

Height functions on Whitney umbrellas

By

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Abstract

We study the singularities of the members of the family of height functions on Whitney umbrellas, which is also known as cross-caps, and show that the family of the height functions is a versal unfolding. Moreover, we study local intersections of a Whitney umbrella with a hyperplane through its singular point.

§ 1. Introduction

H. Whitney [22] investigated singularities which are unavoidable by small perturbation, and found the singularity type, which is called Whitney umbrellas (or cross-caps). This singularity type is very important, since it is the only singularity of a map of a surface to 3-dimensional Euclidean space which is stable under small deformations. It is natural to investigate Whitney umbrellas as a subject of differential geometry, and there are several articles [3, 5, 6, 7, 8, 9, 14, 15, 16, 18, 20] in this direction.

In the authors' previous work [5], we introduce some differential geometric ingredients (principal curvatures, ridge, sub-parabolic points, etc.) for Whitney umbrellas and investigate singularities and versal unfoldings of distance squared functions on Whitney umbrellas in terms of these ingredients. To develop the differential geometric ingredients, we consider the double oriented blowing-up ([10, example (a) in p. 221])

$$\tilde{\Pi} : \mathbb{R} \times S^1 \rightarrow \mathbb{R}^2, \quad (r, \theta) \mapsto (r \cos \theta, r \sin \theta),$$

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and a map, which is usually called a blow-up,

$$\Pi : \mathcal{M} \rightarrow \mathbb{R}^2, \quad [(r, \theta)] \rightarrow (r \cos \theta, r \sin \theta),$$

where \mathcal{M} denotes the quotient space of $\mathbb{R} \times S^1$ with identification $(r, \theta) \sim (-r, \pi + \theta)$. We remark that \mathcal{M} is topologically a Möbius strip. There is a well-defined unit normal vector via $\tilde{\Pi}$, thus we obtain the information on the asymptotic behavior of the principal curvatures and the principal directions. Moreover, we also obtain the configuration of the parabolic line (or ridge lines, or sub-parabolic lines) on \mathcal{M} near the exceptional set $E = \Pi^{-1}(0, 0)$.

In this paper, we investigate the relationships between singularities of height functions on Whitney umbrellas and these differential geometric ingredients for Whitney umbrellas, and the versality of the family of height functions. We also study intersections of a Whitney umbrella and hyperplanes through its singular point. Height functions are a fundamental tool for the study of the differential geometry of submanifolds. Several authors have studied the geometrical properties of submanifolds in Euclidean space by analyzing the singularities of height functions on submanifolds (see, for example, [2, 12, 17]).

The intersection of a surface and a hyperplane is intimately related to the height function on the surface. In particular, for a regular surface in \mathbb{R}^3 , the contact between the surface and a hyperplane is measured by the singularity type of the height function on the surface. For instance, when the contact of a point p is A_1 type (i.e., the height function has an A_1 singularity), the intersection of the surface and the hyperplane at p , which is the tangent plane at p , is locally an isolated point or a pair of transverse curves (see, for example, [2]).

In Section 2, we recall the differential geometric ingredients developed in [5]. In Section 3, we investigate singularities and versal unfoldings of height functions on Whitney umbrella in terms of the differential geometry which is discussed in Section 2. Theorem 3.1 shows that the relationship of the singularities of the height functions and the differential geometric ingredients for Whitney umbrellas. In Section 4, we study the local intersections of a Whitney umbrella and hyperplanes through the singular point. To detect local singularity types of the intersections of a surface with hyperplanes, we must investigate how the zero set of the height function on the surface is mapped to the surface. For a regular surface, it is trivial, since the contact of the surface between hyperplanes is measured by the singularities of the height function on the surface. On the other hand, for a Whitney umbrella, we must investigate how the zero set of the height function on the Whitney umbrella contacts with the double point locus \mathcal{D} in the parameter space. Theorem 4.2 shows that if the hyperplane \mathcal{P} through the Whitney umbrella singularity does not contain the tangent line to \mathcal{D} on the surface then the contact of the Whitney umbrella and \mathcal{P} at the singularity is measured by the type of singularities

of the height function, and that if \mathcal{P} contains the tangent to \mathcal{D} on the surface then the contact of the Whitney umbrella and \mathcal{P} is measured by the type of singularities of the height function and the contact between the zero set of the height function and \mathcal{D} in parameter space. Hence, Theorem 4.2 is considered to be an analogous theorem of Montaldi [13] for Whitney umbrellas.

§ 2. The differential geometry for Whitney umbrella

In this section, we recall the differential geometry of Whitney umbrellas, which is obtained in [5].

A smooth map germ $g : (\mathbb{R}^2, \mathbf{0}) \rightarrow (\mathbb{R}^3, \mathbf{0})$ is a *Whitney umbrella* if g is \mathcal{A} -equivalent to the map germ $(\mathbb{R}^2, \mathbf{0}) \rightarrow (\mathbb{R}^3, \mathbf{0})$ defined by $(u, v) \mapsto (u, uv, v^2)$. Here, map germs $f_1, f_2 : (\mathbb{R}^2, \mathbf{0}) \rightarrow (\mathbb{R}^3, \mathbf{0})$ are \mathcal{A} -equivalent if there exist diffeomorphism map germs $\phi_1 : (\mathbb{R}^2, \mathbf{0}) \rightarrow (\mathbb{R}^2, \mathbf{0})$ and $\phi_2 : (\mathbb{R}^3, \mathbf{0}) \rightarrow (\mathbb{R}^3, \mathbf{0})$ such that $f_1 \circ \phi_1 = \phi_2 \circ f_2$.

