrow[rr, "g"] \arrow[dr, "q_g"'] & & N \\ & W_g \arrow[ur, "\bar{g}"'] & \\ & {|V|} \arrow[u, "\Theta"'] \arrow[dr, "|\psi|"] & \\ {|K|} \arrow[uuu, "\lambda"] \arrow[ur, "|\varphi|"] \arrow[rr, "|f|"] & & {|L|}. \arrow[uuu, "\mu"'] \end{tikzcd}$$

We call φ, ψ and V *triangulations* of q_g, $\bar{g}$ and W_g, respectively.

Our first result of this paper is the following.

Theorem 2.5. *Let $g \colon M \to N$ be a proper continuous map between locally compact topological spaces M and N. If g is triangulable, then so is the Stein factorization of g.*

Theorem 2.5 will be proved in Section 3.

Shiota shows in [22] that proper Thom maps, which constitute quite a large class of C^∞ maps between smooth manifolds, are triangulable. In particular, topologically stable proper C^∞ maps are triangulable. Therefore, we have the following.

Corollary 2.6. *Let $g \colon M \to N$ be a proper C^∞ map between smooth manifolds. If g is a Thom map, then the Stein factorization of g is triangulable.*

Corollary 2.7. *For smooth manifolds M and N, in the space of proper C^∞ maps $C^\infty_{\mathrm{pr}}(M,N)$, the set of those maps whose Stein factorizations are triangulable contains an open and dense subset.*

§ 3. Triangulating Stein factorizations

In order to construct a triangulation of the Stein factorization of a given triangulable map, we begin by studying the fibers $|f|^{-1}(y)$ for a simplicial map $f\colon K \to L$ between simplicial complexes for $y \in |L|$.

The following lemma is well-known (for example, see [13, Lemma 20.5]).

Lemma 3.1. *Let $f\colon K \to L$ be a simplicial map between simplicial complexes, $\tau \in f(K)$ a simplex of L, and b_τ the barycenter of τ. Then, there exists a homeomorphism $\Omega_\tau\colon |f|^{-1}(\mathring{\tau}) \to |f|^{-1}(b_\tau)\times \mathring{\tau}$ making the following diagram commutative:*

$$\begin{tikzcd}
|f|^{-1}(\mathring{\tau}) \arrow[rr, "\Omega_\tau"] \arrow[dr, "|f|"'] & \circlearrowleft & |f|^{-1}(b_\tau)\times \mathring{\tau} \arrow[dl, "p_2"] \\
& \mathring{\tau}\,, &
\end{tikzcd}$$

where $\mathring{\tau}$ is the interior of the simplex τ and p_2 is the projection to the second factor.

Let $\widetilde{f}\colon \widetilde{K} \to \widetilde{L}$ be a simplicial map between locally finite simplicial complexes.

Definition 3.2. Suppose that L is a barycentric subdivision of $\widetilde{L}$. Then by [8, Lemma 1.8], there exist a subdivision K of $\widetilde{K}$ and a simplicial map $f\colon K \to L$ such that $|f| = |\widetilde{f}|$. We call f a *subdivision* of $\widetilde{f}$.

We will define an abstract simplicial complex $\mathcal{V}$ and show that its associated polyhedron is homeomorphic to $W_{|\widetilde{f}|} = W_{|f|}$. Note that K and L are locally finite, since so are $\widetilde{K}$ and $\widetilde{L}$. Let us assume that $|f| = |\widetilde{f}|\colon |K| \to |L|$ is proper.

Let $K^{(0)}$ denote the set of vertices of K. For $v, w \in K^{(0)}$, we say that v is f-*equivalent* to w, written as $v \sim w$, if $f(v) = f(w)$ and there exist a finite sequence of vertices $v_0, v_1, \dots, v_s$ of $f^{-1}(f(v)) = f^{-1}(f(w)) \subset K$ such that $v = v_0$, $w = v_s$, and there are 1-simplices $\langle v_i v_{i+1}\rangle$ in $f^{-1}(f(v))$ connecting v_i and v_{i+1}, $i = 0, 1, \dots, s-1$. This clearly defines an equivalence relation. Note that if $v \sim w$, then they belong to the same connected component of $|f|^{-1}(|f|(v)) = |f|^{-1}(|f|(w))$.

We define the abstract simplicial complex $\mathcal{V}$ as follows. The set of vertices $\mathcal{V}^{(0)}$ is the f-equivalence classes $K^{(0)}/\sim$ of vertices of K. Distinct f-equivalence classes $[v_0], [v_1], \dots, [v_k]$ of $\mathcal{V}^{(0)}$ define a k-simplex of $\mathcal{V}$ if for each $i = 0, 1, \dots, k$, there exists a

$v_i' \in [v_i]$ such that $\langle v_0'v_1' \cdots v_k'\rangle$ defines a k-simplex of K. In this case, we denote the k-simplex by $\{[v_0], [v_1], \ldots, [v_k]\}$. It is easy to verify that $\mathcal{V}$ defines an abstract simplicial complex.

We define the map $\varphi\colon K \to \mathcal{V}$ by $\varphi(\langle v_0v_1 \cdots v_k\rangle) = \{[v_0], [v_1], \ldots, [v_k]\}$. Note that φ is a simplicial map, although $\varphi(\langle v_0v_1 \cdots v_k\rangle)$ may have dimension strictly smaller than k.

Define $\psi\colon \mathcal{V} \to L$ by $\psi(\{[v_0], [v_1], \ldots, [v_k]\}) = \langle f(v_0)f(v_1) \cdots f(v_k)\rangle$. Note that ψ is well defined: in fact $f(v)$ does not depend on the choice of a representative v of $[v]$ and $\langle f(v_0)f(v_1) \cdots f(v_k)\rangle$ defines a simplex of L.

A simplicial map is said to be *non-degenerate* if it preserves the dimension of each simplex.

Lemma 3.3. *The map ψ is simplicial and non-degenerate. Furthermore, the following diagram is commutative:*

$$\begin{tikzcd} K \arrow[rr, "f"] \arrow[dr, "\varphi"'] & \circlearrowleft & L \\ & \mathcal{V}. \arrow[ur, "\psi"'] & \end{tikzcd} \tag{3.1}$$

Proof. It is straightforward to show that ψ is simplicial. In order to show that ψ is non-degenerate, let $\sigma = \{[v_0], [v_1]\}$ be a 1-simplex of $\mathcal{V}$. There exist $v_i' \in [v_i]$, $i = 0, 1$, such that $\langle v_0'v_1'\rangle \in K$. Since ψ does not depend on the representatives, we have $\psi(\{[v_0], [v_1]\}) = \langle f(v_0')f(v_1')\rangle$. Suppose, by contradiction, that $f(v_0') = f(v_1')$, i.e., $\psi(\sigma)$ is a 0-simplex of L. From $\langle v_0'v_1'\rangle \in K$ and $f(v_0') = f(v_1')$ it follows that $\langle v_0'v_1'\rangle \subset f^{-1}(f(v_0')) = f^{-1}(f(v_1'))$. Then, we have $v_0' \sim v_1'$, which is a contradiction, since $[v_0'] \neq [v_1']$. Therefore, ψ is non-degenerate.

The commutativity of diagram (3.1) follows immediately from the definition of ψ. $\square$

Lemma 3.4. *For the PL maps $|\varphi|\colon |K| \to |\mathcal{V}|$ and $|\psi|\colon |\mathcal{V}| \to |L|$ associated with the simplicial maps $\varphi\colon K \to \mathcal{V}$ and $\psi\colon \mathcal{V} \to L$, respectively, $|\varphi|^{-1}(x)$ is contained in a connected component of $|f|^{-1}(|\psi|(x))$ for all $x \in |\mathcal{V}|$.*

Proof. There exists a unique simplex $\sigma = \{[v_0], [v_1], \ldots, [v_k]\} \in \mathcal{V}$ such that $x \in \mathring{\sigma}$. The simplex $\tau = \psi(\sigma) \in L$ can be written as

$$\langle \psi([v_0])\psi([v_1]) \cdots \psi([v_k])\rangle = \langle f(v_0)f(v_1) \cdots f(v_k)\rangle.$$

As ψ is non-degenerate by Lemma 3.3, we have $|\psi|(x) \in \mathring{\tau}$. Since L is a barycentric subdivision of $\widetilde{L}$, there exists a unique $\widetilde{\tau} \in \widetilde{L}$ such that $\mathring{\widetilde{\tau}} \supset \mathring{\tau}$ and $\dim \widetilde{\tau} \geq \dim \tau$ (see

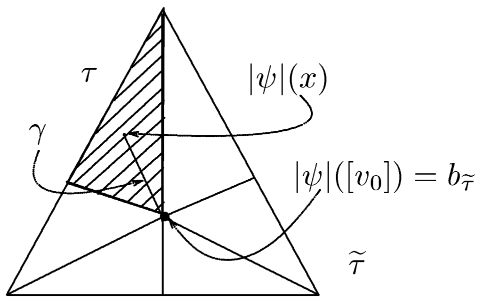

Figure 2. The simplices τ and $\widetilde{\tau}$

Fig. 2). (Note that we have $\dim \widetilde{\tau} > \dim \tau$ when τ is the cone over a simplex contained in $\partial\widetilde{\tau}$ with non-maximal dimension, with vertex the barycenter $b_{\widetilde{\tau}}$ of $\widetilde{\tau}$.) We may assume, without loss of generality, that $|\psi|([v_0])$ is the barycenter of $\widetilde{\tau}$, i.e., $|\psi|([v_0]) = b_{\widetilde{\tau}}$.

