On line degenerated torus curves and weak Zariski pairs

By

Kawashima Masayuki *

Abstract

Let $C = \{ f = 0 \}$ be an affine plane curve. We are interested in a form of the defining polynomial $f$. In this paper, we study line degenerations of torus curves. Line degenerations of torus type are divided into two types which are called visible or invisible degenerations. We construct a pair of plane curves of degree $2p - 2$ such that they have the same configuration of singularities. If $p$ is even, their complements in $\mathbb{P}^2$ have different topologies. Thus they give a weak Zariski pair.

§1. Introduction

Let $\mathbb{P}^2$ be a complex projective space of dimension 2 with homogeneous coordinates $[X_0, X_1, X_2]$ and let $\mathbb{C}^2 = \mathbb{P}^2 \setminus \{X_2 = 0\}$ be the affine space with coordinates $(x, y) = (X_0/X_2, X_1/X_2)$. We study reduced plane curves in $\mathbb{P}^2$ and $\mathbb{C}^2$. Let $\mathcal{M}(d)$ and $\mathcal{M}^a(d)$ be the set of projective and affine plane curves of degree $d$ respectively. For a given curve $C \in \mathcal{M}(d)$ or $\mathcal{M}^a(d)$, we are interested in forms of the defining polynomial of $C$.

Let $p$ and $q$ be positive integers such that $p > q \geq 2$. We say that $C = \{ f = 0 \} \in \mathcal{M}^a(d)$ is a torus curve of type $(p, q)$ if $f$ is written as $f = f_a^p + f_b^q$ where $f_j$ is a polynomial in $\mathbb{C}[x, y]$ of degree $j$. Put $\mathcal{T}(p, q; d)$ as the set of curves of $(p, q)$ torus type of degree $d$.

We also consider another class of plane curves which are called quasi torus curves of type $(p, q)$ (c.f [7], [2]). We say that $C = \{ f = 0 \} \in \mathcal{M}^a(d)$ quasi torus curve of type $(p, q)$ if there exist three polynomials $f_a$, $f_b$ and $f_c$ such that they do not have same
components and they satisfy the following relation:

\[ f_{c}^{pq}f = f_{a}^{p} + f_{b}^{q} \quad \text{in } \mathbb{C}[x, y] \quad \deg f_{j} = j \]

where \( \deg f_{j} \) is the degree of \( f_{j} \). Put \( \mathcal{Q}\mathcal{T}(p, q; d) \) as the set of curves of \((p, q)\) quasi torus type of degree \( d \).

For a given curve \( C \in \mathcal{M}^{a}(d) \), we say that \( C \) has a torus decomposition (resp. quasi torus decomposition) if \( C \) is in \( \mathcal{T}(p, q; d) \) (resp. \( \mathcal{Q}\mathcal{T}(p, q; d) \)) for some \((p, q)\).

**Example 1.1.** The following example is the motivation of this work. Let \( Q = \{f = 0\} \in \mathcal{M}^{a}(4) \) be a 3-cuspidal quartic. Then \( Q \) has at least two torus and one quasi torus decompositions ([6]):

\[ f = f_{1}^{3} + f_{2}^{2}, \quad f = g_{2}^{3} + g_{3}^{2}, \quad h_{1}^{6}f = h_{3}^{3} + h_{5}^{2} \]

where \( \deg f_{i} = i, \deg g_{i} = i \) and \( \deg h_{i} = i \).

To construct these torus decompositions, we used line degenerated torus curves. Now we recall line degeneration of torus curves which are defined by M. Oka in [8].

**Definition 1.2.** Let \( C = \{F = F_{q}^{p} + F_{p}^{q} = 0\} \in \mathcal{M}(pq) \) be a projective \((p, q)\) torus curve. Suppose that \( F \) has the following form:

\[ F(X_{0}, X_{1}, X_{2}) = X_{2}^{j}G(X_{0}, X_{1}, X_{2}) \]

where \( G(X, Y, Z) \) is a reduced homogeneous polynomial of degree \( pq - j \). We call a curve \( D = \{G = 0\} \) a line degenerated torus curve of type \((p, q)\) of order \( j \) and the line \( L_{\infty} = \{X_{2} = 0\} \) the limit line of the degeneration ([8]).

Put \( \mathcal{L}\mathcal{T}_{j}(p, q; d) \) as the set of line degenerated torus curves of type \((p, q)\) of order \( j \). and \( \mathcal{L}\mathcal{T}(p, q) \) is the union of \( \mathcal{L}\mathcal{T}_{j}(p, q; d) \) with respect to \( j \).

We divide the situations (1.2) into two cases which are called visible degenerations and invisible degenerations. Put the integer \( r_{k} := \max\{r \in \mathbb{Z} | X_{2}^{r} | F_{k}\} \) for \( k = p, q \).

