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On line degenerated torus curves and weak Zariski pairs

By

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Abstract

Let $C = \{f = 0\}$ be an affine plane curve. We are interested in a form of the defining polynomial f. In this paper, we study line degenerations of torus curves. Line degenerations of torus type are divided into two types which are called visible or invisible degenerations. We construct a pair of plane curves of degree $2p-2$ such that they have the same configuration of singularities. If p is even, their complements in $\mathbb{P}^2$ have different topologies. Thus they give a weak Zariski pair.

§ 1. Introduction

Let $\mathbb{P}^2$ be a complex projective space of dimension 2 with homogeneous coordinates $[X_0, X_1, X_2]$ and let $\mathbb{C}^2 = \mathbb{P}^2 \setminus \{X_2 = 0\}$ be the affine space with coordinates $(x, y) = (X_0/X_2, X_1/X_2)$. We study reduced plane curves in $\mathbb{P}^2$ and $\mathbb{C}^2$. Let $\mathcal{M}(d)$ and $\mathcal{M}^a(d)$ be the set of projective and affine plane curves of degree d respectively. For a given curve $C \in \mathcal{M}(d)$ or $\mathcal{M}^a(d)$, we are interested in forms of the defining polynomial of C.

Let p and q be positive integers such that $p > q \geq 2$. We say that $C = \{f = 0\} \in \mathcal{M}^a(d)$ is *a torus curve of type* (p, q) if f is written as $f = f_a^p + f_b^q$ where f_j is a polynomial in $\mathbb{C}[x, y]$ of degree j. Put $\mathcal{T}(p, q; d)$ as the set of curves of (p, q) torus type of degree d.

We also consider another class of plane curves which are called *quasi torus curves of type* (p, q) (c.f [7], [2]). We say that $C = \{f = 0\} \in \mathcal{M}^a(d)$ *quasi torus curve of type* (p, q) if there exist three polynomials f_a, f_b and f_c such that they do not have same

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components and they satisfy the following relation:

$$f_c^{pq} f = f_a^p + f_b^q \quad \text{in} \quad \mathbb{C}[x,y] \quad \deg f_j = j$$

where $\deg f_j$ is the degree of f_j. Put $\mathcal{QT}(p,q;d)$ as the set of curves of (p,q) quasi torus type of degree d.

For a given curve $C \in \mathcal{M}^a(d)$, we say that C has *a torus decomposition* (resp. *quasi torus decomposition*) if C is in $\mathcal{T}(p,q;d)$ (resp. $\mathcal{QT}(p,q;d)$) for some (p,q).

Example 1.1. The following example is the motivation of this work. Let $Q = \{f = 0\} \in \mathcal{M}^a(4)$ be a 3-cuspidal quartic. Then Q has at least two torus and one quasi torus decompositions ([6]):

$$f = f_1^3 + f_2^2, \quad f = g_2^3 + g_3^2, \quad h_1^6 f = h_3^3 + h_5^2 \tag{1.1}$$

where $\deg f_i = i$, $\deg g_i = i$ and $\deg h_i = i$.

To construct these torus decompositions, we used *line degenerated torus curves.* Now we recall line degeneration of torus curves which are defined by M. Oka in [8].

Definition 1.2. Let $C = \{F = F_q^p + F_p^q = 0\} \in \mathcal{M}(pq)$ be a projective (p,q) torus curve. Suppose that F has the following form:

$$F(X_0, X_1, X_2) = X_2^j\, G(X_0, X_1, X_2) \tag{1.2}$$

where $G(X,Y,Z)$ is a reduced homogeneous polynomial of degree $pq - j$. We call a curve $D = \{G = 0\}$ *a line degenerated torus curve of type* (p,q) *of order* j and the line $L_\infty = \{X_2 = 0\}$ *the limit line of the degeneration* ([8]).

Put $\mathcal{LT}_j(p,q;d)$ as the set of line degenerated torus curves of type (p,q) of order j. and $\mathcal{LT}(p,q)$ is the union of $\mathcal{LT}_j(p,q;d)$ with respect to j.

We divide the situations (1.2) into two cases which are called *visible degenerations* and *invisible degenerations.* Put the integer $r_k := \max\{r \in \mathbb{Z} \mid X_2^r \mid F_k\}$ for $k = p, q$.