For a regular surface in \mathbb{R}^3 , at any point on the surface we can take the z -axis as the normal direction. After suitable rotation if necessary, the surface can be locally expressed in Monge form:

$$(u, v) \mapsto (u, v, \frac{1}{2}(k_1u^2 + k_2v^2) + O(u, v)^3).$$

For a Whitney umbrella, we have the similar normal form to Monge form.

Proposition 2.1. *Let $g : (\mathbb{R}^2, \mathbf{0}) \rightarrow (\mathbb{R}^3, \mathbf{0})$ be a Whitney umbrella. Then there are $T \in \text{SO}(3)$ and a diffeomorphism $\phi : (\mathbb{R}^2, \mathbf{0}) \rightarrow (\mathbb{R}^2, \mathbf{0})$ so that*

$$(2.1) \quad T \circ g \circ \phi(u, v) = \left(u, uv + \sum_{i=3}^k \frac{b_i}{i!} v^i + O(u, v)^{k+1}, \sum_{m=2}^k A_m(u, v) + O(u, v)^{k+1} \right),$$

where

$$A_m(u, v) = \sum_{i+j=m} \frac{a_{ij}}{i!j!} u^i v^j \quad (a_{02} \neq 0).$$

This result was first obtained in [20]. A proof can also be found in [5]. Without loss of generality, we may assume that $a_{02} > 0$ (see [9, Introduction]). The form (2.1) with such an assumption is called the *normal form of Whitney umbrella*.

We suppose that $g : (\mathbb{R}^2, \mathbf{0}) \rightarrow (\mathbb{R}^3, \mathbf{0})$ is given in the normal form of Whitney umbrella. We shall introduce some differential geometric ingredients (the unit normal vector, principal curvatures, and principal vectors etc.) for Whitney umbrellas (see [5] for details).

Unit normal vectors. We denote the unit normal vector of g by $\mathbf{n}(u, v)$. The unit normal vector $\tilde{\mathbf{n}} = \mathbf{n} \circ \tilde{\Pi}$ in the coordinates (r, θ) is expressed as follows:

$$\tilde{\mathbf{n}}(r, \theta) = \frac{(0, -a_{11} \cos \theta - a_{02} \sin \theta, \cos \theta)}{\mathcal{A}(\theta)} + O(r),$$

where $\mathcal{A}(\theta) = \sqrt{\cos^2 \theta + (a_{11} \cos \theta + a_{02} \sin \theta)^2}$.

Principal curvatures and principal directions. We denote by κ_1 and κ_2 the principal curvatures. We also denote by \mathbf{v}_1 and \mathbf{v}_2 the principal vectors associated with κ_1 and κ_2 , respectively. The principal curvatures $\tilde{\kappa}_i = \kappa_i \circ \tilde{\Pi}$ in the coordinates (r, θ) are expressed as follows:

$$(2.2) \quad \tilde{\kappa}_1(r, \theta) = \frac{(a_{20} \cos^2 \theta - a_{02} \sin^2 \theta) \sec \theta}{\mathcal{A}(\theta)} + O(r),$$

$$(2.3) \quad \tilde{\kappa}_2(r, \theta) = \frac{1}{r^2} \left(\frac{a_{02} \cos \theta}{\mathcal{A}(\theta)^3} + O(r) \right).$$

The (unit) principal vectors $\tilde{\mathbf{v}}_i$ in the coordinates (r, θ) are expressed as follows:

$$\begin{aligned} \tilde{\mathbf{v}}_1(r, \theta) &= (\sec \theta + O(r)) \frac{\partial}{\partial r} + \left(\frac{-2\tilde{A}_{3v}(\theta) + b_3 \tilde{A}_{2v}(\theta) \sin \theta \tan \theta}{2a_{02}} + O(r) \right) \frac{\partial}{\partial \theta}, \\ \tilde{\mathbf{v}}_2(r, \theta) &= \frac{1}{r^2} \left[\left(\frac{\sin \theta}{\mathcal{A}(\theta)} r + O(r^2) \right) \frac{\partial}{\partial r} + \left(\frac{\cos \theta}{\mathcal{A}(\theta)} + O(r) \right) \frac{\partial}{\partial \theta} \right]. \end{aligned}$$

Here, $\tilde{A}_{2v}(\theta) = A_{2v}|_{(u,v)=(\cos \theta, \sin \theta)}$, $\tilde{A}_{3v}(\theta) = A_{3v}|_{(u,v)=(\cos \theta, \sin \theta)}$ and so on.

Gaussian curvature. Since the Gaussian curvature K is the product of the principal curvatures, it does not depend on the choice of the unit normal vector. From (2.2) and (2.3), the Gaussian curvature \tilde{K} of g in the coordinates (r, θ) is expressed as follows:

$$(2.4) \quad \tilde{K}(r, \theta) = K \circ \Pi(r, \theta) = \frac{1}{r^2} \left(\frac{a_{02}(a_{20} \cos^2 \theta - a_{02} \sin^2 \theta)}{\mathcal{A}(\theta)^4} + O(r) \right).$$

By this expression, we say that a point $(0, \theta_0)$ on \mathcal{M} is *elliptic*, *hyperbolic*, or *parabolic point over Whitney umbrella* if $r^2 \tilde{K}(0, \theta_0)$ is positive, negative, or zero, respectively. We often omit the phrase “over Whitney umbrella” if no confusion is possible from the context.