**Visible case.** Suppose that \( r_{p}, r_{q} \neq 0 \) and \( qr_{p} \neq pr_{q} \). Then \( F_{q} \) and \( F_{p} \) are written as follows:

\[ F_{q}(X_{0}, X_{1}, X_{2}) = F_{q-r_{q}}'(X_{0}, X_{1}, X_{2})X_{2}^{r_{q}}, \quad F_{p}(X_{0}, X_{1}, X_{2}) = F_{p-r_{p}}'(X_{0}, X_{1}, X_{2})X_{2}^{r_{p}}. \]

Putting \( j := \min\{qr_{p}, pr_{q}\} \), we can factor \( F \) as \( F(X_{0}, X_{1}, X_{2}) = X_{2}^{j}G(X_{0}, X_{1}, X_{2}) \).

Then \( G \) is written using \( F_{p-r_{p}}' \) and \( F_{q-r_{q}}' \) as

\[ G(X_{0}, X_{1}, X_{2}) = \begin{cases} F_{q-r_{q}}'(X_{0}, X_{1}, X_{2})^{p} + F_{p-r_{p}}'(X_{0}, X_{1}, X_{2})^{q}X_{2}^{qr_{p}-pr_{q}} & \text{if } j = pr_{q}, \\ F_{q-r_{q}}'(X_{0}, X_{1}, X_{2})^{p}X_{2}^{pr_{p}-qr_{p}} + F_{p-r_{p}}'(X_{0}, X_{1}, X_{2})^{q} & \text{if } j = qr_{p}. \end{cases} \]
Such a factorization is called a visible factorization and \( D \) is called a visible degeneration of \((p,q)\) torus curves. We denote the set of visible degenerations of order \( j \) by \( \mathcal{L}T_j^V(p,q;d) \).

**Invisible case.** Either \( r_p = 0 \) or \( r_q = 0 \) but \( F \) can be written as (1.2). Then \( D \) is called an invisible degeneration of \((p,q)\) torus curves. In this case, write \( F = F_p^q + F_q^p \). Then \( A_j(X_0, X_1) = 0 \) for \( i < j - 1 \) and therefore \( X_2 \parallel F \). We denote the set of invisible degenerations of order \( j \) by \( \mathcal{L}T_j^I(p,q;d) \).

Using these terminologies, we will show that torus decompositions (1.1) satisfy:

\[
\{f_1^3 + f_2^2 = 0\} \in \mathcal{L}T_2^V(3,2;4), \quad \{g_2^3 + g_3^2 = 0\} \in \mathcal{L}T_2^I(3,2;4).
\]

Thus \( Q = \{f = 0\} \) is in \( \mathcal{L}T_2^V(3,2;4) \cap \mathcal{L}T_2^I(3,2;4) \).

We consider whether such phenomena occur or not for other curves. Before we consider this problem, we study line degenerated torus curves. More precisely, we look for a pair of curves \( \{C, D\} \) such that \( C \in \mathcal{L}T_j^V(p,q;d) \) and \( D \in \mathcal{L}T_j^I(p,q;d) \) such that \( \text{Sing } C = \text{Sing } D \). Here \( \text{Sing } C \) is the configuration of the singularities. If there exists such a pair \( (C,D) \), then we discuss if the topologies of \( C \) and \( D \) are the same or not.

**Definition 1.3.** A pair of plane curves \( (C_1, C_2) \) is called a weak Zariski pair if they have the same degree and configuration of singularities, while the complements \( \mathbb{P}^2 \setminus C_1 \) and \( \mathbb{P}^2 \setminus C_2 \) are not homeomorphic to each other ([9, 5]).

To express singularities of curves, we use an important class of singularities which is called Brieskorn-Pham singularities:

\[
B_{n,m} : x^n + y^m = 0, \quad n, m \geq 2.
\]

**Theorem 1.4.** For each \( p \geq 3 \), there is a pair of plane curves \( (C,D) \in \mathcal{L}T_2^V(p,2;2p-2) \times \mathcal{L}T_2^I(p,2;2p-2) \) with

\[
\text{Sing } C = \text{Sing } D = \{pA_{p-1}, A_{p-3}, B_{p-2,2(p-2)}\}.
\]

If \( p \) is even, then \( (C,D) \) is a weak Zariski pair.

§ 2. Preliminaries

In section 2, we follow the terminologies in [3] and [4].

Let \( p : \Sigma_d \to \mathbb{P}^1 \) be a Hirzebruch surface of degree \( d \) and let \( \Delta_{\infty,d} \) be the exceptional section with the self-intersection multiplicity \( \Delta_{\infty,d}^2 = -d \). Let \( (X_0, X_1, X_2) \) and \( (Y_0, Y_1) \) be homogeneous coordinates of \( \mathbb{P}^2 \) and \( \mathbb{P}^1 \) respectively. Using these coordinates, \( \Sigma_d \) is defined as

\[
\Sigma_d := \{((X_0, X_1, X_2), (Y_0, Y_1)) \in \mathbb{P}^2 \times \mathbb{P}^1 \mid X_1Y_1^d = X_2Y_0^d\}
\]
and \( p : \Sigma_d \to \mathbb{P}^1 \) is the canonical projection. There are four affine coordinates which cover \( \Sigma_d \). We use two affine spaces \( W_d^1, W_d^2 \subset \Sigma_d \) with coordinates \( (y_d, \tau_d) \) and \( (z_d, \tau_d) \) respectively where

\[
y_d = \frac{X_2}{X_0}, \quad z_d = \frac{X_0}{X_2}, \quad \tau_d = \frac{Y_0}{Y_1}
\]

and they are glued by the relation \( y_d z_d = 1 \). Putting \( V_1 = \{(Y_0, Y_1) \in \mathbb{P}^1 \mid Y_1 \neq 0\} \), they satisfy \( p^{-1}(V_1) = W_d^1 \cup W_d^2 \).