Consider the line segment γ in τ joining $|\psi|(x)$ and $|\psi|([v_0])$. Note that γ is entirely contained in the interior of $\widetilde{\tau}$. Since $|f| = |\widetilde{f}|$, applying Lemma 3.1, we have a homeomorphism $H_\gamma \colon |f|^{-1}(\gamma) \to |f|^{-1}(b_{\widetilde{\tau}}) \times \gamma$ which makes the following diagram commutative:

$$\begin{tikzcd}
|f|^{-1}(\gamma) \arrow[rr, "H_\gamma"] \arrow[dr, "|f|"'] & \arrow[d, phantom, "\circlearrowleft"] & |f|^{-1}(b_{\widetilde{\tau}}) \times \gamma \arrow[dl, "p_2"] \\
& \gamma, &
\end{tikzcd} \tag{3.2}$$

where p_2 is the projection to the second factor.

Let y and z be arbitrary two points of $|\varphi|^{-1}(x)$. We will show that they belong to the same connected component of $|f|^{-1}(|\psi|(x))$. There exist two simplices $\sigma_y = \langle u_0 u_1 \cdots u_\ell \rangle$ and $\sigma_z = \langle w_0 w_1 \cdots w_m \rangle$ of K such that $y \in \mathring{\sigma}_y$, $z \in \mathring{\sigma}_z$, and $\varphi(\sigma_y) = \sigma = \varphi(\sigma_z)$. Let $u_0, u_1, \ldots, u_{i_0}$ (or $w_0, w_1, \ldots, w_{j_0}$) be the vertices of σ_y (resp. σ_z) which are mapped to $[v_0]$ by φ. Then $u_0 \sim \cdots \sim u_{i_0} \sim w_0 \sim \cdots \sim w_{j_0}$, i.e., they belong to the same connected component of $|f|^{-1}(|f|(v_0)) = |f|^{-1}(|\psi|([v_0])) = |f|^{-1}(b_{\widetilde{\tau}})$. Consequently they belong to the same connected component of $|f|^{-1}(\gamma)$.

Let γ_y be the line segment in σ_y joining u_0 to y, and γ_z the line segment in σ_z joining w_0 to z. We have $|f|(\gamma_y) = |f|(\gamma_z) = \gamma$. Then, u_0 and y are in the same connected component of $|f|^{-1}(\gamma)$, and w_0 and z are also in the same connected component of $|f|^{-1}(\gamma)$.

As a consequence, y and z are in the same connected component C of $|f|^{-1}(\gamma)$. By the commutative diagram (3.2), we have that C is homeomorphic to $c \times \gamma$, where c is the connected component of $|f|^{-1}(b_{\widetilde{\tau}})$ that contains u_0 and w_0. In particular, $|f|^{-1}(|\psi|(x)) \cap C$ is homeomorphic to c, which is connected. Moreover, we have $y, z \in |f|^{-1}(|\psi|(x)) \cap C$;

therefore, y and z belong to the same connected component of $|f|^{-1}(|\psi|(x))$. $\square$

We will show that the underlying space $|\mathcal{V}|$ associated to the abstract simplicial complex $\mathcal{V}$ is homeomorphic to $W_{|f|}$. For this, we define the map $\alpha\colon |\mathcal{V}| \to W_{|f|}$ by $\alpha(x) = q_{|f|}(|\varphi|^{-1}(x))$, $x \in |\mathcal{V}|$, where $q_{|f|}\colon |K| \to W_{|f|}$ is the quotient map associated with $|f|\colon |K| \to |L|$. By virtue of Lemma 3.4, the map α is well defined.

We have the following diagram:

$$
\begin{tikzcd}
{|K|} \arrow[rr, "|f|"] \arrow[dr, "q_{|f|}"'] \arrow[ddr, "|\varphi|"'] & \arrow[d, phantom, "\mathrm{(I)}"] & {|L|} \\
\arrow[r, phantom, "\mathrm{(II)}"] & W_{|f|} \arrow[ur, "\overline{|f|}"'] \arrow[r, phantom, "\mathrm{(III)}"] & {} \\
& {|\mathcal{V}|.} \arrow[u, "\alpha"] \arrow[uur, "|\psi|"'] &
\end{tikzcd}
\tag{3.3}
$$

Part (I) in the above diagram is commutative. Part (II) is also commutative by the definition of α. For the commutativity of (III), by the definition of α and commutativity of (I) we have, for $x \in |\mathcal{V}|$,

$$\overline{|f|}(\alpha(x)) = \overline{|f|}(q_{|f|}(|\varphi|^{-1}(x))) = |f|(|\varphi|^{-1}(x)) = |\psi|(x),$$

where the last equality follows from the commutative diagram (3.1). Therefore, part (III) is also commutative.

In order to define the inverse map of α, we need the following.

Lemma 3.5. *For all $x \in W_{|f|}$, $|\varphi|(q_{|f|}^{-1}(x))$ consists of a single point.*

Proof. Let y and z be any two points in $q_{|f|}^{-1}(x)$. By commutativity of part (I) of diagram (3.3), we have

$$|f|(y) = \overline{|f|} \circ q_{|f|}(y) = \overline{|f|}(x) = \overline{|f|} \circ q_{|f|}(z) = |f|(z).$$

There exists a unique $\tau = \langle v_0 v_1 \cdots v_k \rangle \in L$ such that $|f|(y) = |f|(z) \in \mathring{\tau}$. Furthermore, there exist $\sigma_y = \langle u_0 u_1 \cdots u_\ell \rangle$ and $\sigma_z = \langle w_0 w_1 \cdots w_m \rangle \in K$ with $\ell, m \geq k$ such that $y \in \mathring{\sigma}_y$, $z \in \mathring{\sigma}_z$ and $f(\sigma_y) = f(\sigma_z) = \tau$.

We may assume

$$
\begin{array}{rcl}
f(u_0) = \cdots = f(u_{i_0}) & = v_0 = & f(w_0) = \cdots = f(w_{j_0}), \\
f(u_{i_0+1}) = \cdots = f(u_{i_1}) & = v_1 = & f(w_{j_0+1}) = \cdots = f(w_{j_1}), \\
\vdots & \vdots & \vdots \\
f(u_{i_{k-1}+1}) = \cdots = f(u_\ell) & = v_k = & f(w_{j_{k-1}+1}) = \cdots = f(w_m).
\end{array}
$$

Since $u_0, u_1, \dots, u_\ell$ are vertices of $\sigma_y \in K$, we have

$$u_0 \sim \cdots \sim u_{i_0},\ u_{i_0+1} \sim \cdots \sim u_{i_1},\ \dots,\ u_{i_{k-1}+1} \sim \cdots \sim u_\ell.$$

Similarly, we have

$$w_0 \sim \cdots \sim w_{j_0},\ w_{j_0+1} \sim \cdots \sim w_{j_1},\ \dots,\ w_{j_{k-1}+1} \sim \cdots \sim w_m.$$

Suppose, for the moment, we have the equivalences

$$u_0 \sim w_0,\ u_{i_0+1} \sim w_{j_0+1},\ \dots,\ u_{i_{k-1}+1} \sim w_{j_{k-1}+1}. \tag{3.4}$$

We can write y and z in barycentric coordinates:

$$y = \sum_{r=0}^{k} \sum_{i=i_{r-1}+1}^{i_r} \alpha_i u_i \quad \text{and} \quad z = \sum_{r=0}^{k} \sum_{j=j_{r-1}+1}^{j_r} \beta_j w_j,$$

where $i_{-1} = -1 = j_{-1}$, $i_k = \ell$, $j_k = m$, $\alpha_i > 0, \beta_j > 0$,

$$\sum_{i=0}^{\ell} \alpha_i = 1 \quad \text{and} \quad \sum_{j=0}^{m} \beta_j = 1.$$

Then we have

$$\begin{aligned} |\varphi|(y) &= \sum_{r=0}^{k} \sum_{i=i_{r-1}+1}^{i_r} \alpha_i |\varphi|(u_i) \\ &= \sum_{r=0}^{k} \sum_{i=i_{r-1}+1}^{i_r} \alpha_i [u_{i_{r-1}+1}] \\ &= \sum_{r=0}^{k} \left(\sum_{i=i_{r-1}+1}^{i_r} \alpha_i \right) [u_{i_{r-1}+1}]. \end{aligned}$$

Similarly, we have

$$|\varphi|(z) = \sum_{r=0}^{k} \left(\sum_{j=j_{r-1}+1}^{j_r} \beta_j \right) [w_{j_{r-1}+1}].$$

Since

$$\begin{aligned} |f|(y) &= \sum_{r=0}^{k} \sum_{i=i_{r-1}+1}^{i_r} \alpha_i f(u_i) = \sum_{r=0}^{k} \left(\sum_{i=i_{r-1}+1}^{i_r} \alpha_i \right) v_r, \\ |f|(z) &= \sum_{r=0}^{k} \sum_{j=j_{r-1}+1}^{j_r} \beta_j f(w_j) = \sum_{r=0}^{k} \left(\sum_{j=j_{r-1}+1}^{j_r} \beta_j \right) v_r, \end{aligned}$$

and $|f|(y) = |f|(z)$, from the uniqueness of barycentric coordinates it follows that

$$\sum_{i=i_{r-1}+1}^{i_r} \alpha_i = \sum_{j=j_{r-1}+1}^{j_r} \beta_j$$

for all r. Then, since $[u_{i_{r-1}+1}] = [w_{j_{r-1}+1}]$ for all r by our assumption (3.4), we have $|\varphi|(y) = |\varphi|(z)$.

In order to show the equivalences (3.4), we need the following.