Visible case. Suppose that $r_p \cdot r_q \neq 0$ and $qr_p \neq pr_q$. Then F_q and F_p are written as follows:

$$F_q(X_0,X_1,X_2) = F'_{q-r_q}(X_0,X_1,X_2)X_2^{r_q}, \quad F_p(X_0,X_1,X_2) = F'_{p-r_p}(X_0,X_1,X_2)X_2^{r_p}.$$

Putting $j := \min\{qr_p, pr_q\}$, we can factor F as $F(X_0,X_1,X_2) = X_2^j G(X_0,X_1,X_2)$. Then G is written using F'_{p-r_p} and F'_{q-r_q} as

$$G(X_0,X_1,X_2) = \begin{cases} F'_{q-r_q}(X_0,X_1,X_2)^p + F'_{p-r_p}(X_0,X_1,X_2)^q X_2^{qr_p - pr_q} & \text{if } j = pr_q, \\ F'_{q-r_q}(X_0,X_1,X_2)^p X_2^{pr_q - qr_p} + F'_{p-r_p}(X_0,X_1,X_2)^q & \text{if } j = qr_p. \end{cases} \tag{1.3}$$

Such a factorization is called *a visible factorization* and D is called *a visible degeneration of* (p,q) *torus curves.* We denote the set of visible degenerations of order j by $\mathcal{LT}_j^V(p,q;d)$.

Invisible case. Either $r_p = 0$ or $r_q = 0$ but F can be written as (1.2). Then D is called *an invisible degeneration of* (p,q) *torus curves.* In this case, write $F_p^q + F_q^p = \sum_{i=0}^{pq} A_i(X_0,X_1)X_2^i$. Then $A_j(X_0,X_1) = 0$ for $i \le j-1$ and therefore $X_2^j \mid F$. We denote the set of invisible degenerations of order j by $\mathcal{LT}_j^I(p,q;d)$.

Using these terminologies, we will show that torus decompositions (1.1) satisfy:

$$\{f_1^3 + f_2^2 = 0\} \in \mathcal{LT}_2^V(3,2;4), \quad \{g_2^3 + g_3^2 = 0\} \in \mathcal{LT}_2^I(3,2;4).$$

Thus $Q = \{f = 0\}$ is in $\mathcal{LT}_2^V(3,2;4) \cap \mathcal{LT}_2^I(3,2;4)$.

We consider whether such phenomena occur or not for other curves. Before we consider this problem, we study line degenerated torus curves. More precisely, we look for a pair of curves $\{C, D\}$ such that $C \in \mathcal{LT}_j^V(p,q;d)$ and $D \in \mathcal{LT}_j^I(p,q;d)$ such that $\operatorname{Sing} C = \operatorname{Sing} D$. Here $\operatorname{Sing} C$ is the configuration of the singularities. If there exists such a pair (C,D), then we discuss if the topologies of C and D are the same or not.

Definition 1.3. A pair of plane curves (C_1, C_2) is called *a weak Zariski pair* if they have the same degree and configuration of singularities, while the complements $\mathbb{P}^2 \setminus C_1$ and $\mathbb{P}^2 \setminus C_2$ are not homeomorphic to each other ([9, 5]).

To express singularities of curves, we use an important class of singularities which is called Brieskorn-Pham singularities:

$$B_{n,m} \ : \ x^n + y^m = 0, \quad n,m \ge 2.$$

Theorem 1.4. For each $p \ge 3$, there is a pair of plane curves $(C,D) \in \mathcal{LT}_2^V(p,2;2p-2) \times \mathcal{LT}_2^I(p,2;2p-2)$ with

$$\operatorname{Sing} C = \operatorname{Sing} D = \{pA_{p-1}, A_{p-3}, B_{p-2,2(p-2)}\}.$$

If p is even, then (C,D) is a weak Zariski pair.

§ 2. Preliminaries

In section 2, we follow the terminologies in [3] and [4].

Let $p : \Sigma_d \to \mathbb{P}^1$ be a Hirzebruch surface of degree d and let $\Delta_{\infty,d}$ be the exceptional section with the self-intersection multiplicity $\Delta_{\infty,d}^2$ is $-d$. Let (X_0,X_1,X_2) and (Y_0,Y_1) be homogeneous coordinates of $\mathbb{P}^2$ and $\mathbb{P}^1$ respectively. Using these coordinates, Σ_d is defined as

$$\Sigma_d := \{((X_0,X_1,X_2),(Y_0,Y_1)) \in \mathbb{P}^2 \times \mathbb{P}^1 \mid X_1Y_1^d = X_2Y_0^d\}$$

and $p : \Sigma_d \to \mathbb{P}^1$ is the canonical projection. There are four affine coordinates which cover Σ_d. We use two affine spaces $W_d^1, W_d^2 \subset \Sigma_d$ with coordinates (y_d, τ_d) and (z_d, τ_d) respectively where

$$y_d = X_2/X_0,\ z_d = X_0/X_2,\ \tau_d = Y_0/Y_1$$

and they are glued by the relation $y_d z_d = 1$. Putting $V_1 = \{(Y_0, Y_1) \in \mathbb{P}^1 \mid Y_1 \neq 0\}$, they satisfy $p^{-1}(V_1) = W_d^1 \cup W_d^2$.

We denote the fiber over $\tau_d = 0$ in Σ_d by F_∞ and the origin of the affine space W_d^i by $O_{i,d} := (0,0) \in W_d^i$. We put the affine line $F_\infty^\circ := F_\infty \setminus \Delta_{\infty,d} = F_\infty \cap W_d^2$.