We denote the fiber over \( \tau_d = 0 \) in \( \Sigma_d \) by \( F_{\infty}^\circ \) and the origin of the affine space \( W_d^i \) by \( O_{i,d} := (0,0) \in W_d^i \). We put the affine line \( F_{\infty} := F_{\infty}^\circ \setminus \Delta_{\infty,d} = F_{\infty} \cap W_d^2 \).

§ 2.1. \( p \)-gonal curves

Let \( B \subset \Sigma_d \) be a reduced curve such that \( B \) does not contain the exceptional section \( \Delta_{\infty,d} \). If \( B \) intersects with a generic fiber at \( p \) points, then we call \( B \) a generalized \( p \)-gonal curve. A generalized \( p \)-gonal curve \( B \) is called a \( p \)-gonal curve if \( B \) disjoint from the exceptional section \( \Delta_{\infty,d} \).

Let \( f_i \) be a defining equation of \( B \) on \( W_d^i \) and then we have the equality \( f_1(y_d, \tau_d) = y_d^p f_2(z_d, \tau_d) \) on \( W_d^1 \cap W_d^2 \). Using affine coordinates \( (z_d, \tau_d) \in W_d^2 \), the local equation \( f_2(z_d, \tau_d) \) is written as

\[
f_2(z_d, \tau_d) = \sum_{i=0}^{p} b_i(\tau_d)z_d^i, \quad \deg b_i(\tau_d) \leq d(p - i).
\]

The exceptional section \( \Delta_{\infty,d} \) is defined as \( \{y_d = 0\} \) in the affine coordinates \( (y_d, \tau_d) \in W_d^1 \).

§ 2.2. Nagata transformations

Let \( P \) be a fixed point in \( \Sigma_2 \setminus \Delta_{\infty,2} \) and let \( F \) be the fiber which passes through \( P \). A Nagata transformation \( N : \Sigma_2 \to \Sigma_1 \) is a birational transformation which consists of the blowing-up at \( P \notin \Delta_{\infty,2} \) and the blowing-down the strict transform \( F^* \) of \( F \). We observe that the exceptional section \( \Delta_{\infty,1} \) of \( \Sigma_1 \) is the image \( N(\Delta_{\infty,2}) \).

We express a Nagata transformation using local coordinates \( (z_2, \tau_2) \) and \( (z_1, \tau_1) \) assuming \( P = O_{2,2} \in W_2^2 \). Let \( \mu_1 : \bar{W}_2^2 \to W_2^2 \) and \( \mu_2 : \bar{W}_1^1 \to W_1^1 \) be blowing-ups centered at \( O_{2,2} \) and \( O_{1,1} \) respectively. There is an affine coordinate \( \bar{W} \) with coordinates \( (s, t) \) such that \( \mu_1(s, t) = (t, ts) \) and \( \mu_2(s, t) = (s, st) \). Note that \( \{t = 0\} \) defines the exceptional curve of \( \mu_1 \) and \( \{s = 0\} \) defines the exceptional curve of \( \mu_2 \). Then we have:

\[
N(z_2, \tau_2) = (z_1, \tau_1) = \left( \frac{z_2}{\tau_2}, \tau_2 \right).
\]

Let \( B \) be a \( p \)-gonal curve in \( \Sigma_2 \) which is defined by \( \{f_2(z_2, \tau_2) = 0\} \) in \( W_2^2 \). We consider the defining equation of the image of a \( p \)-gonal curve by a Nagata transformation. By the definition of a Nagata transformation, \( B' := N(B) \subset \Sigma_1 \) is defined as
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(2.1) \[ B' : f_2'(z_1, \tau_1) = \frac{1}{\tau_1} f_2(z_1 \tau_1, \tau_1) = 0 \]

where \( M \) is the multiplicity of \( B \) at \( P \). As \( B \) is assumed to be \( p \)-gonal, \( B' \cap \Delta_{\infty, 1} \) is \( \{O_{1,1}\} \). Thus \( B' \) is a generalized \( p \)-gonal curve.

§2.3. Contraction of \( p \)-gonal curves from \( \Sigma_2 \) to \( \mathbb{P}^2 \)

We recall that a Hirzebruch surface \( \Sigma_1 \) is obtained as a blowing-up at an any point in \( \mathbb{P}^2 \). In this section, we consider the defining polynomial of a plane curve which is obtained as the image of the composition of a Nagata transformation and a blowing-up.