Lemma 3.6. *For $r = 0, 1, \ldots, k$, $|f|^{-1}(v_r)$ is a strong deformation retract of $|f|^{-1}(\{v_r\} \cup \mathring{\tau})$.*

Proof. Let θ be a simplex contained in $|f|^{-1}(\tau)$ such that $f(\theta) = \tau$. Let θ_0 be the maximal face of θ such that $|f|(\theta_0) = v_r$, and let θ_1 be the complementary face of θ formed by the vertices not belonging to θ_0. Each point $a \in |f|^{-1}(\{v_r\} \cup \mathring{\tau}) \cap \theta$ can be uniquely expressed as

$$a = \lambda_0 t_0 + \lambda_1 t_1$$

for some $t_0 \in \theta_0$, $t_1 \in \theta_1$, $\lambda_0 > 0$, $\lambda_1 \geq 0$ and $\lambda_0 + \lambda_1 = 1$. Set $\gamma_r(a) = t_0$. Then, we can show that $\gamma_r \colon |f|^{-1}(\{v_r\} \cup \mathring{\tau}) \to |f|^{-1}(v_r)$ is well defined and defines a strong deformation retract. $\square$

Let us now show the equivalences (3.4). For each r there exist

(i) a line segment ω_1 in σ_y joining $u_{i_{r-1}+1}$ and y, where the line segment ω_1 is contained in $|f|^{-1}(\{v_r\} \cup \mathring{\tau})$,

(ii) a path ω_2 in $|f|^{-1}(\overline{|f|}(x))$ joining y and z, since y and z belong to $q_{|f|}^{-1}(x)$, where the path ω_2 is contained in $|f|^{-1}(\mathring{\tau})$, and

(iii) a line segment ω_3 in σ_z joining $w_{j_{r-1}+1}$ and z, where the line segment ω_3 is contained in $|f|^{-1}(\{v_r\} \cup \mathring{\tau})$.

Let ω be the path obtained by connecting ω_1, ω_2 and $\overline{\omega}_3$, where $\overline{\omega}_3$ is the path ω_3 with the opposite orientation. The path ω is contained in $|f|^{-1}(\{v_r\} \cup \mathring{\tau})$ and connects $u_{i_{r-1}+1}$ and $w_{j_{r-1}+1}$. Then $\gamma_r \circ \omega$ is a path in $|f|^{-1}(v_r)$ joining $u_{i_{r-1}+1}$ and $w_{j_{r-1}+1}$. Now it is easy to modify the path $\gamma_r \circ \omega$, fixing the end points, so that it passes through only 1-dimensional simplices of $f^{-1}(v_r)$.

We conclude that $u_{i_{r-1}+1} \sim w_{j_{r-1}+1}$ for all $r = 0, 1, \ldots, k$. $\square$

Let us define the map $\beta \colon W_{|f|} \to |\mathcal{V}|$ by $\beta(x) = |\varphi|(q_{|f|}^{-1}(x))$, $x \in W_{|f|}$. This is well defined by Lemma 3.5, and we have the following diagram:

(3.5)
$$\begin{tikzcd}
{|K|} \arrow[rr, "{|f|}"] \arrow[dr, "q_{|f|}"] \arrow[ddr, "{|\varphi|}"'] & \arrow[d, phantom, "\text{(I)}" near start] & {|L|} \\
& W_{|f|} \arrow[ur, "\overline{|f|}"'] \arrow[d, "\beta"] \arrow[d, phantom, "\text{(II)}", xshift=-4ex] \arrow[d, phantom, "\text{(III)}", xshift=4ex] & \\
& {|\mathcal{V}|}. \arrow[uur, "{|\psi|}"'] &
\end{tikzcd}$$

Part (II) of the above diagram is clearly commutative. As to part (III), take a point $w \in W_{|f|}$. Since $|\psi| \circ |\varphi| = |f|$ and $\overline{|f|} \circ q_{|f|} = |f|$, we have $|\psi| \circ |\varphi| = \overline{|f|} \circ q_{|f|}$. Then

$$\begin{aligned}
&(|\psi| \circ \beta)(w) = |\psi|(\beta(w)) = |\psi|(|\varphi|(q_{|f|}^{-1}(w))) \\
&= (|\psi| \circ |\varphi|)(q_{|f|}^{-1}(w)) = (\overline{|f|} \circ q_{|f|})(q_{|f|}^{-1}(w)) = \overline{|f|}(w).
\end{aligned}$$

Therefore, part (III) is also commutative.

To prove the continuity of α we need the following.

Lemma 3.7. *If $|f|$ is proper and K and L are locally finite, then $|\varphi|$ is a closed proper map.*

Proof. Let D be a compact subset of $|\mathcal{V}|$. Let us show that $|\varphi|^{-1}(D)$ is a compact subset of $|K|$. By the commutativity of diagram (3.5), we have $|\varphi|^{-1}(D) \subset |f|^{-1}(|\psi|(D))$. Since $|\psi|$ is continuous and $|f|$ is proper, we see that $|f|^{-1}(|\psi|(D))$ is a compact subset of $|K|$. On the other hand, $|\varphi|^{-1}(D)$ is a closed subset, since $|\varphi|$ is continuous. Since it is contained in the compact set $|f|^{-1}(|\psi|(D))$, it is compact. Therefore, $|\varphi|$ is a proper map.

Now let us show that $\mathcal{V}$ is locally finite. Let $[v]$ be a vertex of $\mathcal{V}$. If a simplex $\{[v_0], [v_1], \ldots, [v_m]\}$ of $\mathcal{V}$ has $[v]$ as one of its vertices, then there exist vertices $v_i' \sim v_i$ of K such that $\langle v_0' v_1' \cdots v_m' \rangle \in K$, and $v \sim v_i'$ for some i. Since $|f|$ is proper, there exist at most finitely many vertices w of K such that $v \sim w$. For each w, we have only finitely many simplices of K that have w as one of its vertices, since K is locally finite. Then only a finite number of simplices of $\mathcal{V}$ have $[v]$ as one of its vertices. Therefore, $\mathcal{V}$ is locally finite.

Then, by [15, Lemma 2.6, p. 11], $|K|$ and $|\mathcal{V}|$ are locally compact. By [15, Lemma 2.4, p. 10], they are Hausdorff. Since $|\varphi|$ is proper, by [2, Proposition 11.5, p. 33], $|\varphi|$ is a closed map. $\square$

Now we are able to show the following.

Lemma 3.8. *Under the hypothesis above, α is the inverse map of β. Furthermore, α and β are continuous.*

Proof. Let w be an arbitrary point of $W_{|f|}$. Then we have

$$\begin{aligned}(\alpha \circ \beta)(w) = \alpha(\beta(w)) \; &= \; \alpha(|\varphi|(q_{|f|}^{-1}(w))) \\ &= q_{|f|}(|\varphi|^{-1}(|\varphi|(q_{|f|}^{-1}(w)))).\end{aligned}$$

By virtue of Lemmas 3.4 and 3.5, we have that $q_{|f|}(|\varphi|^{-1}(|\varphi|(q_{|f|}^{-1}(w))))$ consists of a single point. Since $q_{|f|}$ is surjective, $q_{|f|}(|\varphi|^{-1}(|\varphi|(q_{|f|}^{-1}(w))))$ contains w, and we have $q_{|f|}(|\varphi|^{-1}(|\varphi|(q_{|f|}^{-1}(w)))) = w$. Therefore, we have $\alpha \circ \beta(w) = w$, $\forall w \in W_{|f|}$, and consequently $\alpha \circ \beta = \mathrm{id}_{W_{|f|}}$.

Consider now an arbitrary point y of $|\mathcal{V}|$. We have

$$(\beta \circ \alpha)(y) = \beta(\alpha(y)) = \beta(q_{|f|}(|\varphi|^{-1}(y))) = |\varphi|(q_{|f|}^{-1}(q_{|f|}(|\varphi|^{-1}(y)))).$$

By Lemmas 3.4 and 3.5, $|\varphi|(q_{|f|}^{-1}(q_{|f|}(|\varphi|^{-1}(y))))$ consists of a single point of $|\mathcal{V}|$. On the other hand, $|\varphi|(q_{|f|}^{-1}(q_{|f|}(|\varphi|^{-1}(y))))$ contains y, since $|\varphi|$ is surjective. Hence, we have $(\beta \circ \alpha)(y) = y$, $\forall y \in |\mathcal{V}|$, and consequently α is the inverse map of β.

If $B \subset |\mathcal{V}|$ is an open subset of $|\mathcal{V}|$, then by the definition of the quotient topology on $W_{|f|}$, $\beta^{-1}(B)$ is an open subset of $W_{|f|}$ if and only if $q_{|f|}^{-1}(\beta^{-1}(B))$ is an open subset of $|K|$. In fact, $q_{|f|}^{-1}(\beta^{-1}(B)) = (\beta \circ q_{|f|})^{-1}(B) = |\varphi|^{-1}(B)$ is an open subset of $|K|$, since $|\varphi|$ is continuous. Therefore, β is continuous.

Since α is the inverse of β, in order to show that α is continuous, we have only to show that β is a closed map. Let D be a closed subset of $W_{|f|}$. Since $q_{|f|}$ is continuous, $q_{|f|}^{-1}(D)$ is a closed subset of $|K|$. By Lemma 3.7, $|\varphi|$ is a closed map. Thus, $|\varphi|(q_{|f|}^{-1}(D))$ is closed in $|\mathcal{V}|$. On the other hand, $|\varphi|(q_{|f|}^{-1}(D)) = \beta(D)$, since $q_{|f|}$ is surjective. Therefore, α is continuous. □

Summarizing the above argument, we have proved the following.

Proposition 3.9. *Let $\widetilde{f} \colon \widetilde{K} \to \widetilde{L}$ be a simplicial map between two locally finite simplicial complexes such that $|\widetilde{f}| \colon |\widetilde{K}| \to |\widetilde{L}|$ is proper. Let $f \colon K \to L$ be a barycentric subdivision of $\widetilde{f}$. Then, there exist a simplicial complex $\mathcal{V}$, a simplicial map $\varphi \colon K \to \mathcal{V}$, a non-degenerate simplicial map $\psi \colon \mathcal{V} \to L$ and a homeomorphism $\alpha \colon |\mathcal{V}| \to W_{|f|}$, making the following diagrams commutative:*

(3.6)
$$\begin{tikzcd} K \arrow[rr, "f"] \arrow[dr, "\varphi"'] \arrow[rr, phantom, bend right=20, "\circlearrowleft"] & & L \\ & \mathcal{V}, \arrow[ur, "\psi"'] & \end{tikzcd} \qquad \begin{tikzcd} {|K|} \arrow[rr, "|f|"] \arrow[dr, "q_{|f|}"] \arrow[ddr, "|\varphi|"'] & & {|L|} \\ & W_{|f|} \arrow[ur, "\overline{|f|}"] & \\ & {|\mathcal{V}|.} \arrow[u, "\alpha"] \arrow[uur, "|\psi|"'] & \end{tikzcd}$$

Remark 3.10. In Proposition 3.9, the first diagram can be regarded as a triangulation of the Stein factorization of $|\widetilde{f}|\colon |\widetilde{K}| \to |\widetilde{L}|$ (see Definition 2.4). In particular, the quotient space $W_{|f|}$ is homeomorphic to a polyhedron.