Parabolic lines. It is shown in [20] that Whitney umbrellas are generically classified into two types: the *elliptic Whitney umbrella*, whose parabolic line, in the parameter space, has an A_1^- singularity (locally two transversally intersecting curves); and the

hyperbolic Whitney umbrella, whose parabolic line has an A_1^+ singularity (locally an isolated point). At the transition between two types, there is a Whitney umbrella whose parabolic line has an A_2 singularity (locally a cusp). Such a Whitney umbrella is called a *parabolic Whitney umbrella* (see [14, 15]). The parabolic line of g in \mathcal{M} is expressed as follows:

$$0 = a_{02}(a_{20} \cos^2 \theta - a_{02} \sin^2 \theta) + \left[(a_{30}a_{02} + a_{12}a_{20}) \cos^3 \theta + (2a_{21}a_{02} + a_{03}a_{20} - a_{20}a_{11}b_3) \cos^2 \theta \sin \theta - a_{02}(a_{03} - a_{11}b_3) \sin^3 \theta \right] r + O(r^2).$$

From this form, we obtain the following proposition providing criteria for the type of Whitney umbrella.

Proposition 2.2.

- (1) *The map g is an elliptic (resp. hyperbolic) Whitney umbrella if and only if $a_{20} > 0$ (resp. < 0).*
- (2) *The map g is a parabolic Whitney umbrella if and only if $a_{20} = 0$ and $a_{30} \neq 0$.*

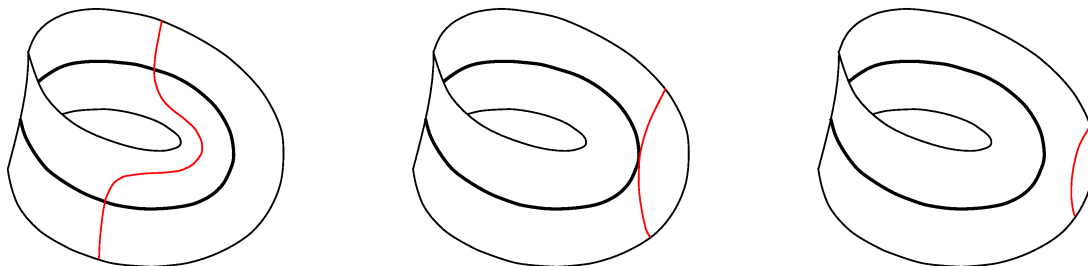


Figure 1. Parabolic lines on \mathcal{M} : elliptic Whitney umbrella (left), parabolic Whitney umbrella (center), hyperbolic Whitney umbrella (right).

Ridge and sub-parabolic points. Ridge points of regular surfaces in \mathbb{R}^3 were first studied by Porteous [17] in terms of singularities of distance squared functions. The ridge points are points where one principal curvature has an extremal value along lines of the same principal curvature. The ridge points are also correspond to singularities on the focal surface.

The directional derivative of $\widetilde{\kappa}_1$ in the direction $\widetilde{\mathbf{v}}_1$ is expressed as follows:

$$\widetilde{\mathbf{v}}_1 \widetilde{\kappa}_1(r, \theta) = \frac{\left(6\widetilde{A}_3(\theta) \cos \theta - b_3 \widetilde{A}_{2v}(\theta) \sin^3 \theta \right) \sec^3 \theta}{\mathcal{A}(\theta)} + O(r),$$

We say that a point (r_0, θ_0) is a *ridge point relative to $\widetilde{\mathbf{v}}_1$* if $\widetilde{\mathbf{v}}_1 \widetilde{\kappa}_1(r_0, \theta_0) = 0$. In particular, if the ridge point (r_0, θ_0) is in the exceptional set $E = \Pi^{-1}(0, 0)$ (that is,

$r_0 = 0$), the point is said to be the *ridge point over Whitney umbrella*. Moreover, a point (r_0, θ_0) is a n -th order ridge point relative to $\widetilde{\mathbf{v}}_1$ if $\widetilde{\mathbf{v}}_1^{(k)} \widetilde{\kappa}_1(r_0, \theta_0) = 0$ ($1 \leq k \leq n$) and $\widetilde{\mathbf{v}}_1^{(n+1)} \widetilde{\kappa}_1(r_0, \theta_0) \neq 0$, where $\widetilde{\mathbf{v}}_1^{(k)} \widetilde{\kappa}_1$ denotes the k -times directional derivative of $\widetilde{\kappa}_1$ in $\widetilde{\mathbf{v}}_1$. The twice directional derivative of $\widetilde{\kappa}_1$ in $\widetilde{\mathbf{v}}_1$ is expressed as follows:

$$\widetilde{\mathbf{v}}_1^2 \widetilde{\kappa}_1(r, \theta) = \frac{\left(\widetilde{A}_{3vv}(\theta) \cos \theta - b_3 \widetilde{A}_{2v} \right) \widetilde{\mathbf{v}}_1 \widetilde{\kappa}_1(0, \theta) \sec^2 \theta}{a_{02}} + \frac{\Gamma(\theta) \sec^5 \theta}{a_{02} \mathcal{A}(\theta)} + O(r)$$

where

$$\begin{aligned} \Gamma(\theta) = & 24a_{02} \widetilde{A}_4(\theta) \cos^2 \theta - 12a_{02} \left(\widetilde{A}_2(\theta)^2 + \cos^2 \theta \sin^2 \theta \right) (a_{20} \cos^2 \theta - a_{02} \sin^2 \theta) \\ & - 3 \left(2\widetilde{A}_{3v}(\theta) \cos \theta - b_3 \widetilde{A}_{2v}(\theta) \sin^2 \theta \right)^2 - a_{02} b_4 \widetilde{A}_{2v}(\theta) \cos \theta \sin^4 \theta. \end{aligned}$$

Sub-parabolic points of regular surfaces in \mathbb{R}^3 were first studied in terms of folding maps in [4] and [21]. The sub-parabolic points are points where one principal curvature has an extremal value along lines of the other principal curvature. The sub-parabolic points are also correspond to parabolic points on the focal surface.