§ 2.1. p-gonal curves

Let $B \subset \Sigma_d$ be a reduced curve such that B does not contain the exceptional section $\Delta_{\infty,d}$. If B intersects with a generic fiber at p points, then we call B *a generalized p-gonal curve.* A generalized p-gonal curve B is called *a p-gonal curve* if B disjoint from the exceptional section $\Delta_{\infty,d}$.

Let f_i be a defining equation of B on W_d^i and then we have the equality $f_1(y_d, \tau_d) = y_d^p f_2(z_d, \tau_d)$ on $W_d^1 \cap W_d^2$. Using affine coordinates $(z_d, \tau_d) \in W_d^2$, the local equation $f_2(z_d, \tau_d)$ is written as

$$f_2(z_d, \tau_d) = \sum_{i=0}^{p} b_i(\tau_d) z_d^i, \qquad \deg b_i(\tau_d) \leq d(p-i).$$

The exceptional section $\Delta_{\infty,d}$ is defined as $\{y_d = 0\}$ in the affine coordinates $(y_d, \tau_d) \in W_d^1$.

§ 2.2. Nagata transformations

Let P be a fixed point in $\Sigma_2 \setminus \Delta_{\infty,2}$ and let F be the fiber which passes through P. *A Nagata transformation* $N : \Sigma_2 \dashrightarrow \Sigma_1$ is a birational transformation which consists of the blowing-up at $P \notin \Delta_{\infty,2}$ and the blowing-down the strict transform F^* of F. We observe that the exceptional section $\Delta_{\infty,1}$ of Σ_1 is the image $N(\Delta_{\infty,2})$.

We express a Nagata transformation using local coordinates (z_2, τ_2) and (z_1, τ_1) assuming $P = O_{2,2} \in W_2^2$. Let $\mu_1 : \tilde{W}_2^2 \to W_2^2$ and $\mu_2 : \tilde{W}_1^1 \to W_1^1$ be blowing-ups centered at $O_{2,2}$ and $O_{1,1}$ respectively. There is an affine coordinate $\tilde{W}$ with coordinates (s, t) such that $\mu_1(s,t) = (t, ts)$ and $\mu_2(s,t) = (s, st)$. Note that $\{t = 0\}$ defines the exceptional curve of μ_1 and $\{s = 0\}$ defines the exceptional curve of μ_2. Then we have:

$$N(z_2, \tau_2) = (z_1, \tau_1) = \left(\frac{z_2}{\tau_2}, \tau_2\right).$$

Let B be a p-gonal curve in Σ_2 which is defined by $\{f_2(z_2, \tau_2) = 0\}$ in W_2^2. We consider the defining equation of the image of a p-gonal curve by a Nagata transformation. By the definition of a Nagata transformation, $B' := N(B) \subset \Sigma_1$ is defined as

$$B' : \ f_2'(z_1, \tau_1) = \tfrac{1}{\tau_1^M} f_2(z_1\tau_1, \tau_1) = 0 \tag{2.1}$$

where M is the multiplicity of B at P. As B is assumed to be p-gonal, $B' \cap \Delta_{\infty,1}$ is $\{O_{1,1}\}$. Thus B' is a generalized p-gonal curve.

§ 2.3. Contraction of p-gonal curves from Σ_2 to $\mathbb{P}^2$

We recall that a Hirzebruch surface Σ_1 is obtained as a blowing-up at an any point in $\mathbb{P}^2$. In this section, we consider the defining polynomial of a plane curve which is obtained as the image of the composition of a Nagata transformation and a blowing-up.

Let $B = \{f_2(z_2, \tau_2) = 0\}$ be a p-gonal curve in W_2^2 and let $B' = \{f_2'(z_1, \tau_1) = 0\} \subset W_1^2$ be the image of B by a Nagata transformation $N : \Sigma_2 \dashrightarrow \Sigma_1$ at $O_{2,2}$. Put m the intersection multiplicity of B' and $\Delta_{\infty,1}$ at $O_{1,1}$. Let U_1 be the affine coordinate chart $\mathbb{P}^2 \setminus \{X_1 = 0\}$ with the coordinate $(x_0, x_2) = (X_0/X_1, X_2/X_1)$. Let $\pi : \tilde{U}_1 \to U_1$ be a blowing-up at $(0,0) \in U_1$. We naturally identify $\tilde{U}_1$ with Σ_1 as follows: Let $\tilde{U}_{10}$ and $\tilde{U}_{11}$ be two affine coordinates of $\tilde{U}$ and let (s,t) be the affine coordinate of $\tilde{U}_{11}$. Then π is defined as $\pi(s,t) = (x_0, x_2) = (s, st)$ on $\tilde{U}_{11}$. We identify $\tilde{U}_{11}$ with W_1^1 as $(s,t) \mapsto (y_1, \tau_1)$.