Let \( B = \{ f_2(z_2, \tau_2) = 0 \} \) be a \( p \)-gonal curve in \( W_2^2 \) and let \( B' = \{ f_2'(z_1, \tau_1) = 0 \} \subset W_1^2 \) be the image of \( B \) by a Nagata transformation \( N : \Sigma_2 \rightarrow \Sigma_1 \) at \( O_{2,2} \). Put \( m \) the intersection multiplicity of \( B' \) and \( \triangle_{\infty, 1} \) at \( O_{1,1} \).

Let \( U_1 \) be the affine coordinate chart \( \mathbb{P}^2 \backslash \{ X_1 = 0 \} \) with the coordinate \( (x_0, x_2) = (X_0/X_1, X_2/X_1) \). Let \( \pi : \tilde{U}_1 \rightarrow U_1 \) be a blowing-up at \( (0, 0) \in U_1 \). We naturally identify \( \tilde{U}_1 \) with \( \Sigma_1 \) as follows: Let \( \tilde{U}_{10} \) and \( \tilde{U}_{11} \) be two affine coordinates of \( \tilde{U}_1 \) and let \( (s, t) \) be the affine coordinate of \( \tilde{U}_{11} \). Then \( \pi \) is defined as \( \pi(s, t) = (x_0, x_2) = (s, st) \) on \( \tilde{U}_{11} \). We identify \( \tilde{U}_{11} \) with \( W_1^1 \) as \( (s, t) \mapsto (y_1, \tau_1) \).

By the definition of \( \pi : \Sigma_1 \rightarrow U_1 \) and the equality (2.1), the defining polynomial \( f \) of \( C := (\pi \circ N)(B) \subset U_1 \) as

\[
(2.2) \quad f(x_0, x_2) = \frac{x_0^{M+m+p}}{x_2^M} f_2\left( \frac{x_2}{x_0^2}, \frac{x_2}{x_0} \right).
\]

Indeed, let \( f_1'(y_1, \tau_1) \) be the defining equation of \( B' \) in \( W_1^1 \) which is written as

\[
f_1'(y_1, \tau_1) = y_1^{p} f_2'(1/y_1, \tau_1) = \frac{y_1^p}{\tau_1^M} f_2(\tau_1/y_1, \tau_1).
\]

where we use (2.1) for the second equality. And \( f \) must satisfy \( f(y_1, y_1 \tau_1) = y_1^p f_1'(y_1, \tau_1) \). Using these equalities and \( \pi(y_1, \tau_1) = (x_0, x_2) = (y_1, y_1 \tau_1) \), we have the equality (2.2).

Next we consider singularities of \( B' \) and \( C \). Assume that \( B \) satisfies the following conditions:

- \( B \) has an \( A_{\ell-1} = B_{\ell,2} \) singularity at \( O_{2,2} \in W_2^2 \) and its tangent cone is transverse to the fiber \( F_\infty = \{ \tau_2 = 0 \} \).

- \( B \) intersects transversely at \( p - 2 \) distinct points with \( F_\infty \) outside of \( O_{2,2} \in W_2^2 \).

Under the above conditions, the intersection \( B \cap (F_\infty \backslash \{O_{2,2}\}) \) consists of distinct \( p - 2 \) points and \( B' \) intersects with \( F_\infty \) so that
• If \( \ell = 2 \), then \( B' \) intersects transversely with \( F_\infty^\circ \) at two points.

• If \( \ell = 3 \), then \( B' \) is tangent to \( F_\infty^\circ \) with the intersection multiplicity 2.

• If \( \ell > 3 \), then \( B' \) has \( A_{\ell-3} = B_{\ell-2,2} \) singularity.

Observation. If \( B \) is a trigonal curve (\( p = 3 \)), then \( B' \) is smooth and intersects transversely with \( \Delta_\infty,1 \) at \( O_{1,1} \). If \( p \) is greater than 3, then \( B' \) has \( B_{p-2,p-2} \) singularity at \( (0,0) \in U_1 \).

Proof. The first assertion is obvious. Assume \( p > 3 \). The defining equation \( f_{1}'(y_1, \tau_1) \) of \( B' \) in \( W_1^{1} \) is written as:

\[
f_{1}'(y_1, \tau_1) = c \prod_{i=1}^{p-2} (y_1 - \alpha_i \tau_1) + \text{(higher terms)}, \quad c \neq 0, \quad \alpha_i \neq \alpha_j \ (i \neq j).
\]

Now we use the equality \( f(x_0, x_2) = x_0^{p-2} f_{1}'(x_0, x_2/x_0) \) which is obtained from (2.2). Then we have

\[
f(x_0, x_2) = x_0^{p-2} f_{1}'(x_0, x_2/x_0) = \prod_{i=1}^{p-2} (x_0^2 - \alpha_i x_2) + \text{(higher terms)}.
\]

Thus \( C \) has \( B_{p-2,2(p-2)} \) singularity at \((0,0) \in U_1 \). \( \square \)

§ 3. \( p \)-gonal curves of \( (p, 2) \) torus type

Let \( B \) be a \( p \)-gonal curve in \( \Sigma_2 \). We say that \( B \) is torus curve of type \( (p, 2) \) if the defining equation \( f_2 \) of \( B \) in the affine space \( (W_2^{2}, (z_2, \tau_2)) \) is written as

\[
f_2(z_2, \tau_2) = k(z_2, \tau_2)^p - h(z_2, \tau_2)^2.
\]

We assume further that

\[
\begin{align*}
k(z_2, \tau_2) &= z_2 + b_2(\tau_2), \\
h(z_2, \tau_2) &= b_{p-2}(\tau_2)z_2 + b_p(\tau_2),
\end{align*}
\]

\( \deg b_i(\tau_2) = i \).