The directional derivative of $\widetilde{\kappa}_1$ in the direction $\widetilde{\mathbf{v}}_2$ is expressed as follows:

$$(2.5) \quad \widetilde{\mathbf{v}}_2 \widetilde{\kappa}_1(r, \theta) = \frac{1}{r^2} \left(-\frac{2a_{02} \left(\widetilde{A}_2(\theta) \widetilde{A}_{2v}(\theta) + \cos^2 \theta \sin \theta \right)}{\mathcal{A}(\theta)^4} + O(r) \right).$$

We say that a point (r_0, θ_0) is a *sub-parabolic point relative to $\widetilde{\mathbf{v}}_2$* if $r^2 \widetilde{\mathbf{v}}_2 \widetilde{\kappa}_1(r_0, \theta_0) = 0$. In particular, if the sub-parabolic point (r_0, θ_0) is in the exceptional set E , the point is said to be the *sub-parabolic point over Whitney umbrella*.

Remark. The map g is a parabolic Whitney umbrella if and only if g has a unique parabolic point $(r, \theta) = (0, 0)$ which is not a ridge point over Whitney umbrella. In this case, $(0, 0)$ is a sub-parabolic point relative to $\widetilde{\mathbf{v}}_2$ over Whitney umbrella (cf. Proposition 2.2).

Focal conics. In [5], we introduce the focal conic as the counterpart of the focal point. We define a family of distance squared functions $D : (\mathbb{R}^2, \mathbf{0}) \times \mathbb{R}^3 \rightarrow \mathbb{R}$ on a Whitney umbrella g by $D(u, v, x, y, z) = \|(x, y, z) - g(u, v)\|^2$. It is shown in [5] that the function D has an A_k singularity ($k \geq 2$) at $(0, 0)$ for (x, y, z) if and only if (x, y, z) is on a conic in the normal plane, where the *normal plane* is the plane passing through $g(0, 0)$ perpendicular to the image of $dg_{(0,0)}$. Such a conic is called a *focal conic*. The focal conic of the normal form of Whitney umbrella is given by the equation

$$y^2 + 2a_{11}yz - (a_{20}a_{02} - a_{11}^2)z^2 + a_{02}z = 0.$$

This implies that the focal conic is an ellipse, hyperbola, or parabola if and only if $a_{20} < 0$, $a_{20} > 0$, or $a_{20} = 0$, respectively. Hence, we obtain the following criteria for the (generic) type of Whitney umbrellas as corollary of Proposition 2.2.

Corollary 2.3. *The map g is an elliptic (resp. hyperbolic) Whitney umbrella if and only if its focal conic is a hyperbola (resp. ellipse).*

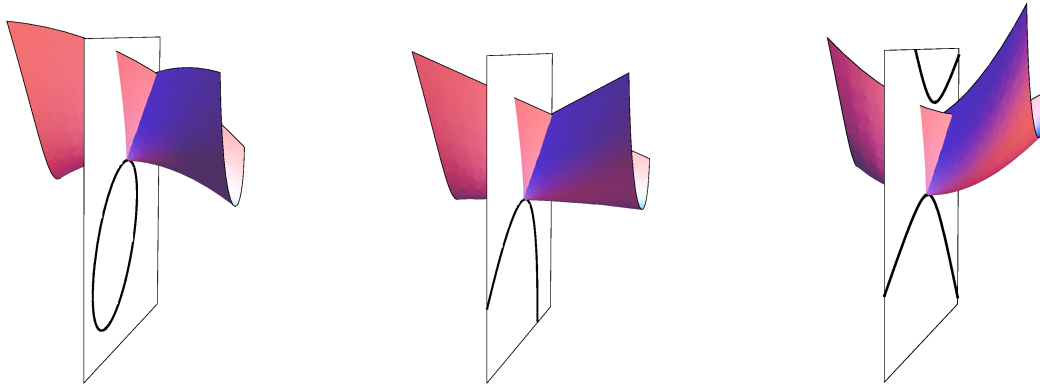


Figure 2. The three types of focal conics: focal ellipse (left), focal parabola (center), focal hyperbola (right).

§ 3. Height functions on a Whitney umbrella

In this section, we investigate singularities and \mathcal{R}^+ -versal unfoldings of height functions on Whitney umbrellas.