By the definition of $\pi : \Sigma_1 \to U_1$ and the equality (2.1), the defining polynomial f of $C := (\pi \circ N)(B) \subset U_1$ as

$$f(x_0, x_2) = \frac{x_0^{M+m+p}}{x_2^M} f_2\left(\frac{x_2}{x_0^2}, \frac{x_2}{x_0}\right). \tag{2.2}$$

Indeed, let $f_1'(y_1, \tau_1)$ be the defining equation of B' in W_1^1 which is written as

$$f_1'(y_1, \tau_1) = y_1^p f_2'(1/y_1, \tau_1) = \frac{y_1^p}{\tau_1^M} f_2(\tau_1/y_1, \tau_1).$$

where we use (2.1) for the second equality. And f must satisfy $f(y_1, y_1\tau_1) = y_1^m f_1'(y_1, \tau_1)$. Using these equalities and $\pi(y_1, \tau_1) = (x_0, x_2) = (y_1, y_1\tau_1)$, we have the equality (2.2).

Next we consider singularities of B' and C. Assume that B satisfies the following conditions:

- B has an $A_{\ell-1} = B_{\ell,2}$ singularity at $O_{2,2} \in W_2^2$ and its tangent cone is transverse to the fiber $F_\infty = \{\tau_2 = 0\}$.
- B intersects transversely at $p-2$ distinct points with F_∞ outside of $O_{2,2} \in W_2^2$.

Under the above conditions, the intersection $B \cap (F_\infty \setminus \{O_{2,2}\})$ consists of distinct $p-2$ points and B' intersects with F_∞° so that

- If $\ell = 2$, then B' intersects transversely with F_∞° at two points.
- If $\ell = 3$, then B' is tangent to F_∞° with the intersection multiplicity 2.
- If $\ell > 3$, then B' has $A_{\ell-3} = B_{\ell-2,2}$ singularity.

Observation. If B is a trigonal curve ($p = 3$), then B' is smooth and intersects transversely with $\Delta_{\infty,1}$ at $O_{1,1}$. If p is greater than 3, then B' has $B_{p-2,p-2}$ singularity at $O_{1,1}$ and $C = \pi(B')$ has $B_{p-2,2(p-2)}$ singularity at $(0,0) \in U_1$.

Proof. The first assertion is obvious. Assume $p > 3$. The defining equation $f_1'(y_1, \tau_1)$ of B' in W_1^1 is written as:

$$f_1'(y_1, \tau_1) = c \prod_{i=1}^{p-2} (y_1 - \alpha_i \tau_1) + (\text{higher terms}), \quad c \neq 0,\ \alpha_i \neq \alpha_j\ (i \neq j).$$

Now we use the equality $f(x_0, x_2) = x_0^{p-2} f_1'(x_0, x_2/x_0)$ which is obtained from (2.2). Then we have

$$f(x_0, x_2) = x_0^{p-2} f_1'(x_0, x_2/x_0) = \prod_{i=1}^{p-2} (x_0^2 - \alpha_i x_2) + (\text{higher terms}).$$

Thus C has $B_{p-2,2(p-2)}$ singularity at $(0,0) \in U_1$. $\square$

§ 3. p-gonal curves of $(p, 2)$ torus type

Let B be a p-gonal curve in Σ_2. We say that B is *torus curve of type* $(p, 2)$ if the defining equation f_2 of B in the affine space $(W_2^2, (z_2, \tau_2))$ is written as

$$f_2(z_2, \tau_2) = k(z_2, \tau_2)^p - h(z_2, \tau_2)^2.$$

We assume further that

$$\begin{cases} k(z_2, \tau_2) = z_2 + b_2(\tau_2), \\ h(z_2, \tau_2) = b_{p-2}(\tau_2) z_2 + b_p(\tau_2), \end{cases} \qquad \deg b_i(\tau_2) = i.$$

§ 3.1. Singularities of $(p, 2)$ torus type

We consider curves $K := \{k = 0\}$ and $H := \{h = 0\}$ in W_2^2 where h and k are as above. Let $P \in B$ be a singular point. If $P \in K \cap H$, we call P an *inner singularity.* Otherwise P is called an *outer singularity.* We put $\Delta_1(\tau_2) := h(-b_2(\tau_2), \tau_2) = b_p(\tau_2) - b_{p-2}(\tau_2) b_2(\tau_2)$ and take an inner singular point $P \in K \cap H$. Then P is written as $(-b_2(s), s)$ for some $s \in \mathbb{C}$ with $\Delta_1(s) = 0$ and the multiplicity of $\Delta_1(\tau_2)$ at s, say ι, is equal to the intersection multiplicity of K and H at P.

By a similar argument as that in Lemma 1 in [1], we have the following.