§ 3.1. Singularities of \( (p, 2) \) torus type

We consider curves \( K := \{k = 0\} \) and \( H := \{h = 0\} \) in \( W_2^{2} \) where \( h \) and \( k \) are as above. Let \( P \in B \) be a singular point. If \( P \in K \cap H \), we call \( P \) an inner singularity. Otherwise \( P \) is called an outer singularity. We put \( \Delta_1(\tau_2) := h(-b_2(\tau_2), \tau_2) = b_p(\tau_2) - b_{p-2}(\tau_2)b_2(\tau_2) \) and take an inner singular point \( P \in K \cap H \). Then \( P \) is written as \( (-b_2(s), s) \) for some \( s \in \mathbb{C} \) with \( \Delta_1(s) = 0 \) and the multiplicity of \( \Delta_1(\tau_2) \) at \( s \), say \( \iota \), is equal to the intersection multiplicity of \( K \) and \( H \) at \( P \).

By a similar argument as that in Lemma 1 in [1], we have the following.
Lemma 3.1. Let \( B \) be the \( p \)-gonal curve as above in \( \Sigma_2 \). Suppose that \( s \) is a root of \( \Delta_1(\tau) \) and let \( P = (-b_q(s), s) \in B \) be an inner singular point with the intersection multiplicity \( \iota \). If \( \Delta_2(s) \neq 0 \), then \( B \) has \( B_{p,2} = A_{p-1} \) singularity at \( P \).

\[
\Delta_1(\tau) = 1 - \frac{p}{2} - p\tau_2^{p-2} + (1 - p)\tau_2^p \quad \text{and} \quad p \geq 3,
\]

Then \( f_2 = k^p - h^2 \) has an outer \( A_{p-1} \) singularity at \( O_{2,2} \) and its tangent cone does not contain \( \{\tau_2 = 0\} \). As \( \Delta_1(\tau_2) = 1 - \frac{p}{2} - p\tau_2^{p-2} + (1 - p)\tau_2^p \) and \( p \geq 3 \), \( K \) and \( H \) intersect transversely at distinct \( p \) points and \( K \cap H \cap F_\infty = \emptyset \).

Let \( P \) be an inner \( A_{p-1} \) singular point and let \( Q \) be an outer \( A_{p-1} \) singular point of \( B \). Let \( N_1 \) and \( N_2 \) be the Nagata transformations from \( \Sigma_2 \) to \( \Sigma_1 \) at \( P \) and \( Q \) respectively. We consider the defining polynomial of \( C := (\pi \circ N_1)(B) \) and \( D := (\pi \circ N_2)(B) \) where \( \pi : \Sigma_1 \to U_1 \) is the blowing-up at \((0,0) \in U_1\).

\[\begin{array}{c}
\Sigma_2 \\
\downarrow \\
N_2 \\
\downarrow \\
\Sigma_1 \\
\downarrow \\
N_1 \\
\downarrow \\
P^2
\end{array}\]

\[\begin{array}{c}
\Sigma_1 \\
\downarrow \\
\pi \\
\downarrow \\
P^2
\end{array}\]

\[
b_2(\tau_2) = 1 + \tau_2^2, \quad b_p(\tau_2) = 1 + \frac{p}{2} \tau_2^2 + \tau_2^p, \quad b_{p-2}(\tau_2) = \frac{p}{2} + p\tau_2^{p-2}.
\]

§ 4. Proof of Theorem 1.4

Let \( B \subset \Sigma_2 \) be a \( p \)-gonal curve of \((p, 2)\) torus type. As the degree of \( \Delta_1(\tau_2) \) is \( p \), \( B \) has \( pA_{p-1} \) inner singularities by Lemma 3.1. We may assume that \( B \) has an outer \( A_{p-1} \) singularity. For example, we take \( b_2(\tau_2), b_{p-2}(\tau_2) \) and \( b_p(\tau_2) \) as

\[
b_2(\tau_2) = 1 + \tau_2^2, \quad b_p(\tau_2) = 1 + \frac{p}{2} \tau_2^2 + \tau_2^p, \quad b_{p-2}(\tau_2) = \frac{p}{2} + p\tau_2^{p-2}.
\]

Then \( f_2 = k^p - h^2 \) has an outer \( A_{p-1} \) singularity at \( O_{2,2} \) and its tangent cone does not contain \( \{\tau_2 = 0\} \). As \( \Delta_1(\tau_2) = 1 - \frac{p}{2} - p\tau_2^{p-2} + (1 - p)\tau_2^p \) and \( p \geq 3 \), \( K \) and \( H \) intersect transversely at distinct \( p \) points and \( K \cap H \cap F_\infty = \emptyset \).