Proof of Theorem 2.5. Let $g\colon M \to N$ be a continuous map between topological spaces M and N. Suppose M and N are locally compact, and g is proper and triangulable. Let $\widetilde{f}\colon \widetilde{K} \to \widetilde{L}$ be a triangulation of g, i.e., there exist homeomorphisms $\lambda\colon |\widetilde{K}| \to M$ and $\mu\colon |\widetilde{L}| \to N$ such that $\mu \circ |\widetilde{f}| = g \circ \lambda$. From [15, Lemma 2.6, p. 11] it follows that $\widetilde{K}$ and $\widetilde{L}$ are locally finite. Let $f\colon K \to L$ be a barycentric subdivision of $\widetilde{f}$. It follows that K and L are also locally finite. The map $|f|\colon |K| \to |L|$ is proper and $\mu \circ |f| = g \circ \lambda$.

We can define the homeomorphism $\Lambda\colon W_{|f|} \to W_g$ by $\Lambda(x) = q_g(\lambda(q_{|f|}^{-1}(x)))$, for $x \in W_{|f|}$, whose inverse map $\Gamma\colon W_g \to W_{|f|}$ is given by $\Gamma(y) = q_{|f|}(\lambda^{-1}(q_g^{-1}(y)))$, for $y \in W_g$. We have the following commutative diagram:

$$\begin{tikzcd}
M \arrow[rr, "g"] \arrow[dr, "q_g"'] & & N \\
& W_g \arrow[ur, "\bar{g}"'] \arrow[d, shift left, "\Gamma"] & \\
& W_{|f|} \arrow[u, shift left, "\Lambda"] \arrow[dr, "\overline{|f|}"] & \\
|K| \arrow[uuu, "\lambda"] \arrow[ur, "q_{|f|}"] \arrow[rr, "|f|"] & & |L|. \arrow[uuu, "\mu"']
\end{tikzcd} \tag{3.7}$$

Consider now the simplicial complex $\mathcal{V}$, the simplicial maps $\varphi\colon K \to \mathcal{V}$, $\psi\colon \mathcal{V} \to L$ and the homeomorphism $\alpha\colon |\mathcal{V}| \to W_{|f|}$, which exist for f by Proposition 3.9. The commutative diagrams (3.6) and (3.7) give rise to the following commutative diagram:

$$\begin{tikzcd}
M \arrow[rr, "g"] \arrow[dr, "q_g"'] & & N \\
& W_g \arrow[ur, "\bar{g}"'] & \\
& W_{|f|} \arrow[u, "\Lambda"'] \arrow[dr, "\overline{|f|}"] & \\
|K| \arrow[uuu, "\lambda"] \arrow[ur, "q_{|f|}"] \arrow[dr, "|\varphi|"'] & & |L| \arrow[uuu, "\mu"'] \\
& |\mathcal{V}|. \arrow[uu, "\alpha"'] \arrow[ur, "|\psi|"'] &
\end{tikzcd}$$

Then the map $\Theta\colon |\mathcal{V}| \to W_g$ defined by $\Theta = \Lambda \circ \alpha$ is a homeomorphism and satisfies the commutative diagram of Definition 2.4 with V replaced by $\mathcal{V}$. Therefore, it follows that the diagram (3.1) is a triangulation of the Stein factorization of g. $\square$

Remark 3.11. In the above proof, we used a barycentric subdivision of a given triangulation. This procedure is necessary, since without the barycentric subdivision, the space that we get may not be a simplicial complex as can be seen in Fig. 3.

Furthermore, we need that the map is proper, in order to guarantee that the resulting space is locally finite as a simplicial complex. See Fig. 4 for an example of a non-proper simplicial map whose Stein factorization is not locally finite. As another example, consider the map $g\colon \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ defined by $g(x, y) = x$, which is a non-proper submersion. Then its Stein factorization is not even Hausdorff.

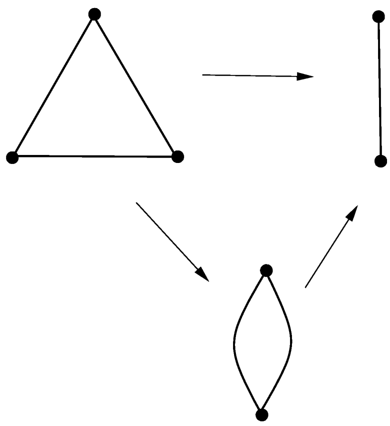

Figure 3. Stein factorization of a simplicial map

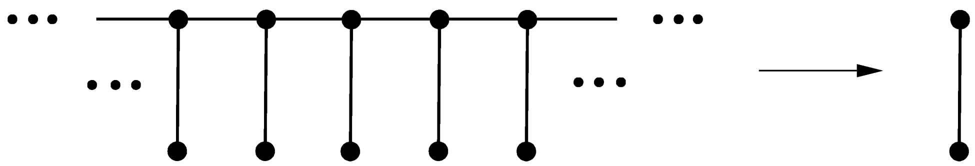

Figure 4. Example of a non-proper simplicial map

§ 4. Euler characteristic and local indices for simplicial maps

Let $g\colon M \to \mathbb{R}^n$ be a C^∞-stable map, where M is a smooth closed manifold. We would like to find formulas relating the number of singularities of g with the Euler characteristics of certain subsets of M, $g(M)$ and W_g.

In [16], Ballesteros and Saeki define local indices for simplicial maps $f\colon K \to L$ between simplicial complexes, where K is finite. They have found a formula relating such indices with the Euler characteristics of the source and the image of f. In this section we refine their formula by using the Stein factorization of f.

Definition 4.1. Let $f\colon X \to Y$ be a simplicial map, where X is a finite simplicial complex.

(i) The closure of the set $\{x \in X \,:\, f^{-1}(f(x)) \neq \{x\}\}$ in X is called the *self-intersection set* of f, denoted by $D(f)$.

(ii) The set $M(f) = f(D(f))$ is called the *multiple-point set* of f.

Note that $D(f)$ and $M(f)$ constitute subcomplexes of X and Y, respectively. Furthermore, if $v \in M(f)$, then $f^{-1}(v) \subset D(f)$.

In the following, for a simplicial complex C and a vertex $u \in C^{(0)}$, we set

$$\chi_u(C) = \sum_{\tau \ni u} \frac{(-1)^{\dim \tau}}{\dim \tau + 1}$$

with τ running over all simplices of C which contain u. Then, we define the index $\mathrm{Ind}_f(v)$ for each vertex v of $M(f)$ by

$$\mathrm{Ind}_f(v) = \chi_v(f(X)) - \sum_{w \in (f^{-1}(v))^{(0)}} \chi_w(X) \tag{4.1}$$

(see [16]). In other words, we have

$$\mathrm{Ind}_f(v) = \sum_{f(X) \supset \tau \ni v} \frac{(-1)^{\dim \tau}}{\dim \tau + 1} - \sum_{w \in (f^{-1}(v))^{(0)}} \sum_{X \supset \sigma \ni w} \frac{(-1)^{\dim \sigma}}{\dim \sigma + 1}. \tag{4.2}$$

Remark 4.2. We can define the index for any vertex $v \in (f(X))^{(0)}$ by (4.1). Then, by definition we have $\mathrm{Ind}_f(v) = 0$ for all $v \in (f(X))^{(0)} \smallsetminus M(f)^{(0)}$. Furthermore, if $v \in M(f)^{(0)}$, then we have

$$\mathrm{Ind}_f(v) = \chi_v(M(f)) - \sum_{w \in (f^{-1}(v))^{(0)}} \chi_w(D(f)).$$

In the following, χ denotes the Euler characteristic with respect to the singular homology. By [16, Theorem 3.3], we have the following.

Theorem 4.3. *Let $f\colon X \to Y$ be a simplicial map between two simplicial complexes, where X is finite. Then, we have*

$$\chi(f(X)) - \chi(X) = \sum_{v \in (f(X))^{(0)}} \mathrm{Ind}_f(v).$$

Let $f\colon X \to Y$ be a simplicial map, where X is finite. Let us assume that it is the subdivision of a simplicial map as in Definition 3.2. By Proposition 3.9 we have the following Stein factorization, where W_f is a finite simplicial complex whose associated polyhedron is identified with $W_{|f|}$, and q_f and $\bar{f}$ are simplicial maps:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "q_f"'] & \arrow[d, phantom, "\circlearrowleft"] & Y \\ & W_f. \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

Note that $f\colon X \to Y$ is proper, since X is a finite simplicial complex and its associated polyhedron is compact.

Since the quotient space W_f is finite, we can apply Theorem 4.3 above three times to the equality $\chi(f(X)) - \chi(X) = (\chi(f(X)) - \chi(W_f)) + (\chi(W_f) - \chi(X))$ to obtain

$$\sum_{v \in f(X)^{(0)}} \mathrm{Ind}_f(v) = \sum_{v \in f(X)^{(0)}} \mathrm{Ind}_{\bar{f}}(v) + \sum_{u \in W_f^{(0)}} \mathrm{Ind}_{q_f}(u).$$

In fact, we have the following.