Lemma 3.1. *Let B be the p-gonal curve as above in Σ_2. Suppose that s is a root of $\Delta_1(\tau)$ and let $P = (-b_q(s), s) \in B$ be an inner singular point with the intersection multiplicity ι. If $\Delta_2(s) \neq 0$, then B has $B_{p\iota,2} = A_{p\iota-1}$ singularity at P.*

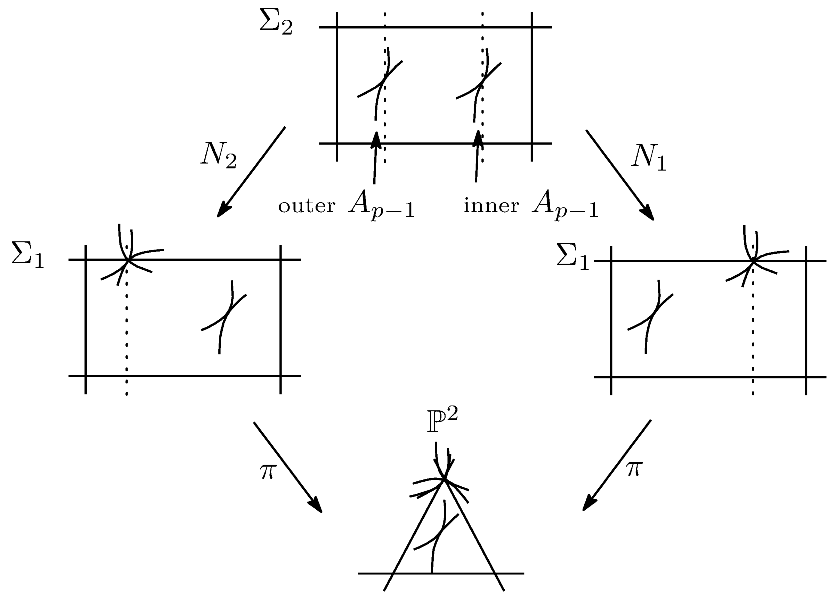
§ 4. Proof of Theorem 1.4

Let $B \subset \Sigma_2$ be a p-gonal curve of $(p, 2)$ torus type. As the degree of $\Delta_1(\tau_2)$ is p, B has pA_{p-1} inner singularities by Lemma 3.1. We may assume that B has an outer A_{p-1} singularity. For example, we take $b_2(\tau_2)$, $b_{p-2}(\tau_2)$ and $b_p(\tau_2)$ as

$$b_2(\tau_2) = 1 + \tau_2^2, \quad b_p(\tau_2) = 1 + \frac{p}{2}\tau_2^2 + \tau_2^p, \quad b_{p-2}(\tau_2) = \frac{p}{2} + p\tau_2^{p-2}.$$

Then $f_2 = k^p - h^2$ has an outer A_{p-1} singularity at $O_{2,2}$ and its tangent cone does not contain $\{\tau_2 = 0\}$. As $\Delta_1(\tau_2) = 1 - \frac{p}{2} - p\tau_2^{p-2} + (1-p)\tau_2^p$ and $p \geq 3$, K and H intersect transversely at distinct p points and $K \cap H \cap F_\infty = \emptyset$.

Let P be an inner A_{p-1} singular point and let Q be an outer A_{p-1} singular point of B. Let N_1 and N_2 be the Nagata transformations from Σ_2 to Σ_1 at P and Q respectively. We consider the defining polynomial of $C := (\pi \circ N_1)(B)$ and $D := (\pi \circ N_2)(B)$ where $\pi : \Sigma_1 \to U_1$ is the blowing-up at $(0, 0) \in U_1$.



§ 4.1. Construction of a visible degeneration

Hereafter we assume that K and H intersect transversely at p points. Assume that $P = O_{2,2}$ in the affine space W_2^2. Let $f_2(z_2, \tau_2) = k(z_2, \tau_2)^p - h(z_2, \tau_2)^2$ be the defining equation of B where

$$k(z_2,\tau_2)=z_2+b_2(\tau_2),\quad h(z_2,\tau_2)=b_{p-2}(\tau_2)z_2+b_p(\tau_2),\quad \deg b_i(\tau_2)=i.$$

As $k(0,0)=h(0,0)=0$, we can write $b_2(\tau_2)$ and $b_p(\tau_2)$ as

$$b_2(\tau_2)=\tau_2 b_1(\tau_2),\quad b_p(\tau_2)=\tau_2 b_{p-1}(\tau_2),\quad \deg b_i=i.$$

Let f be the defining polynomial of C and using (2.2), we have

$$f(x_0,x_2)=\frac{x_0^{2p}}{x_2^2}f_2\left(\frac{x_2}{x_0^2},\frac{x_2}{x_0}\right).$$