Firstly, we recall the definition of unfoldings and its \mathcal{R}^+ -versality. See [1, Section 8 and 19] for details. See also [19, Section 3]. Let $f : (\mathbb{R}^n, \mathbf{0}) \rightarrow (\mathbb{R}, \mathbf{0})$ be a smooth function germ. We say that a smooth function germ $F : (\mathbb{R}^n \times \mathbb{R}^r, \mathbf{0}) \rightarrow (\mathbb{R}, \mathbf{0})$ is an r -parameter unfolding of f if $F|_{\mathbb{R}^n \times \{\mathbf{0}\}} = f$. Let $F : (\mathbb{R}^n \times \mathbb{R}^r, \mathbf{0}) \rightarrow (\mathbb{R}, \mathbf{0})$ be an unfolding of f . The unfolding F is an \mathcal{R}^+ -versal unfolding of f if for every unfolding $G : (\mathbb{R}^n \times \mathbb{R}^s, \mathbf{0}) \rightarrow (\mathbb{R}, \mathbf{0})$ of f there exist a smooth map germ $\Phi : (\mathbb{R}^n \times \mathbb{R}^s, \mathbf{0}) \rightarrow (\mathbb{R}^n, \mathbf{0})$ with $\Phi(\mathbf{u}, \mathbf{0}) = \mathbf{u}$, a smooth map germ $\varphi : (\mathbb{R}^s, \mathbf{0}) \rightarrow (\mathbb{R}^r, \mathbf{0})$, and a smooth function germ $\lambda : (\mathbb{R}^s, \mathbf{0}) \rightarrow (\mathbb{R}, \mathbf{0})$ such that $G(\mathbf{u}, \mathbf{x}) = F(\Phi(\mathbf{u}, \mathbf{x}), \varphi(\mathbf{x})) + \lambda(\mathbf{x})$, where $\mathbf{u} \in \mathbb{R}^n$ and $\mathbf{x} \in \mathbb{R}^s$. The set of smooth function germs $(\mathbb{R}^n, \mathbf{0}) \rightarrow \mathbb{R}$ is denoted by \mathcal{E}_n . The set \mathcal{E}_n is the local ring with the unique maximal ideal $\mathcal{M}_n = \{f \in \mathcal{E}_n; f(\mathbf{0}) = 0\}$. A smooth function germ f is said to be *right- k -determined* if all function germs with a given k -jet are right-equivalent. If $f \in \mathcal{M}_n$ is right- k -determined, then an unfolding $F : (\mathbb{R}^n \times \mathbb{R}^r, \mathbf{0}) \rightarrow (\mathbb{R}, \mathbf{0})$ of f is an \mathcal{R}^+ -versal unfolding if and only if

$$\mathcal{E}_n = \left\langle \frac{\partial f}{\partial u_1}, \dots, \frac{\partial f}{\partial u_n} \right\rangle_{\mathcal{E}_n} + \left\langle \frac{\partial F}{\partial x_1} \Big|_{\mathbb{R}^n \times \{\mathbf{0}\}}, \dots, \frac{\partial F}{\partial x_r} \Big|_{\mathbb{R}^n \times \{\mathbf{0}\}} \right\rangle_{\mathbb{R}} + \langle 1 \rangle_{\mathbb{R}} + \langle u, v \rangle_{\mathcal{E}_n}^{k+1}$$

(cf. [11, p. 149]).

Let $g : (\mathbb{R}^2, \mathbf{0}) \rightarrow (\mathbb{R}^3, \mathbf{0})$ be a smooth map which defines a surface in \mathbb{R}^3 . We set a family of functions

$$H : (\mathbb{R}^2 \times S^2, (\mathbf{0}, \mathbf{w}_0)) \rightarrow \mathbb{R}$$

by $H(u, v, \mathbf{w}) = \langle g(u, v), \mathbf{w} \rangle$, where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product in \mathbb{R}^3 . We define $h(u, v) = H(u, v, \mathbf{w}_0)$. The function h is the *height function* on g in the direction \mathbf{w}_0 and the family H is a 2-parameter unfolding of h .

The following theorem is our main theorem.

Theorem 3.1. *Let $g : (\mathbb{R}^2, \mathbf{0}) \rightarrow (\mathbb{R}^3, \mathbf{0})$ be given in the normal form of Whitney umbrella, and let h be the height function on g in the direction $\mathbf{w}_0 = \pm \tilde{\mathbf{n}}(0, \theta_0)$, where $\theta_0 \in (-\pi/2, \pi/2]$. Then h has a singularity at $(0, 0)$. Moreover, we have the following assertions.*

- (1) *The height function h has an A_1^+ (resp. A_1^-) singularity at $(0, 0)$ if and only if $(0, \theta_0)$ is an elliptic (resp. hyperbolic) point over Whitney umbrella. In this case, H is an \mathcal{R}^+ -versal unfolding of h .*
- (2) *The height function h has an A_2 singularity at $(0, 0)$ if and only if $(0, \theta_0)$ is a parabolic point and not a ridge point relative to $\tilde{\mathbf{v}}_1$ over Whitney umbrella. In this case, H is an \mathcal{R}^+ -versal unfolding of h .*
- (3) *The height function h has an A_3^+ (resp. A_3^-) singularity at $(0, 0)$ if and only if $(0, \theta_0)$ is a parabolic point and first order ridge point relative to $\tilde{\mathbf{v}}_1$ over Whitney umbrella, and $\tilde{\mathbf{v}}_1^2 \tilde{\kappa}_1(0, \theta_0) > 0$ (resp. < 0). In this case, H is an \mathcal{R}^+ -versal unfolding of h if and only if g is an elliptic Whitney umbrella.*
- (4) *The height function h has an A_k singularity ($k \geq 4$) at $(0, 0)$ if and only if $(0, \theta_0)$ is a parabolic point and second or higher order ridge point relative to $\tilde{\mathbf{v}}_1$ over Whitney umbrella.*
- (5) *The height function h does not have a singularity of type D_4 or worse (that is, the Hessian matrix of h never be of rank 0) at $(0, 0)$.*

Proof. We now remark that

$$\mathbf{w}_0 = \pm \tilde{\mathbf{n}}(0, \theta_0) = \pm \left(0, \frac{-a_{11} \cos \theta_0 - a_{02} \sin \theta_0}{\mathcal{A}(\theta_0)}, \frac{\cos \theta_0}{\mathcal{A}(\theta_0)} \right).$$

We have $g_u(0, 0) = (1, 0, 0)$ and $g_v(0, 0) = (0, 0, 0)$, and thus $H_u(0, 0, \mathbf{w}) = H_v(0, 0, \mathbf{w}) = 0$ if and only if \mathbf{w} is on the yz -plane. Hence, h has a singularity at $(0, 0)$.