Let \( P \) be an inner \( A_{p-1} \) singular point and let \( Q \) be an outer \( A_{p-1} \) singular point of \( B \). Let \( N_1 \) and \( N_2 \) be the Nagata transformations from \( \Sigma_2 \) to \( \Sigma_1 \) at \( P \) and \( Q \) respectively. We consider the defining polynomial of \( C := (\pi \circ N_1)(B) \) and \( D := (\pi \circ N_2)(B) \) where \( \pi : \Sigma_1 \to U_1 \) is the blowing-up at \((0,0) \in U_1\).

§ 4.1. Construction of a visible degeneration

Hereafter we assume that \( K \) and \( H \) intersect transversely at \( p \) points. Assume that \( P = O_{2,2} \) in the affine space \( W_2^2 \). Let \( f_2(z_2, \tau_2) = k(z_2, \tau_2)^p - h(z_2, \tau_2)^2 \) be the defining
equation of $B$ where

$$k(z_2, \tau_2) = z_2 + b_2(\tau_2), \quad h(z_2, \tau_2) = b_{p-2}(\tau_2)z_2 + b_p(\tau_2), \quad \deg b_i(\tau_2) = i.$$  

As $k(0, 0) = h(0, 0) = 0$, we can write $b_2(\tau_2)$ and $b_p(\tau_2)$ as

$$b_2(\tau_2) = \tau_2 b_1(\tau_2), \quad b_p(\tau_2) = \tau_2 b_{p-1}(\tau_2), \quad \deg b_i = i.$$  

Let $f$ be the defining polynomial of $C$ and using (2.2), we have

$$f(x_0, x_2) = \frac{x_0^{2p}}{x_2^2} f_2 \left( \frac{x_2}{x_0}, \frac{x_2}{x_0} \right).$$  

We calculate the above equation as the following:

$$x_2^2 f(x_0, x_2) = x_0^{2p} \left( k \left( \frac{x_2}{x_0}, \frac{x_2}{x_0} \right)^p - h \left( \frac{x_2}{x_0}, \frac{x_2}{x_0} \right)^2 \right)$$

$$= x_0^{2p} \left( \left( \frac{x_2}{x_0} + \frac{x_2}{x_0} b_1(\frac{x_2}{x_0}) \right)^p - \left( \frac{b_{p-2}(\frac{x_2}{x_0}) x_2}{x_0} + \frac{x_2}{x_0} b_{p-1}(\frac{x_2}{x_0}) \right)^2 \right)$$

$$= x_2^p \left( 1 + x_0 b_1(\frac{x_2}{x_0}) \right)^p - x_2^2 \left( x_0^{p-2} b_{p-2}(\frac{x_2}{x_0}) + x_0^{p-1} b_{p-1}(\frac{x_2}{x_0}) \right)^2$$

$$= f_1(x_0, x_2)^p x_2^p - f_{p-1}(x_0, x_2)^2 x_2^2.$$  

and then where

$$f_1(x_0, x_2) := 1 + c_1(x_0, x_2), \quad f_{p-1}(x_0, x_2) := c_{p-2}(x_0, x_2) + c_{p-1}(x_0, x_2).$$  

Note that $c_i(x_0, x_2) := x_0^i b_i(x_2/x_0)$ is a polynomial for $i = 1, p-2$ and $p-1$. Hence we have

$$x_2^2 f(x_0, x_2) = (f_1(x_0, x_2)x_2)^p - (f_{p-1}(x_0, x_2)x_2)^2.$$  

Thus the above equation shows that $C := \{ f = 0 \}$ is a visible line degeneration of order 2 of $(p, 2)$ torus type.

§ 4.2. Construction of an invisible degeneration

Assume that $Q = O_{2,2}$ in the affine space $W_2^2$. Let $f_2(z_2, \tau_2) = k(z_2, \tau_2)^p - h(z_2, \tau_2)^2$ be the defining equation of $B$ where

$$k(z_2, \tau_2) = z_2 + b_2(\tau_2), \quad h(z_2, \tau_2) = b_{p-2}(\tau_2)z_2 + b_p(\tau_2), \quad \deg b_i(\tau_2) = i.$$  