Lemma 4.4. *Let $f\colon X \to Y$ be a simplicial map between two simplicial complexes, where X is finite. Then, for all $v \in f(X)^{(0)}$, we have*

$$\mathrm{Ind}_f(v) = \mathrm{Ind}_{\bar{f}}(v) + \sum_{u \in (\bar{f}^{-1}(v))^{(0)}} \mathrm{Ind}_{q_f}(u).$$

Proof. The formula follows from the following calculation:

$$\begin{aligned}
\mathrm{Ind}_f(v) &= \sum_{f(X) \supset \tau \ni v} \frac{(-1)^{\dim \tau}}{\dim \tau + 1} - \sum_{w \in (f^{-1}(v))^{(0)}} \sum_{X \supset \sigma \ni w} \frac{(-1)^{\dim \sigma}}{\dim \sigma + 1} \\
&= \sum_{\bar{f}(W_f) \supset \tau \ni v} \frac{(-1)^{\dim \tau}}{\dim \tau + 1} - \sum_{u \in (\bar{f}^{-1}(v))^{(0)}} \sum_{W_f \supset \delta \ni u} \frac{(-1)^{\dim \delta}}{\dim \delta + 1} \\
&\quad + \sum_{u \in (\bar{f}^{-1}(v))^{(0)}} \sum_{W_f \supset \delta \ni u} \frac{(-1)^{\dim \delta}}{\dim \delta + 1} - \sum_{w \in (f^{-1}(v))^{(0)}} \sum_{X \supset \sigma \ni w} \frac{(-1)^{\dim \sigma}}{\dim \sigma + 1} \\
&= \mathrm{Ind}_{\bar{f}}(v) + \sum_{u \in (\bar{f}^{-1}(v))^{(0)}} \sum_{W_f \supset \delta \ni u} \frac{(-1)^{\dim \delta}}{\dim \delta + 1} \\
&\quad - \sum_{u \in (\bar{f}^{-1}(v))^{(0)}} \sum_{w \in (q_f^{-1}(u))^{(0)}} \sum_{X \supset \sigma \ni w} \frac{(-1)^{\dim \sigma}}{\dim \sigma + 1} \\
&= \mathrm{Ind}_{\bar{f}}(v) + \sum_{u \in (\bar{f}^{-1}(v))^{(0)}} \mathrm{Ind}_{q_f}(u).
\end{aligned}$$

$\square$

§ 5. Stable maps of 3-manifolds into $\mathbb{R}^2$

Let M be a closed orientable 3-dimensional manifold. In this section, by applying Corollary 2.6 and Theorem 4.3 to a C^∞-stable map $g\colon M \to \mathbb{R}^2$ and its Stein factorization, we obtain a formula relating the singularities of g to its topological invariants.

In the following, for a topological space X, $\chi^c(X)$ denotes the Euler characteristic of X defined by using the Borel–Moore homology $H^c_*(X;\mathbb{Z})$ (see [1]), or the homology of infinite chains (see [23, Chap. 6, §3]). Note that when X is compact, $\chi^c(X)$ coincides with the Euler characteristic $\chi(X)$ defined by using the usual singular homology.

The following well-known proposition gives us conditions for a C^∞ map $g\colon M \to \mathbb{R}^2$ to be C^∞-stable. For details, see [5, 11, 12].

Proposition 5.1. *Let M be a closed 3-dimensional manifold and $g\colon M \to \mathbb{R}^2$ a C^∞ map. Then g is C^∞-stable if and only if the following local and global conditions are satisfied.*

(i) *For all $p \in M$, there exist local coordinates (x,y,z) and (X,Y) around $p \in M$ and $g(p) \in \mathbb{R}^2$, respectively, such that*

$$(X \circ g, Y \circ g) = \begin{cases} (x,y), & p: \text{ regular point,} \\ (x, y^2+z^2), & p: \text{ definite fold point,} \\ (x, y^2-z^2), & p: \text{ indefinite fold point,} \\ (x, -z^2+yx+y^3), & p: \text{ cusp point.} \end{cases}$$

Note that then the set $S(g)$ of singular points of g forms a closed submanifold of M of dimension 1.

(ii) *For every cusp point $p \in S(g)$, we have $g^{-1}(g(p)) \cap S(g) = \{p\}$, and the map $g|_{S(g)\smallsetminus\{\text{cusp points}\}}$ is an immersion with normal crossings.*

Let M be a closed orientable 3-dimensional manifold and $g\colon M \to \mathbb{R}^2$ a C^∞-stable map. The *singular set* $S(g)$ of g is the union of the set S_0 of the definite fold points, the set S_1 of the indefinite fold points, and the set C of the cusp points.

Let us consider the Stein factorization of g:

$$\begin{tikzcd} M \arrow[rr, "g"] \arrow[dr, "q_g"'] & \circlearrowleft & \mathbb{R}^2 \\ & W_g. \arrow[ur, "\bar{g}"'] & \end{tikzcd}$$

Every point p of W_g admits one of the canonical neighborhoods as shown in Fig. 5, where the thick lines represent the image of the singular set $S(g)$ of g by q_g (see [11, 12]). A

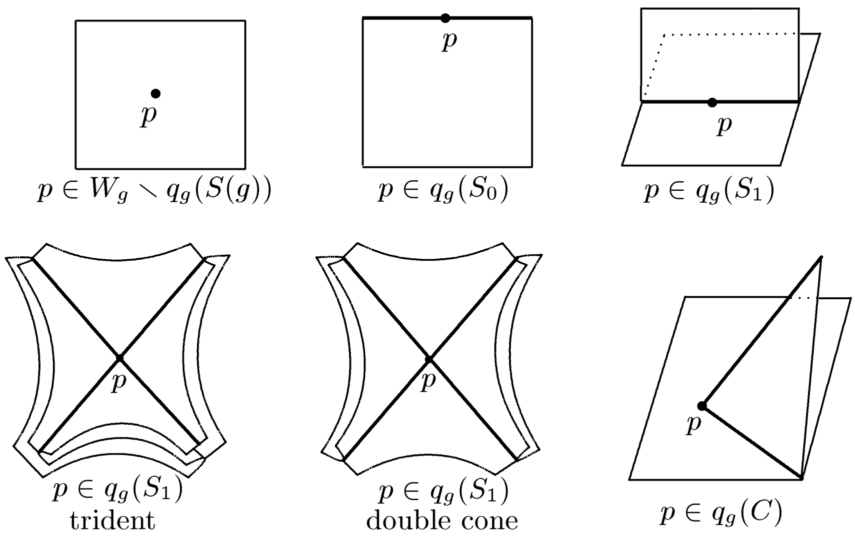

Figure 5. Local structures of W_g

point $p \in S_1$ with $q_g(p)$ having a neighborhood as in the upper-right figure of Fig. 5 is said to be *simple*, and a point $p \in S_1$ with $q_g(p)$ having a trident or a double cone as its neighborhood is said to be *non-simple*.

Remark 5.2. The images of the canonical neighborhoods as above by $\bar{g}$ are as shown in Fig. 6 (see [12]).

From [25, 26] it follows that g is triangulable and by Theorem 2.5 or by Corollary 2.6, W_g is homeomorphic to a polyhedron. Let $f\colon K \to L$ be a triangulation of g, and let q_f and $\bar{f}$ be triangulations of q_g and $\bar{g}$, respectively. They constitute a triangulation of the Stein factorization of g:

$$\begin{tikzcd} K \arrow[rr, "f"] \arrow[dr, "q_f"'] & \arrow[d, phantom, "\circlearrowleft"] & L \\ & W_f. \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

We denote by $S(f)$, S_0, S_1 and C the subsets of K corresponding to $S(g)$, S_0, S_1 and C, respectively, as long as there is no risk of confusion.

Remark 5.3. For $r \in \bar{f}(W_f) = f(K)$, we can easily describe the multi-germ $\bar{f}\colon (W_f, \bar{f}^{-1}(r)) \to (L, r)$. For example, when $f^{-1}(r) \cap S(f) = f^{-1}(r) \cap S_1 = \{p, p'\}$ with $p \neq p'$ and $q_f(p) = q_f(p')$, we have the two possibilities: the multi-germs are described as in either Fig. 7 or Fig. 8.

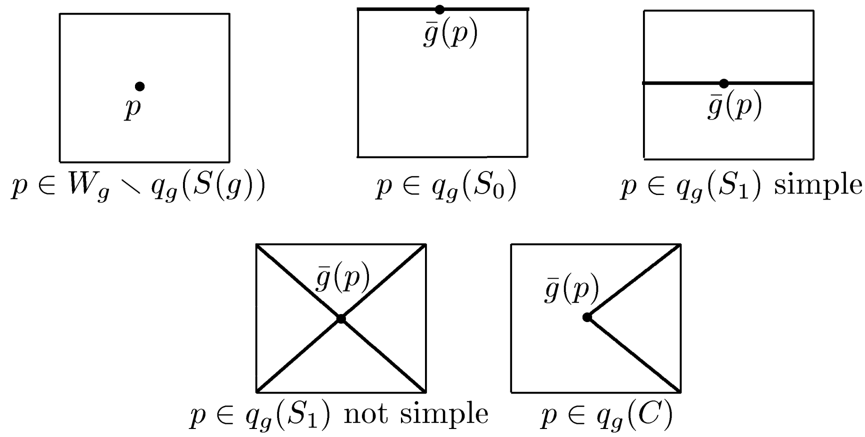

Figure 6. Images of canonical neighborhoods by $\bar{g}$

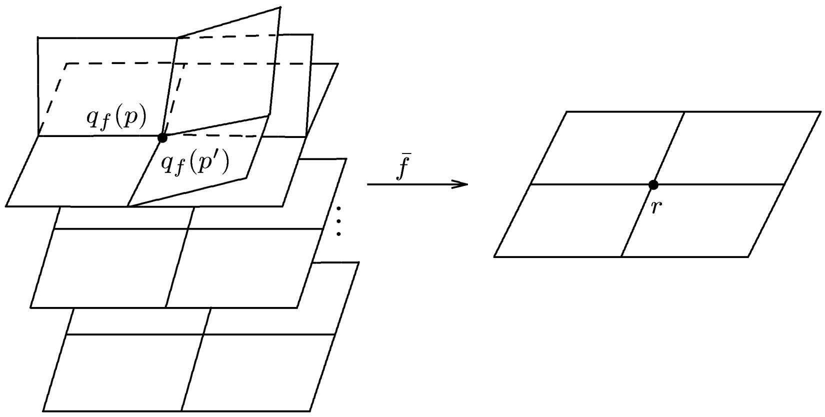

Figure 7. Multi-germ corresponding to a trident

Later we will use the representation above to compute the local index for each vertex of $M(\bar{f})$.