We calculate the above equation as the following:

$$\begin{aligned}
x_2^2 f(x_0,x_2) &= x_0^{2p}\left(k\left(\frac{x_2}{x_0^2},\frac{x_2}{x_0}\right)^p - h\left(\frac{x_2}{x_0^2},\frac{x_2}{x_0}\right)^2\right)\\
&= x_0^{2p}\left(\left(\frac{x_2}{x_0^2}+\frac{x_2}{x_0}b_1\left(\frac{x_2}{x_0}\right)\right)^p - \left(b_{p-2}\left(\frac{x_2}{x_0}\right)\frac{x_2}{x_0^2}+\frac{x_2}{x_0}b_{p-1}\left(\frac{x_2}{x_0}\right)\right)^2\right)\\
&= x_2^p\left(1+x_0 b_1\left(\frac{x_2}{x_0}\right)\right)^p - x_2^2\left(x_0^{p-2}b_{p-2}\left(\frac{x_2}{x_0}\right)+x_0^{p-1}b_{p-1}\left(\frac{x_2}{x_0}\right)\right)^2\\
&= f_1(x_0,x_2)^p x_2^p - f_{p-1}(x_0,x_2)^2 x_2^2.
\end{aligned}$$

and then where

$$f_1(x_0,x_2):=1+c_1(x_0,x_2),\quad f_{p-1}(x_0,x_2):=c_{p-2}(x_0,x_2)+c_{p-1}(x_0,x_2).$$

Note that $c_i(x_0,x_2):=x_0^i b_i(x_2/x_0)$ is a polynomial for $i=1$, $p-2$ and $p-1$. Hence we have

$$x_2^2 f(x_0,x_2)=(f_1(x_0,x_2)x_2)^p-(f_{p-1}(x_0,x_2)x_2)^2.$$

Thus the above equation shows that $C:=\{f=0\}$ is a visible line degeneration of order 2 of $(p,2)$ torus type.

§ 4.2. Construction of an invisible degeneration

Assume that $Q=O_{2,2}$ in the affine space W_2^2. Let $f_2(z_2,\tau_2)=k(z_2,\tau_2)^p-h(z_2,\tau_2)^2$ be the defining equation of B where

$$k(z_2,\tau_2)=z_2+b_2(\tau_2),\quad h(z_2,\tau_2)=b_{p-2}(\tau_2)z_2+b_p(\tau_2),\quad \deg b_i(\tau_2)=i.$$

Let g be the defining polynomial of D and using (2.2) in §2.3, we have

$$g(x_0,x_2)=\frac{x_0^{2p}}{x_2^2}f_2\left(\frac{x_2}{x_0^2},\frac{x_2}{x_0}\right).$$

We calculate the above equation:

$$\begin{aligned}
x_2^2 g(x_0,x_2) &= x_0^{2p}\left(k\left(\frac{x_2}{x_0^2},\frac{x_2}{x_0}\right)^p - h\left(\frac{x_2}{x_0^2},\frac{x_2}{x_0}\right)^2\right)\\
&= x_0^{2p}\left(\left(\frac{x_2}{x_0^2}+b_2\left(\frac{x_2}{x_0}\right)\right)^p - \left(b_{p-2}\left(\frac{x_2}{x_0}\right)\frac{x_2}{x_0^2}+b_p\left(\frac{x_2}{x_0}\right)\right)^2\right)\\
&= \left(x_2+x_0^2 b_2\left(\frac{x_2}{x_0}\right)\right)^p - \left(x_0^{p-2}b_{p-2}\left(\frac{x_2}{x_0}\right)x_2 + x_0^p b_p\left(\frac{x_2}{x_0}\right)\right)^2\\
&= g_2(x_0,x_2)^p - g_p(x_0,x_2)^2
\end{aligned}$$

where the polynomials g_2 and g_p are defined as

$$g_2(x_0,x_2) := x_2 + d_2(x_0,x_2) \qquad g_p(x_0,x_2) := d_{p-2}(x_0,x_2)x_2 + d_p(x_0,x_2)$$

where $d_i(x_0,x_2) := x_0^i b_i\left(x_2/x_0\right)$ for $i = 2$, $p-2$ and p. Thus the above equation shows that $D := \{g = 0\}$ is an invisible line degeneration of order 2 of $(p,2)$ torus type:

$$x_2^2 g(x_0,x_2) = g_2(x_0,x_2)^p - g_p(x_0,x_2)^2.$$

§ 4.3. Singularities of constructed curves

We consider singularities of C and D. By our constructions and the argument in §2.3, we have the following:

- $\operatorname{Sing} C$ and $\operatorname{Sing} D$ are the same:

$$\operatorname{Sing} C = \operatorname{Sing} D = \{pA_{p-1}, A_{p-3}, B_{p-2,2(p-2)}\}.$$

- C has $(p-1)A_{p-1}$ and A_{p-3} singularities as inner and one A_{p-1} singularity as outer.
- D has pA_{p-1} singularities as are inner and A_{p-3} singularity as outer.

Thus we have a pair (C,D) which satisfy the statement of the first part of Theorem 1.4.