Let $g$ be the defining polynomial of $D$ and using (2.2) in §2.3, we have

$$g(x_0, x_2) = \frac{x_0^{2p}}{x_2^2} f_2 \left( \frac{x_2}{x_0}, \frac{x_2}{x_0} \right).$$
We calculate the above equation:
\[
x_{2}^{2}g(x_{0}, x_{2}) = x_{0}^{2p} \left( k \left( \frac{x_{2}}{x_{0}^{2}}, \frac{x_{2}}{x_{0}} \right)^{p} - h \left( \frac{x_{2}}{x_{0}^{2}}, \frac{x_{2}}{x_{0}} \right)^{2} \right)
\]
\[
= x_{0}^{2p} \left( \left( \frac{x_{2}}{x_{0}^{2}} + b_{2} \left( \frac{x_{2}}{x_{0}} \right) \right)^{p} - \left( b_{p-2} \left( \frac{x_{2}}{x_{0}} \right)^{p-2} b_{p} \left( \frac{x_{2}}{x_{0}} \right) \right)^{2} \right)
\]
\[
= \left( x_{2} + x_{0}^{2} b_{2} \left( \frac{x_{2}}{x_{0}} \right) \right)^{p} - \left( x_{0}^{p-2} b_{p-2} \left( \frac{x_{2}}{x_{0}} \right) + x_{0}^{p} b_{p} \left( \frac{x_{2}}{x_{0}} \right) \right)^{2}
\]
\[
= g_{2}(x_{0}, x_{2})^{p} - g_{p}(x_{0}, x_{2})^{2}
\]
where the polynomials $g_{2}$ and $g_{p}$ are defined as
\[
g_{2}(x_{0}, x_{2}) := x_{2} + d_{2}(x_{0}, x_{2}) \quad g_{p}(x_{0}, x_{2}) := d_{p-2}(x_{0}, x_{2}) x_{2} + d_{p}(x_{0}, x_{2})
\]
where $d_{i}(x_{0}, x_{2}) := x_{0}^{i} b_{i}(x_{2}/x_{0})$ for $i = 2, p-2$ and $p$. Thus the above equation shows that $D := \{g = 0\}$ is an invisible line degeneration of order 2 of $(p, 2)$ torus type:
\[
x_{2}^{2}g(x_{0}, x_{2}) = g_{2}(x_{0}, x_{2})^{p} - g_{p}(x_{0}, x_{2})^{2}.
\]

§ 4.3. Singularities of constructed curves

We consider singularities of $C$ and $D$. By our constructions and the argument in §2.3, we have the following:

- Sing $C$ and Sing $D$ are the same:
  \[\text{Sing } C = \text{Sing } D = \{pA_{p-1}, A_{p-3}, B_{p-2,2(p-2)}\}\]

- $C$ has $(p-1)A_{p-1}$ and $A_{p-3}$ singularities as inner and one $A_{p-1}$ singularity as outer.
- $D$ has $pA_{p-1}$ singularities as are inner and $A_{p-3}$ singularity as outer.

Thus we have a pair $(C, D)$ which satisfy the statement of the first part of Theorem 1.4.

§ 4.4. The case $p$ is even

In this section, we suppose that $p$ is even. We will show that the pair $(C, D)$ is a weak Zariski pair. Recall that the defining polynomials $f$ and $g$ of $C$ and $D$ satisfy
\[
f(x_{0}, x_{2}) = f_{1}(x_{0}, x_{2})^{p}x_{2}^{p-2} - f_{p-1}(x_{0}, x_{2})^{2}
\]
\[
x_{2}^{2} g(x_{0}, x_{2}) = g_{2}(x_{0}, x_{2})^{p} - g_{p}(x_{0}, x_{2})^{2}.
\]
As $p$ is even, $C$ is decomposed as $C = C_{1} \cup C_{2}$ where $\deg C_{1} = \deg C_{2} = p - 1$. 
Lemma 4.1. $D$ is decomposed as $D = D_{p-2} \cup D_p$ where $\deg D_{p-2} = p-2$ and $D_p = p$.

Proof. Put $p = 2s$. Let $B = \{f_2 = k^{2s} - h^2 = 0\}$ be a 2s-gonal curve in $\Sigma_2$ and let $\pi \circ N_2 : \Sigma_2 \rightarrow \mathbb{P}^2$ be a birational map which are considered in the proof of Theorem 1.4. Then we can factorize $f_2(z_2, \tau_2)$ as

$$f_2(z_2, \tau_2) = (k(z_2, \tau_2)^s - h(z_2, \tau_2))(k(z_2, \tau_2)^s + h(z_2, \tau_2)) = k_1(z_2, \tau_2) k_2(z_2, \tau_2)$$

where

$$k_1(z_2, \tau_2) = k(z_2, \tau_2)^s - h(z_2, \tau_2), \quad k_2(z_2, \tau_2) = k(z_2, \tau_2)^s + h(z_2, \tau_2).$$

As we assumed that $O_{2,2}$ is an outer singular point of $B$, we may assume that $O_{2,2}$ is in $\{k_1 = 0\} \setminus \{k_2 = 0\}$. Then, using (2.2) in §2.3, the defining polynomial $w_1$ of $\pi \circ N_2(\{k_1 = 0\})$ is given by

$$w_1(x_0, x_2) = \frac{x_0^{2s}}{x_0^2} k_1(x_0^2, x_2) = \frac{1}{x_0^2} (g_2(x_0, x_2)^s - g_p(x_0, x_2)).$$

As $w_1$, $g_2$ and $g_p$ are polynomials and $\deg g_2 = 2$ and $\deg g_p = p$, the degree $w_1$ must be $p-2$. Note that $\{w_1 = 0\}$ has $A_{p-3}$ singularity. As $g$ is obtained as

$$g(x_0, x_2) = \frac{x_0^{4s}}{x_2^2} f_2(x_2, x_0) w_2(x_0, x_2)$$

where $w_2 := x_0^{2s}k_2$. As $\deg g = 2p - 2$ and $\deg w_1 = p - 2$, the degree $w_2$ must be $p$. □