Notation 5.4. In the following, for an equivalence class of a singular fiber in the sense of [19, Remark 3.14], say I^0, the symbol $\mathrm{I}^0(g)$ denotes the set of points $y \in \mathbb{R}^2$ such that the fiber $g^{-1}(y)$ over y is equivalent to I^0 and some copies of a fiber of the trivial circle bundle. We also use the notation $\mathrm{I}^0(f)$, etc., for the corresponding subcomplexes of $f(S(f))$. Furthermore, for a finite set X, $|X|$ denotes the number of its elements.

Remark 5.5. The number of vertices in $f(K) \smallsetminus f(S(f))$ and that in $\mathrm{I}^*(f)$ depend on the choice of a triangulation $f\colon K \to L$ of g, while the numbers $|\mathrm{II}^*(f)|$ do not.

Definition 5.6. For $y \in f(K)$, we call $k = |\bar{f}^{-1}(y)|$ the *multiplicity* of y. Similarly, for $z \in g(M)$, we call $k = |\bar{g}^{-1}(z)|$ the *multiplicity* of z. For each positive integer

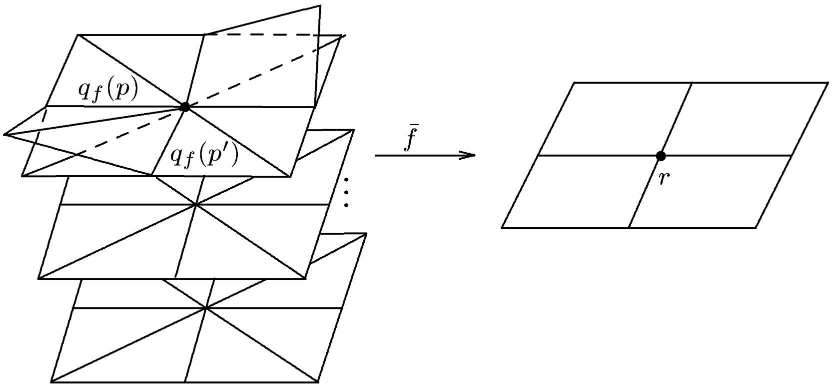

Figure 8. Multi-germ corresponding to a double cone

k, the set of points in $f(K) \smallsetminus f(S(f))$ (or in $g(M) \smallsetminus g(S(g))$) with multiplicity k is denoted by $M_k(\bar{f})$ (resp. $M_k(\bar{g})$).

Remark 5.7. The multiplicities of $y \in f(K)$ and $z \in g(M)$ are positive integers, since $\bar{f}$ is a non-degenerate simplicial map and $\bar{g}^{-1}(z)$ constitutes a finite set of points which correspond to the connected components of $g^{-1}(z)$.

Set $\Delta = f(S(f))$. Consider the following decomposition of the set $f(K)$ into the union of disjoint subsets:

$$f(K) = \left(\bigcup_{k \geq 1} M_k(\bar{f}) \right) \cup \Delta.$$

For $v \in \Delta^{(0)}$, let $\mathrm{Ind}_{\bar{f}_\Delta}(v)$ denote the index of v with respect to the map $\bar{f}_\Delta = \bar{f}|_{\bar{f}^{-1}(\Delta)} \colon \bar{f}^{-1}(\Delta) \to \Delta$ in the sense of (4.1).

Using Remark 5.3 and considering a neighborhood of each vertex $v \in \Delta^{(0)}$, we can compute the index $\mathrm{Ind}_{\bar{f}_\Delta}(v)$ as follows. Let k be the multiplicity of a vertex $v \in \Delta^{(0)}$. From equation (4.2) it follows

$$\mathrm{Ind}_{\bar{f}_\Delta}(v) = \sum_{\tau \ni v} \frac{(-1)^{\dim \tau}}{\dim \tau + 1} - \sum_{w \in (\bar{f}_\Delta^{-1}(v))^{(0)}} \sum_{\sigma \ni w} \frac{(-1)^{\dim \sigma}}{\dim \sigma + 1}.$$

Then we obtain Table 1. For notations related to singular fibers, refer to [19, §3.1].

In the following, for each positive integer k, we denote by $\mathrm{II}^{00}_k(f)$ the set of points in $\mathrm{II}^{00}(f)$ of multiplicity k. Similarly, we use the notations $\mathrm{II}^{01}_k(f)$, $\mathrm{II}^{11}_k(f)$, $\mathrm{II}^{2}_k(f)$ and $\mathrm{II}^{3}_k(f)$. We also use the notation $\mathrm{II}^{*}_k(g)$ similarly.

Theorem 5.8. *Let $g \colon M \to \mathbb{R}^2$ be a C^∞-stable map of a closed orientable* 3-

	$\mathrm{Ind}_{\bar{f}_\Delta}(v)$
$v \in \mathrm{I}^0(f)$	0
$v \in \mathrm{I}^1(f)$	0
$v \in \mathrm{II}^{00}(f)$	$k-2$
$v \in \mathrm{II}^{01}(f)$	$k-1$
$v \in \mathrm{II}^{11}(f)$	k
$v \in \mathrm{II}^{2}(f)$	k
$v \in \mathrm{II}^{3}(f)$	$k-1$
$v \in \mathrm{II}^{a}(f)$	$1/2$

Table 1. Index of each vertex $v \in \Delta^{(0)}$

dimensional manifold into the plane. Then, we have

$$\begin{aligned}\chi(g(M)) = \chi(W_g) &+ \sum_{k\geq 2}(1-k)\chi^c(M_k(\bar{g})) + \sum_{k\geq 3}(k-2)|\mathrm{II}^{00}_k(g)| \\ &+ \sum_{k\geq 2}(k-1)|\mathrm{II}^{01}_k(g)| + \sum_{k\geq 2} k|\mathrm{II}^{11}_k(g)| + \sum_{k\geq 1} k|\mathrm{II}^{2}_k(g)| \\ &+ \sum_{k\geq 2}(k-1)|\mathrm{II}^{3}_k(g)| + \frac{|\mathrm{II}^a(g)|}{2}.\end{aligned}$$

Proof. Since $f(K) = \bar{f}(W_f)$ and $W_f = \bar{f}^{-1}(f(K))$, we have

$$\begin{aligned}&\chi(\bar{f}(W_f)) - \chi(W_f) \\ &= \chi\left(\left(\bigcup_{k\geq 1} M_k(\bar{f})\right) \cup \Delta\right) - \chi\left(\bar{f}^{-1}\left(\bigcup_{k\geq 1} M_k(\bar{f})\right) \cup \bar{f}^{-1}(\Delta)\right) \\ &= \chi^c\left(\bigcup_{k\geq 1} M_k(\bar{f})\right) + \chi(\Delta) - \left(\chi^c\left(\bar{f}^{-1}\left(\bigcup_{k\geq 1} M_k(\bar{f})\right)\right) + \chi\left(\bar{f}^{-1}(\Delta)\right)\right) \\ &= \sum_{k\geq 1}\chi^c\left(M_k(\bar{f})\right) + \chi(\Delta) - \sum_{k\geq 1} k\chi^c\left(M_k(\bar{f})\right) - \chi\left(\bar{f}^{-1}(\Delta)\right) \\ &= \sum_{k\geq 2}(1-k)\chi^c\left(M_k(\bar{f})\right) + \chi(\Delta) - \chi\left(\bar{f}^{-1}(\Delta)\right).\end{aligned}$$

Applying Theorem 4.3 to $\bar{f}_\Delta = \bar{f}|_{\bar{f}^{-1}(\Delta)} \colon \bar{f}^{-1}(\Delta) \to \Delta$, we obtain

$$\chi(\Delta) - \chi(\bar{f}^{-1}(\Delta)) = \sum_{v\in\Delta^{(0)}} \mathrm{Ind}_{\bar{f}_\Delta}(v).$$

Then we have

$$\chi(\bar{f}(W_f)) - \chi(W_f) = \sum_{k\geq 2}(1-k)\chi^c(M_k(\bar{f})) + \sum_{v\in\Delta^{(0)}} \mathrm{Ind}_{\bar{f}_\Delta}(v).$$

Using the indices calculated in Table 1, we have

$$\begin{aligned}
&\chi(\bar{f}(W_f)) - \chi(W_f) \\
=&\sum_{k\geq 2}(1-k)\chi^c(M_k(\bar{f})) + \sum_{v\in \mathrm{II}^{00}(f)} \mathrm{Ind}_{\bar{f}_\Delta}(v) + \sum_{v\in \mathrm{II}^{01}(f)} \mathrm{Ind}_{\bar{f}_\Delta}(v) \\
&\quad + \sum_{v\in \mathrm{II}^{11}(f)} \mathrm{Ind}_{\bar{f}_\Delta}(v) + \sum_{v\in \mathrm{II}^{2}(f)} \mathrm{Ind}_{\bar{f}_\Delta}(v) + \sum_{v\in \mathrm{II}^{3}(f)} \mathrm{Ind}_{\bar{f}_\Delta}(v) \\
&\quad + \sum_{v\in \mathrm{II}^{a}(f)} \mathrm{Ind}_{\bar{f}_\Delta}(v) \\
=&\sum_{k\geq 2}(1-k)\chi^c(M_k(\bar{f})) + \sum_{k\geq 3}(k-2)|\mathrm{II}^{00}_k(f)| \\
&\quad + \sum_{k\geq 2}(k-1)|\mathrm{II}^{01}_k(f)| + \sum_{k\geq 2}k|\mathrm{II}^{11}_k(f)| + \sum_{k\geq 1}k|\mathrm{II}^{2}_k(f)| \\
&\quad + \sum_{k\geq 2}(k-1)|\mathrm{II}^{3}_k(f)| + \frac{|\mathrm{II}^a(f)|}{2}.
\end{aligned}$$

Since f is a triangulation of g and $\bar{g}(W_g) = g(M)$, we get the desired result. □

We have the following immediate corollary, which is originally due to Thom [24].