§ 4.4. The case p is even

In this section, we suppose that p is even. We will show that the pair (C,D) is a weak Zariski pair. Recall that the defining polynomials f and g of C and D satisfy

$$\begin{aligned}
f(x_0,x_2) &= f_1(x_0,x_2)^p x_2^{p-2} - f_{p-1}(x_0,x_2)^2\\
x_2^2\, g(x_0,x_2) &= g_2(x_0,x_2)^p - g_p(x_0,x_2)^2.
\end{aligned}$$

As p is even, C is decomposed as $C = C_1 \cup C_2$ where $\deg C_1 = \deg C_2 = p-1$.

Lemma 4.1. *D is decomposed as $D = D_{p-2} \cup D_p$ where $\deg D_{p-2} = p-2$ and $D_p = p$.*

Proof. Put $p = 2s$. Let $B = \{f_2 = k^{2s} - h^2 = 0\}$ be a $2s$-gonal curve in Σ_2 and let $\pi \circ N_2 : \Sigma_2 \dashrightarrow \mathbb{P}^2$ be a birational map which are considered in the proof of Theorem 1.4. Then we can factorize $f_2(z_2, \tau_2)$ as

$$\begin{aligned} f_2(z_2,\tau_2) &= (k(z_2,\tau_2)^s - h(z_2,\tau_2))(k(z_2,\tau_2)^s + h(z_2,\tau_2)) \\ &= k_1(z_2,\tau_2)\, k_2(z_2,\tau_2) \end{aligned}$$

where

$$k_1(z_2,\tau_2) = k(z_2,\tau_2)^s - h(z_2,\tau_2), \quad k_2(z_2,\tau_2) = k(z_2,\tau_2)^s + h(z_2,\tau_2).$$

As we assumed that $O_{2,2}$ is an outer singular point of B, we may assume that $O_{2,2}$ is in $\{k_1 = 0\} \setminus \{k_2 = 0\}$. Then, using (2.2) in §2.3, the defining polynomial w_1 of $\pi \circ N_2(\{k_1 = 0\})$ is given by

$$w_1(x_0,x_2) = \frac{x_0^{2s}}{x_0^2}\, k_1\left(\frac{x_2}{x_0^2}, \frac{x_2}{x_0}\right) = \frac{1}{x_0^2}\left(g_2(x_0,x_2)^s - g_p(x_0,x_2)\right).$$

As w_1, g_2 and g_p are polynomials and $\deg g_2 = 2$ and $\deg g_p = p$, the degree w_1 must be $p-2$. Note that $\{w_1 = 0\}$ has A_{p-3} singularity. As g is obtained as

$$g(x_0,x_2) = \frac{x_0^{4s}}{x_2^2} f_2\left(\frac{x_2}{x_0^2}, \frac{x_2}{x_0}\right) = w_1(x_0,x_2) w_2(x_0,x_2)$$

where $w_2 := x_0^{2s} k_2$. As $\deg g = 2p-2$ and $\deg w_1 = p-2$, the degree w_2 must be p. □

Now we consider the irreducibility of C_1 and C_2. Let $P_1, \dots, P_{p-1}$, Q, R and O^* be the singular points of C such that

$$\begin{aligned} &(C,P_i) \sim A_{p-1}, \quad i = 1,\dots,p-1, \\ &(C,Q) \sim A_{p-3}, \quad (C,R) \sim A_{p-1}, \quad (C,O^*) \sim B_{p-2,2(p-2)} \end{aligned}$$

and P_i and Q are inner singularities and R is an outer singular point of C. As P_i and Q are inner, they are in $\{f_1 = 0\} \cap \{f_{p-1} = 0\}$. Hence P_i and Q are also in $C_1 \cap C_2$. Note that C_1 and C_2 are smooth at P_i and Q. As R is the outer singular point, we may assume that $R \in C_1 \setminus C_2$.

By the form of the defining polynomials of C_1 and C_2, both curves have $B_{\frac{p-2}{2},p-2}$ singularity at O^*. Note that C_1 and C_2 have no other singularities.

Now we assume that C_1 is reducible as $C_1 = E_a \cup E_b$ where $\deg E_i = i$ and $a \leq b$. Assume that $p > 4$. As O^* and R are singular points of C_1, the intersection $E_a \cap E_b$ is one of the following:

$$\{O^*\}, \quad \{R\}, \quad \{O^*, R\}.$$

We consider the cases $E_a \cap E_b = \{O^*\}$ or $\{O^*, R\}$. Let n and m be positive integers such that $(E_a, O^*) \sim B_{n,2n}$ and $(E_b, O^*) \sim B_{m,2m}$. Positive integers (a, b, n, m) must satisfy the following equations:

(1) $a + b = p - 1$.

(2) $2m + 2n = p - 2$.

(3) $a \geq 2n$, $b \geq 2m$.

(4) If $E_a \cap E_b = \{O^*\}$, then $ab = 2mn$.