Now we consider the irreducibility of $C_1$ and $C_2$. Let $P_1, \ldots, P_{p-1}$, $Q$, $R$ and $O^*$ be the singular points of $C$ such that

$$(C, P_i) \sim A_{p-1}, \quad i = 1, \ldots, p-1,$$

$$(C, Q) \sim A_{p-3}, \quad (C, R) \sim A_{p-1}, \quad (C, O^*) \sim B_{p-2,2(p-2)}$$

and $P_i$ and $Q$ are inner singularities and $R$ is an outer singular point of $C$. As $P_i$ and $Q$ are inner, they are in $\{f_1 = 0\} \cap \{f_{p-1} = 0\}$. Hence $P_i$ and $Q$ are also in $C_1 \cap C_2$. Note that $C_1$ and $C_2$ are smooth at $P_i$ and $Q$. As $R$ is the outer singular point, we may assume that $R \in C_1 \setminus C_2$.

By the form of the defining polynomials of $C_1$ and $C_2$, both curves have $B_{\frac{p-2}{2}, p-2}$ singularity at $O^*$. Note that $C_1$ and $C_2$ have no other singularities.

Now we assume that $C_1$ is reducible as $C_1 = E_a \cup E_b$ where $\deg E_i = i$ and $a \leq b$. Assume that $p > 4$. As $O^*$ and $R$ are singular points of $C_1$, the intersection $E_a \cap E_b$ is one of the following:

$$\{O^*\}, \quad \{R\}, \quad \{O^*, R\}.$$
We consider the cases $E_a \cap E_b = \{O^*\}$ or $\{O^*, R\}$. Let $n$ and $m$ be positive integers such that $(E_a, O^*) \sim B_{n,2n}$ and $(E_b, O^*) \sim B_{m,2m}$. Positive integers $(a, b, n, m)$ must satisfy the following equations:

1. $a + b = p - 1$.
2. $2m + 2n = p - 2$.
3. $a \geq 2n$, $b \geq 2m$.
4. If $E_a \cap E_b = \{O^*\}$, then $ab = 2mn$.
5. If $E_a \cap E_b = \{O^*, R\}$, then $ab = \frac{p}{2} + 2mn$.

Equalities (4) and (5) are obtained by Bézout theorem. By simple calculations, there are no positive integers $(a, b, n, m)$ which satisfy the above equations. Hence if $O^* \in E_a \cap E_b$, then $C_1$ is irreducible. By the same argument, we can show that $C_2$ is irreducible because $C_2$ has only a $B_{\frac{p-2}{2},p-2}$ singularity.

Now we consider the case $E_a \cap E_b = \{R\}$. Then $E_a$ and $E_b$ are smooth at $R$ with $I(E_a, E_b; R) = \frac{p}{2}$. As $E_a \cap E_b = \{R\}$, we have $ab = \frac{p}{2}$ by Bézout theorem. The equations $a + b = p - 1$ and $ab = \frac{p}{2}$ are satisfied for the case $(p, a, b) = (4, 1, 2)$ only. Hence if $p > 4$, then $C_1$ and $C_2$ are irreducible. Therefore the pair $(C, D)$ is a weak Zariski pair.

§ 4.5. The case $p = 4$

We suppose that $p = 4$. Then $\deg C = \deg D = 6$ and their singularities are

$\text{Sing } C = \text{Sing } D = \{5A_3, A_1\}$.

By the above argument, $C$ is decomposed as $E_1 \cup E_2 \cup C_2$ and $C_2$ is a smooth cubic. Their intersection points and intersection multiplicities of these curves are the following:

$E_1 \cap E_2 = \{R\}$, \quad $E_2 \cap C_3 = \{P_1, P_2, P_3\}$, \quad $E_1 \cap C_3 = \{P_4, Q\}$

$I(E_1, E_2; R) = 2$, \quad $I(E_1, C_3; Q) = 1$, \quad $I(E_i, C_3; P_k) = 2$, \quad $k = 1, \ldots, 4$. 

![Diagram of intersecting curves](image-url)
On the other hand, $D$ is also decomposed as $D_4 \cup D_1 \cup D'_1$ where $\deg D_4 = 4$ and
$\deg D_1 = \deg D'_1 = 1$. Indeed, outer $A_1$ singularity must be in $D_2$. Hence $D_2$ consists
of two distinct lines. Thus $D$ is decomposed as $D_4 \cup D_1 \cup D'_1$. Note that $D_1$ and $D'_1$
are bitangent lines of $D_4$.

Thus $C$ and $D$ have different irreducible decompositions. Hence the pair $(C, D)$ is
a weak Zariski pair.

§4.6. Observation for the case $p = 3$

By our construction, $C$ and $D$ are 3-cuspidal quartics. As we mentioned in the
introduction, each curve has both torus decompositions. Moreover it is known that the
moduli space of 3-cuspidal quartic is irreducible and hence $C$ and $D$ are in the same
moduli space.

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References

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