Next, we prove the criteria for the type of singularities of h . Calculations show that

$$h_{uu}(0, 0) = \pm \frac{a_{20} \cos \theta_0}{\mathcal{A}(\theta_0)}, \quad h_{uv}(0, 0) = \mp \frac{a_{02} \sin \theta_0}{\mathcal{A}(\theta_0)}, \quad h_{vv}(0, 0) = \pm \frac{a_{02} \cos \theta_0}{\mathcal{A}(\theta_0)},$$

so that the determinant of the Hessian matrix of h at $(0, 0)$ is given by

$$\frac{a_{02}(a_{20} \cos^2 \theta_0 - a_{02} \sin^2 \theta_0)}{\mathcal{A}(\theta_0)^2} = r^2 \mathcal{A}(\theta_0)^2 \widetilde{K}(0, \theta_0).$$

From this, h has an A_1^+ (resp. A_1^-) singularity at $(0, 0)$ if and only if $(0, \theta_0)$ is an elliptic (resp. hyperbolic) point over Whitney umbrella. Moreover, h has a singularity of type A_2 or worse singularity at $(0, 0)$ if and only if $(0, \theta_0)$ is a parabolic point over Whitney umbrella. When $(0, \theta_0)$ is a parabolic point over Whitney umbrella we have $\cos \theta_0 \neq 0$, that is, $h_{vv}(0, 0) \neq 0$. Thus, h does not have a singularity of type D_4 or worse at $(0, 0)$.

We assume that h has an A_k ($k \geq 2$) singularity at $(0, 0)$. Now we regard S^2 as the set of unit vectors in \mathbb{R}^3 and we write $(x, y, z) = \mathbf{w} \in S^2$. For any $\mathbf{w} \in S^2$, we have $x^2 + y^2 + z^2 = 1$. We may assume that $z \neq 0$. We have $z = \pm \sqrt{1 - x^2 - y^2}$. We set

$$b = \frac{h_{11}}{h_{02}}, \quad c = \frac{h_{21}(h_{02})^2 - 2h_{12}h_{11}h_{02} + h_{03}(h_{11})^2}{2(h_{02})^3},$$

where

$$h_{ij} = \frac{\partial^{i+j} h}{\partial u^i \partial v^j}(0, 0).$$

Replacing by v by $v - bu - cv^2$ and writing down H as

$$(3.1) \quad H = ux + \sum_{i+j=2}^4 \frac{1}{i!j!} c_{ij}(x, y) u^i v^j + O(u, v)^5,$$

we have

$$(3.2) \quad h = \frac{1}{2} c_{02}^0 v^2 + \frac{1}{6} (c_{30}^0 u^3 + 3c_{12}^0 uv^2 + c_{03}^0 v^3) + \sum_{i+j=4} \frac{1}{i!j!} c_{ij}^0(x, y) u^i v^j + O(u, v)^5,$$

where

$$c_{ij}(x, y) = \frac{\partial^{i+j} H}{\partial u^i \partial v^j}(0, 0, x, y)$$

and $c_{ij}^0 = c_{ij}(x_0, y_0)$. Note that $c_{02}^0 = h_{vv}(0, 0) \neq 0$. The expansion (3.2) implies that h has A_k singularity at $(0, 0)$ if and only if the following conditions holds:

- $A_2 : c_{30}^0 \neq 0$;
- A_3^+ (resp. A_3^-) : $c_{30}^0 = 0, c_{02}^0 c_{40}^0 > 0$ (resp. < 0);
- $A_{\geq 4} : c_{30}^0 = c_{40}^0 = 0$.

Straightforward calculations show that $c_{30}^0 = 0$ is equivalent to $\widetilde{\mathbf{v}}_1 \widetilde{\kappa}_1(0, \theta_0) = 0$, and that if $c_{30}^0 = 0$ then $c_{02}^0 c_{40}^0 = a_{02} \mathcal{A}(\theta_0) \widetilde{\mathbf{v}}_1^2 \widetilde{\kappa}_1(0, \theta_0) \cos \theta_0$. This conclude the proof of the criteria the type of singularities of h .

Next, we proceed to check the versality of H . We skip the proofs of (1) and (2), since the proofs are similar to that of (3). We assume that h has an A_3 singularity at $(0, 0)$, and we consider the expansions (3.1) and (3.2). We note that $(0, \theta_0)$ is now a parabolic point over Whitney umbrella. Therefore, from (2.4) we have $a_{20} \geq 0$. In order to show the \mathcal{R}^+ -versality of H , we need to verify that the following equality holds:

$$(3.3) \quad \mathcal{E}_2 = \left\langle \frac{\partial h}{\partial u}, \frac{\partial h}{\partial v} \right\rangle_{\mathcal{E}_2} + \left\langle \frac{\partial H}{\partial x} \Big|_{\mathbb{R}^2 \times \{\mathbf{w}_0\}}, \frac{\partial H}{\partial y} \Big|_{\mathbb{R}^2 \times \{\mathbf{w}_0\}} \right\rangle_{\mathbb{R}} + \langle 1 \rangle_{\mathbb{R}} + \langle u, v \rangle_{\mathcal{E}_2}^5.$$