Corollary 5.9. *Let $g: M \to \mathbb{R}^2$ be a C^∞-stable map of a closed orientable 3-dimensional manifold into the plane. Then, the number of cusps is always even.*

Some explicit examples can be found in [6].

§ 6. Stable maps of 4-manifolds into $\mathbb{R}^3$

In this section, we study C^∞-stable maps $g: M \to \mathbb{R}^3$, where M is a closed orientable 4-dimensional manifold. We will obtain an integer formula relating singularities of such a map with some topological invariants.

Proposition 6.1. *Let M be a closed 4-dimensional manifold and $g: M \to \mathbb{R}^3$ a C^∞ map. Then, g is C^∞-stable if and only if the following local and global conditions are satisfied.*

(i) *For every $p \in M$ there exist local coordinates (x, y, z, w) and (X, Y, Z) around $p \in M$ and $g(p) \in \mathbb{R}^3$, respectively, such that*

$$
\begin{aligned}
&(X \circ g, Y \circ g, Z \circ g) \\
&= \begin{cases}
(x, y, z), & p: \ \textit{regular point}, \\
(x, y, z^2 + w^2), & p: \ \textit{definite fold point}, \\
(x, y, z^2 - w^2), & p: \ \textit{indefinite fold point}, \\
(x, y, z^3 + xz - w^2), & p: \ \textit{cusp point}, \\
(x, y, z^4 + xz^2 + yz + w^2), & p: \ \textit{definite swallowtail point}, \\
(x, y, z^4 + xz^2 + yz - w^2), & p: \ \textit{indefinite swallowtail point}.
\end{cases}
\end{aligned}
$$

(ii) *Set $S(g) = \{p \in M : \operatorname{rank} dg_p < 3\}$, which is a closed 2-dimensional submanifold of M under the above condition* (i) *and is called the singular set of g. Then, for every $r \in g(S(g))$, $g^{-1}(r) \cap S(g)$ consists of at most three points and the multi-germ $(g|_{S(g)}, g^{-1}(r) \cap S(g))$ is equivalent to one of the six multi-germs as described in Fig.* 9*:* (1) *represents a single immersion germ which corresponds to a fold point,* (2) *and* (4) *represent normal crossings of two and three immersion germs, respectively, each of which corresponds to a fold point,* (3) *corresponds to a cusp point,* (5) *represents a transverse crossing of a cuspidal edge as in* (3) *and an immersion germ corresponding to a fold point, and* (6) *corresponds to a swallowtail point.*

Let $g \colon M \to \mathbb{R}^3$ be a C^∞-stable map of a closed orientable 4-dimensional manifold M. Then, as in the previous section, we have a triangulation of the Stein factorization of g (see Proposition 3.9):

$$
\begin{tikzcd}
K \arrow[rr, "f"] \arrow[dr, "q_f"'] & \arrow[d, phantom, "\circlearrowleft"] & L \\
& W_f. \arrow[ur, "\bar{f}"'] &
\end{tikzcd}
$$

We also use notations similar to those in the previous section. In particular, for notations related to singular fibers, refer to [19, §3.1].

Remark 6.2. The numbers of vertices in $f(K) \smallsetminus f(S(f))$, $\mathrm{I}^*(f)$ and $\mathrm{II}^*(f)$ depend on the choice of a triangulation $f \colon K \to L$ of g, while the numbers $|\mathrm{III}^*(f)|$ do not.

Set $\Delta = f(S(f)) \smallsetminus \bigl(\mathrm{I}^0(f) \cup \mathrm{I}^1(f)\bigr)$. Note that Δ is the set of points $r \in f(K)$ which correspond to (2)–(6) in Fig. 9 and that Δ is of dimension one. (Note that here we use $f(S(f)) \smallsetminus \bigl(\mathrm{I}^0(f) \cup \mathrm{I}^1(f)\bigr)$ instead of $f(S(f))$ as Δ, since we need to ignore the 2-dimensional simplices in order to simplify the computation.) For $v \in \Delta^{(0)}$, $\mathrm{Ind}_{\bar{f}_\Delta}(v)$ denotes the index of v with respect to the map $\bar{f}_\Delta = \bar{f}|_{\bar{f}^{-1}(\Delta)} \colon \bar{f}^{-1}(\Delta) \to \Delta$. Let Δ_{II} (or Δ_{III}) denote the subset of Δ corresponding to $\mathrm{II}^*(f)$ (resp. $\mathrm{III}^*(f)$).

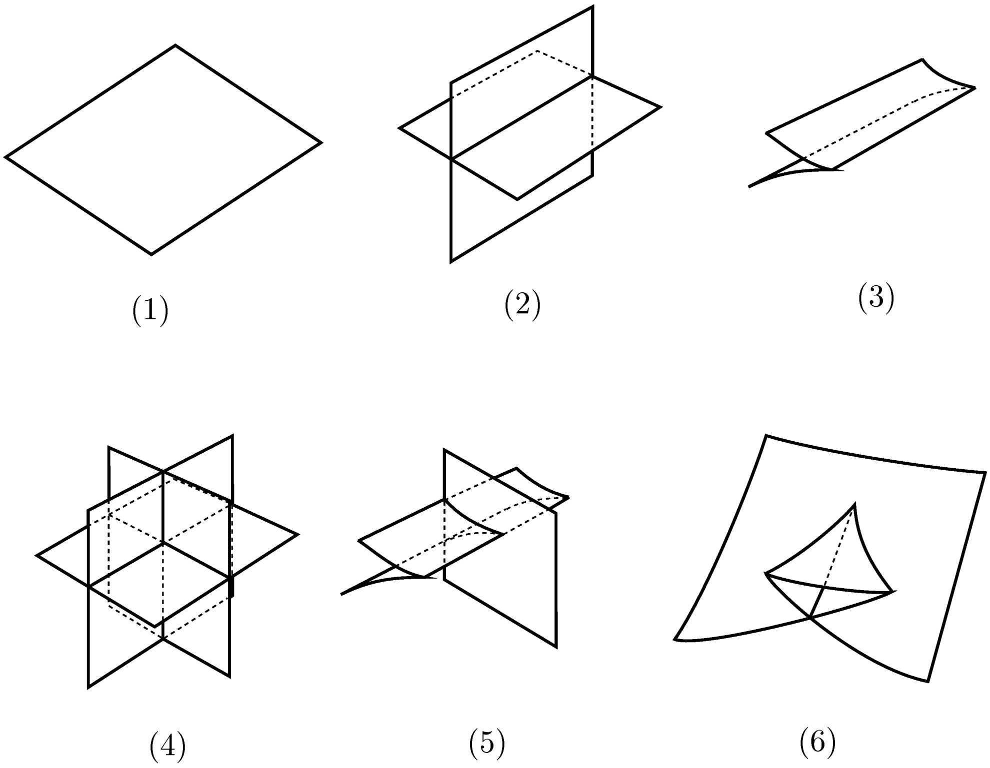

Figure 9. Multi-germs of $g|_{S(g)}$

Proposition 6.3. *The index* $\mathrm{Ind}_{\bar{f}_\Delta}(v)$ *for each vertex* v *of* Δ_{III} *is given as in Table* 2*, where* k *is the multiplicity of* v*. The index* $\mathrm{Ind}_{\bar{f}_\Delta}(v)$ *for each vertex* v *of* Δ_{II} *vanishes.*

Proof. The index $\mathrm{Ind}_{\bar{f}_\Delta}(v)$ can be calculated by analyzing the adjacencies among the sets $\mathrm{I}^*(f)$, $\mathrm{II}^*(f)$ and $\mathrm{III}^*(f)$. For example, for $v \in \mathrm{III}^4(f)$, the fibers near the singular fiber component of $f^{-1}(v)$ are as in Fig. 10 (for details, see [19, §3.1]). Then, we can calculate the index $\mathrm{Ind}_{\bar{f}_\Delta}(v)$ according to its definition. Details are left to the reader. □

In the following, for each positive integer k, we denote by $|\mathrm{III}^*_k(f)|$ (or $|\mathrm{III}^*_k(g)|$) the number of elements of $\mathrm{III}^*(f)$ (resp. $\mathrm{III}^*(g)$) of multiplicity k.

By the same argument as in the proof of Theorem 5.8, we obtain the following.

Theorem 6.4. *Let* $g\colon M \to \mathbb{R}^3$ *be a* C^∞*-stable map of a closed orientable* 4-

	$\mathrm{Ind}_{\bar{f}_\Delta}(v)$
$v \in \mathrm{III}^{000}(f)$	$2k-(7/2)$
$v \in \mathrm{III}^{001}(f)$	$2k-(5/2)$
$v \in \mathrm{III}^{011}(f)$	$2k-(3/2)$
$v \in \mathrm{III}^{111}(f)$	$2k-(1/2)$
$v \in \mathrm{III}^{02}(f)$	$2k-(3/2)$
$v \in \mathrm{III}^{03}(f)$	$2k-(5/2)$
$v \in \mathrm{III}^{12}(f)$	$2k-(1/2)$
$v \in \mathrm{III}^{13}(f)$	$2k-(3/2)$
$v \in \mathrm{III}^{4}(f)$	$2k-(1/2)$
$v \in \mathrm{III}^{5}(f)$	$2k-(1/2)$

	$\mathrm{Ind}_{\bar{f}_\Delta}(v)$
$v \in \mathrm{III}^{6}(f)$	$2k-2$
$v \in \mathrm{III}^{7}(f)$	$2k-(3/2)$
$v \in \mathrm{III}^{8}(f)$	$2k-2$
$v \in \mathrm{III}^{0a}(f)$	$k-1$
$v \in \mathrm{III}^{1a}(f)$	k
$v \in \mathrm{III}^{b}(f)$	k
$v \in \mathrm{III}^{c}(f)$	$k/2$
$v \in \mathrm{III}^{d}(f)$	$(k-1)/2$
$v \in \mathrm{III}^{e}(f)$	$(k+1)/2$