(5) If $E_a \cap E_b = \{O^*, R\}$, then $ab = \frac{p}{2} + 2mn$.

Equalities (4) and (5) are obtained by Bézout theorem. By simple calculations, there are no positive integers (a, b, n, m) which satisfy the above equations. Hence if $O^* \in E_a \cap E_b$, then C_1 is irreducible. By the same argument, we can show that C_2 is irreducible because C_2 has only a $B_{\frac{p-2}{2}, p-2}$ singularity.

Now we consider the case $E_a \cap E_b = \{R\}$. Then E_a and E_b are smooth at R with $I(E_a, E_b; R) = \frac{p}{2}$. As $E_a \cap E_b = \{R\}$, we have $ab = \frac{p}{2}$ by Bézout theorem. The equations $a + b = p - 1$ and $ab = \frac{p}{2}$ are satisfied for the case $(p, a, b) = (4, 1, 2)$ only. Hence if $p > 4$, then C_1 and C_2 are irreducible. Therefore the pair (C, D) is a weak Zariski pair.

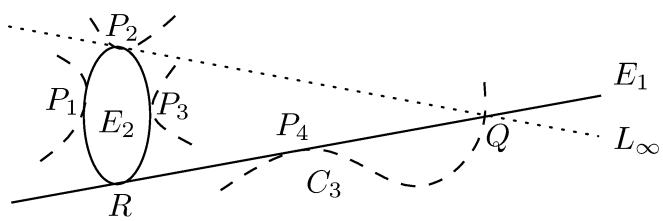
§ 4.5. The case $p = 4$

We suppose that $p = 4$. Then $\deg C = \deg D = 6$ and their singularities are

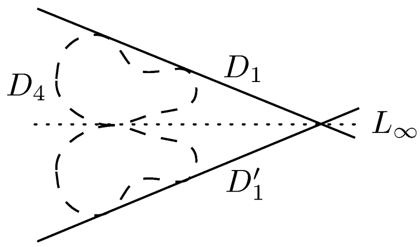
$$\operatorname{Sing} C = \operatorname{Sing} D = \{5A_3, A_1\}.$$

By the above argument, C is decomposed as $E_1 \cup E_2 \cup C_2$ and C_2 is a smooth cubic. Their intersection points and intersection multiplicities of these curves are the following:

$$E_1 \cap E_2 = \{R\}, \quad E_2 \cap C_3 = \{P_1, P_2, P_3\}, \quad E_1 \cap C_3 = \{P_4, Q\}$$
$$I(E_1, E_2; R) = 2, \quad I(E_1, C_3; Q) = 1, \quad I(E_i, C_3; P_k) = 2, \; k = 1, \dots, 4.$$



On the other hand, D is also decomposed as $D_4 \cup D_1 \cup D_1'$ where $\deg D_4 = 4$ and $\deg D_1 = \deg D_1' = 1$. Indeed, outer A_1 singularity must be in D_2. Hence D_2 consists of two distinct lines. Thus D is decomposed as $D_4 \cup D_1 \cup D_1'$. Note that D_1 and D_1' are bitangent lines of D_4.



Thus C and D have different irreducible decompositions. Hence the pair (C, D) is a weak Zariski pair.

§ 4.6. Observation for the case $p = 3$

By our construction, C and D are 3-cuspidal quartics. As we mentioned in the introduction, each curve has both torus decompositions. Moreover it is known that the moduli space of 3-cuspidal quartic is irreducible and hence C and D are in the same moduli space.

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References

[1] B. Audoubert, T. C. Nguyen, and M. Oka. On Alexander polynomials of torus curves. *J. Math. Soc. Japan*, 57(4):935–957, 2005.
[2] J. Cogolludo and A. Libgober. Mordell-Weil groups of elliptic threefolds and the Alexander module of plane curves. *Preprint available at arXiv:1008.2018v2 [math.AG].*
[3] A. Degtyarev. Dihedral coverings of trigonal curves. *Indiana Math. J, to appear.*
[4] A. Degtyarev. Plane sextics via dessins d'enfants. *Geom. Topol.*, 14(1):393–433, 2010.
[5] C. Eyral and M. Oka. π_1-equivalent weak Zariski pairs. *Tokyo J. Math.*, 28(2):499–526, 2005.
[6] M. Kawashima and K. Yoshizaki. On (2, 3) torus of decompositions of QL-configurations. *SUT J. Math.*, 46(1):93–117, 2010.
[7] V. S. Kulikov. On plane algebraic curves of positive Albanese dimension. *Izv. Math*, 59(6):1173–1192, 1995.
[8] M. Oka. Tangential Alexander polynomials and non-reduced degeneration. *Singularities in geometry and topology*, pages 669–704, 2007.
[9] D. T. Pho. Classification of singularities on torus curves of type (2, 3). *Kodai Math. J.*, 24(1):259–284, 2001.