The coefficients of $u^i v^j$ of functions appearing in (3.3) are given by the following table:

	u	v	u^2	uv	v^2	u^3	u^2v	uv^2	v^3	u^4
H_x	1	0	$\alpha_{20}/2$	α_{11}	$\alpha_{02}/2$	$\alpha_{30}/6$	$\alpha_{21}/2$	$\alpha_{12}/2$	$\alpha_{03}/6$	$\alpha_{40}/24$
H_y	0	0	$\beta_{20}/2$	β_{11}	$\beta_{02}/2$	$\beta_{30}/6$	$\beta_{21}/2$	$\beta_{12}/2$	$\beta_{03}/6$	$\beta_{40}/24$
h_u	0	0	$c_{30}^0/2$	0	$c_{21}^0/2$	$c_{40}^0/6$	$c_{31}^0/2$	$c_{22}^0/2$	$c_{13}^0/6$	$c_{50}^0/24$
h_v	0	c_{02}^0	0	c_{12}^0	$c_{03}^0/2$	$c_{31}^0/6$	$c_{22}^0/2$	$c_{13}^0/2$	$c_{04}^0/6$	$c_{41}^0/24$
uh_u	0	0	0	0	0	$c_{30}^0/2$	0	$c_{12}^0/2$	0	$c_{40}^0/6$
uh_v	0	0	0	c_{02}^0	0	0	c_{12}^0	$c_{03}^0/2$	0	$c_{31}^0/6$
vh_v	0	0	0	0	c_{02}^0	0	0	c_{12}^0	$c_{03}^0/2$	0
u^2h_v	0	0	0	0	0	0	c_{02}^0	0	0	0
uvh_v	0	0	0	0	0	0	0	c_{02}^0	0	0
v^2h_v	0	0	0	0	0	0	0	0	c_{02}^0	0

	$u^i v^j \ (i + j \leq 3)$	u^4	u^3v	u^2v^2	uv^3	v^4
u^3h_v	0	0	c_{02}^0	0	0	0
u^2vh_v	0	0	0	c_{02}^0	0	0
uv^2h_v	0	0	0	0	c_{02}^0	0
v^3h_v	0	0	0	0	0	c_{02}^0

Here, $\alpha_{ij} = (\partial c_{ij} / \partial x)(x_0, y_0)$, $\beta_{ij} = (\partial c_{ij} / \partial y)(x_0, y_0)$. We remark that the boxed entries are non-zero. Calculations give

$$\beta_{20} = \frac{r^2 \mathcal{A}(\theta_0)^4 \widetilde{\mathbf{v}}_2 \widetilde{\kappa}_1(0, \theta_0) \sec^3 \theta_0}{a_{02}}.$$

It follows that the matrix presented by this table is full rank, that is, (3.3) holds if and only if $(0, \theta_0)$ is not a sub-parabolic point relative to \widetilde{v}_2 over Whitney umbrella. It now follows from (2.4) and (2.5) that $(0, \theta_0)$ not being a sub-parabolic point over Whitney umbrella is equivalent to $a_{20} \neq 0$. Therefore, from Proposition 2.2, we have completed the proof of the necessary and sufficient condition for H to be \mathcal{R}^+ -versal unfolding of h . \square

Remark. Theorem 3.1 implies that the height function on a hyperbolic Whitney umbrella can have an A_1^- singularity, and it on a parabolic Whitney umbrella can have an A_1^- or A_2 singularity.

§ 4. Local intersections of a Whitney umbrella with hyperplanes

In this section, we study the local intersections of a Whitney umbrella with hyperplanes through the Whitney umbrella singularity.

Let $g : (\mathbb{R}^2, \mathbf{0}) \rightarrow (\mathbb{R}^3, \mathbf{0})$ be given in the normal form of Whitney umbrella, and let (u_1, v_1) and (u_2, v_2) be two points on the double point locus \mathcal{D} with $g(u_1, v_1) = g(u_2, v_2)$. Then we have

$$(4.1) \quad u_1 = u_2,$$

$$(4.2) \quad 0 = u_1 v_1 - u_2 v_2 - \frac{b_3}{6}(v_1^3 - v_2^3) - \frac{b_4}{24}(v_1^4 - v_2^4) + \dots,$$

$$(4.3) \quad 0 = \sum_{i+j=2}^3 \frac{a_{ij}}{i!j!}(u_1^i v_1^j - u_2^i v_2^j) + \dots.$$

From (4.1) and (4.2), we have

$$(4.4) \quad u_1 = -\frac{b_3}{6}(v_1^2 + v_1 v_2 + v_2^2) - \frac{b_4}{24}(v_1^3 + v_1^2 v_2 + v_1 v_2^2 + v_2^3) + \dots.$$

From (4.1), (4.3), and (4.4), we have

$$(4.5) \quad v_2 = -v_1 + \frac{a_{11}b_3 - 3a_{03}}{3a_{02}}v_1^2 + \dots.$$

It follows from (4.4) and (4.5) that the parameterization of the double point locus \mathcal{D} in the domain is given by

$$(4.6) \quad \gamma(v) = \left(-\frac{b_3}{6}v^2 + \frac{(a_{11}b_3 - 3a_{03})b_3}{18a_{02}}v^3 + O(v^4), v \right).$$

Lemma 4.1. *Let $h : (\mathbb{R}^2, \mathbf{0}) \rightarrow \mathbb{R}$ be a height function on g . Then $h(\gamma(v))$ is flat at $v = 0$ (that is, all derivatives of $h(\gamma(v))$ vanish at $v = 0$) or the degree of the first term of $h(\gamma(v))$ is even.*