Table 2. Index of each vertex $v \in \Delta^{(0)}$

dimensional manifold into $\mathbb{R}^3$. Then, we have

$$
\begin{aligned}
&\chi(g(M)) \\
&= \chi(W_g) + \sum_{k\geq 2}(1-k)\chi^c(M_k^3(\bar{g})) + \sum_{k\geq 2}(1-k)\chi^c(M_k^2(\bar{g})) \\
&\quad + \sum_{k\geq 3}\left(2k-\frac{7}{2}\right)|\mathrm{III}_k^{000}(g)| + \sum_{k\geq 3}\left(2k-\frac{5}{2}\right)|\mathrm{III}_k^{001}(g)| \\
&\quad + \sum_{k\geq 3}\left(2k-\frac{3}{2}\right)|\mathrm{III}_k^{011}(g)| + \sum_{k\geq 3}\left(2k-\frac{1}{2}\right)|\mathrm{III}_k^{111}(g)| \\
&\quad + \sum_{k\geq 2}\left(2k-\frac{3}{2}\right)|\mathrm{III}_k^{02}(g)| + \sum_{k\geq 2}\left(2k-\frac{5}{2}\right)|\mathrm{III}_k^{03}(g)| \\
&\quad + \sum_{k\geq 2}\left(2k-\frac{1}{2}\right)|\mathrm{III}_k^{12}(g)| + \sum_{k\geq 2}\left(2k-\frac{3}{2}\right)|\mathrm{III}_k^{13}(g)| \\
&\quad + \sum_{k\geq 1}\left(2k-\frac{1}{2}\right)|\mathrm{III}_k^{4}(g)| + \sum_{k\geq 1}\left(2k-\frac{1}{2}\right)|\mathrm{III}_k^{5}(g)| \\
&\quad + \sum_{k\geq 1}(2k-2)\,|\mathrm{III}_k^{6}(g)| + \sum_{k\geq 1}\left(2k-\frac{3}{2}\right)|\mathrm{III}_k^{7}(g)| + \sum_{k\geq 1}(2k-2)\,|\mathrm{III}_k^{8}(g)| \\
&\quad + \sum_{k\geq 2}(k-1)|\mathrm{III}_k^{0a}(g)| + \sum_{k\geq 2}k|\mathrm{III}_k^{1a}(g)| + \sum_{k\geq 1}k|\mathrm{III}_k^{b}(g)| \\
&\quad + \sum_{k\geq 1}\frac{k}{2}|\mathrm{III}_k^{c}(g)| + \sum_{k\geq 1}\left(\frac{k}{2}-\frac{1}{2}\right)|\mathrm{III}_k^{d}(g)| + \sum_{k\geq 1}\left(\frac{k}{2}+\frac{1}{2}\right)|\mathrm{III}_k^{e}(g)|,
\end{aligned}
$$

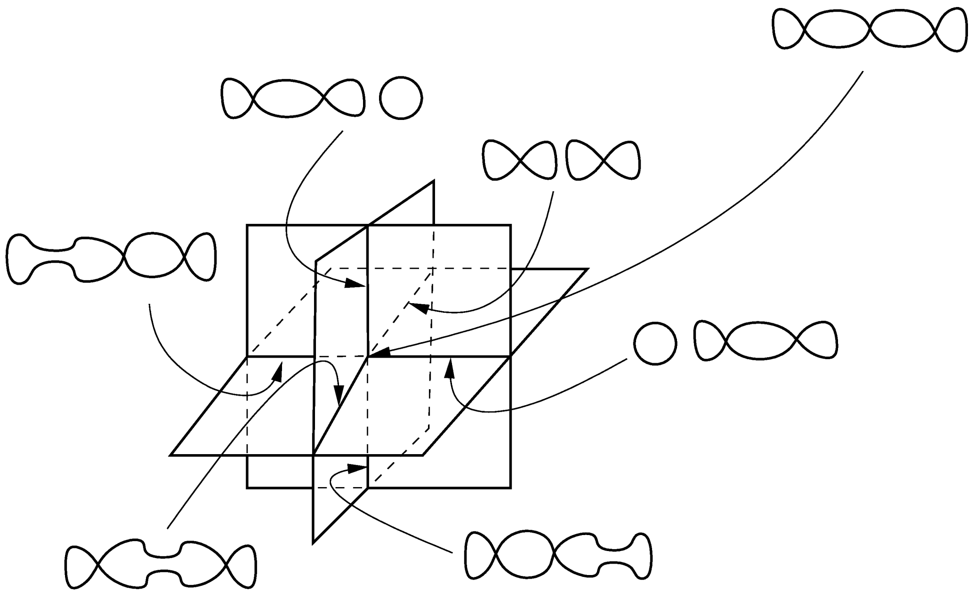

Figure 10. Degeneration of fibers near a III^4-type singular fiber

where $M_k^3(\bar{g}) = \{y \in g(M) \setminus g(S(g)) : |\bar{g}^{-1}(y)| = k\}$, $\Delta = g(S(g)) \setminus (\mathrm{I}^0(g) \cup \mathrm{I}^1(g))$ *and* $M_k^2(\bar{g}) = \{y \in g(S(g)) \setminus \Delta : |\bar{g}^{-1}(y)| = k\}$.

As an immediate corollary, we have the following.

Corollary 6.5. *Let* $g\colon M \to \mathbb{R}^3$ *be a* C^∞*-stable map of a closed orientable* 4*-dimensional manifold into* $\mathbb{R}^3$. *Then, we have*

$$\begin{aligned} &|T(g)| - (|\mathrm{III}^6(g)| + |\mathrm{III}^8(g)|) \\ &\quad + \sum_{k:\mathrm{odd}} |\mathrm{III}^c_k(g)| + \sum_{k:\mathrm{even}} (|\mathrm{III}^d_k(g)| + |\mathrm{III}^e_k(g)|) \equiv 0 \pmod 2, \end{aligned}$$

where $T(g)$ $(\subset \mathbb{R}^3)$ *is the set of triple points of* $g|_{S(g)}$ *in the target as in Fig.* 9 (4).

Note that the above congruence can also be obtained by using the adjacencies of singular fibers. For details, see [19, Remark 4.4].

Example 6.6. Consider the C^∞-stable map $g\colon \mathbb{C}P^2 \sharp 2\overline{\mathbb{C}P^2} \to \mathbb{R}^3$ constructed in [19]. Note that g has only fold points as its singularities. Furthermore, the set $g(S(g))$ is a disjoint union of three spheres, which are the images of definite fold points, and a Boy surface, which is the image of indefinite fold points. Therefore, g has exactly one singular fiber of codimension three and g has neither swallowtails nor cusp points.

For this map, we have $|\mathrm{III}^c_k(g)| = |\mathrm{III}^d_k(g)| = |\mathrm{III}^e_k(g)| = 0$ for all $k > 0$. From Corollary 6.5 it follows that $|T(g)| - (|\mathrm{III}^6(g)| + |\mathrm{III}^8(g)|) \equiv 0 \pmod 2$. This implies that the singular fiber of codimension three must be of type III^6 or III^8, since $|T(g)| = 1$. In fact, g has exactly one singular fiber of type III^8.

Example 6.7. In [9, Theorem 2.1, p. 6], Kobayashi presents a C^∞-stable map

$$g_1 \colon \mathbb{C}P^2 \to \mathbb{R}^3$$

satisfying the following properties.

(1) The singular set $S(g_1)$ is diffeomorphic to $\mathbb{R}P^2 \sharp \mathbb{R}P^2 \sharp \mathbb{R}P^2$ (see [7, Example 5.8] as well).

(2) The map g_1 has six swallowtails and six curves of cusp points.

For the map g_1, we have two triple points of $g_1|_{S(g_1)}$, one of which corresponds to a singular fiber of type III^{000}, and six definite swallowtails. Note that the six swallowtails are of the same type, while three of them have multiplicity one and the other three have multiplicity two. Then, by Corollary 6.5, we have $|\mathrm{III}^6(g_1)| + |\mathrm{III}^8(g_1)| \equiv 1 \pmod 2$. Since $|T(g_1)| = 2$ and $|\mathrm{III}^{000}(g_1)| = 1$, the singular fiber corresponding to the other triple point of $g_1|_{S(g_1)}$ has to be of type III^6 or III^8.

Example 6.8. In [9, Theorem 3.1, p. 8], Kobayashi presents a C^∞-stable map

$$g_2 \colon \mathbb{C}P^2 \to \mathbb{R}^3$$

satisfying the following properties.

(1) The singular set $S(g_2)$ is diffeomorphic to the disjoint union $S^2 \cup \mathbb{R}P^2$.

(2) The map g_2 has a circle of cusp points and some surfaces of fold points, while it does not have any swallowtails.

(3) The singular value set $g_2(S(g_2)) \subset \mathbb{R}^3$ consists of two connected components: the image of $\mathbb{R}P^2$ is a singular surface obtained from an embedded sphere with a cuspidal edge circle, by replacing a small 2-dimensional disk with a punctured Boy surface, and the image of S^2 is embedded so that it surrounds the image of $\mathbb{R}P^2$.

As in Example 6.6, g_2 does not have any swallowtails and $g_2|_{S(g_2)}$ has only one triple point. Therefore, the singular fiber corresponding to this point has to be of type III^6 or III^8.

Remark 6.9. As a consequence of Corollary 6.5, if a C^∞-stable map $g \colon M \to \mathbb{R}^3$ of a closed orientable 4-dimensional manifold M does not have any swallowtails and $g|_{S(g)}$ has only one triple point, then the singular fiber corresponding to this point has to be of type III^6 or III^8. Moreover, by [18, Corollary 5.4], the Euler characteristic $\chi(M)$ is odd.

Remark 6.10. Some related Euler characteristic formulas based on singular fibers of maps are obtained in [7].

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