Proof. If (u_1, v_1) and (u_2, v_2) are two points on \mathcal{D} with $g(u_1, v_1) = g(u_2, v_2)$, then we have $h(\gamma(v_1)) = h(\gamma(v_2))$. We assume that $h(\gamma(v))$ is not flat at $v = 0$. This means

$$\left. \frac{d^m}{dv^m} h(\gamma(v)) \right|_{v=0} \neq 0$$

for some $m > 1$. If m is odd, the sign of h changes near $v = 0$ and we have contradiction. \square

Theorem 4.2. *Let $g : (\mathbb{R}^2, \mathbf{0}) \rightarrow (\mathbb{R}^3, \mathbf{0})$ be given in the normal form of Whitney umbrella, and let $\mathcal{P}_{\mathbf{w}_0}$ denote the hyperplane through the origin with normal vector $\mathbf{w}_0 \in S^2$. Moreover, let $h : (\mathbb{R}^2, \mathbf{0}) \rightarrow \mathbb{R}$ be the height function on g in the direction \mathbf{w}_0 . Then the following hold:*

- (1) *Suppose that $\mathcal{P}_{\mathbf{w}_0}$ contains the tangent line to the image of the double point locus \mathcal{D} under g at the origin, and suppose that $h(\gamma(v))$ is not flat at $v = 0$, where γ is the parameterization of \mathcal{D} in (4.6).*
 - (a) *Suppose that h is non-singular at $(0, 0)$. Then $h^{-1}(0)$ and \mathcal{D} , in the parameter space, have $2k$ -point contact ($k \geq 2$) at $(0, 0)$. Moreover, the local intersection of g with $\mathcal{P}_{\mathbf{w}_0}$ has a singularity of type A_{2k} in $\mathcal{P}_{\mathbf{w}_0}$ as a curve in the plane $\mathcal{P}_{\mathbf{w}_0}$.*
 - (b) *Suppose that h is singular at $(0, 0)$. Then one of the local components of $h^{-1}(0)$ which is not the u -axis and \mathcal{D} , in the parameter space, have $(2k + 1)$ -point contact ($k \geq 1$) at $(0, 0)$. Moreover, the local intersection of g with $\mathcal{P}_{\mathbf{w}_0}$ has a singularity of type D_{2k+3} in $\mathcal{P}_{\mathbf{w}_0}$ as a curve in the plane $\mathcal{P}_{\mathbf{w}_0}$.*
- (2) *Suppose that $\mathcal{P}_{\mathbf{w}_0}$ does not contain the tangent line to image of \mathcal{D} under g , and that h has an A_k^\pm singularity at $(0, 0)$. Then the local intersection of g with $\mathcal{P}_{\mathbf{w}_0}$ has a singularity of type A_{k+2} in $\mathcal{P}_{\mathbf{w}_0}$ as a curve in the plane $\mathcal{P}_{\mathbf{w}_0}$.*

Proof. Now we regard S^2 as the set of unit vectors in \mathbb{R}^3 and we write $\mathbf{w}_0 = (x_0, y_0, z_0)$.

(1) From (4.6), the tangent line to the the image of the double point locus \mathcal{D} , as a space curve, under g at the origin is given by

$$(4.7) \quad \{(-b_3, 0, 3a_{02})t; t \in \mathbb{R}\}.$$

From assumption and (4.7), we have

$$(4.8) \quad 3a_{02}z_0 - b_3x_0 = 0.$$

Consider the parameterization of the double point locus \mathcal{D} given by (4.6).

We start with the case (a). We have

$$(4.9) \quad h(\gamma(v)) = \frac{1}{6}(3a_{02}z_0 - b_3x_0)v^2 + O(v^3).$$

From (4.8) and (4.9), $h^{-1}(0)$ and \mathcal{D} have more than 2-point contact at $(0, 0)$. Lemma 4.1 implies that when $h^{-1}(0)$ is non-singular at $(0, 0)$, $h^{-1}(0)$ and \mathcal{D} have even-point contact at $(0, 0)$. Hence, it follows that $h^{-1}(0)$ and \mathcal{D} have $2k$ -point contact ($k \geq 2$). The double point locus of the standard Whitney umbrella $f : (\mathbb{R}^2, \mathbf{0}) \rightarrow (\mathbb{R}^3, \mathbf{0})$ given by $(u, v) \mapsto (u, uv, v^2)$ is parameterized by $(0, v)$. Since g is \mathcal{A} -equivalent to f , there is a local diffeomorphism ϕ such that $\phi(\gamma(v)) = (0, v)$. We assume that $\gamma(v)$ is written as the form

$$\gamma(v) = (d_2v^2 + \dots + d_{2k-1}v^{2k-1} + d_{2k}v^{2k} + \dots, v) \quad (d_i \in \mathbb{R}).$$

Then $h^{-1}(0)$ is parameterized by

$$\hat{\gamma}(v) = (d_2v^2 + \dots + d_{2k-1}v^{2k-1} + p_{2k}v^{2k} + \dots, v) \quad (p_i \in \mathbb{R}, p_{2k} \neq d_{2k}).$$

We have $\phi(h^{-1}(0)) = \{((p_{2k} - d_{2k})v^{2k} + \dots, v); v \in (\mathbb{R}, 0)\}$ and thus

$$f(\phi(h^{-1}(0))) = \{((p_{2k} - d_{2k})v^{2k} + \dots, (p_{2k} - d_{2k})v^{2k+1} + \dots, v^2); v \in (\mathbb{R}, 0)\}.$$

This is locally diffeomorphic to $g(h^{-1}(0))$, which is the intersection of g with $\mathcal{P}_{\mathbf{w}_0}$. By a suitable change of coordinates in \mathbb{R}^3 , $(2k + 1)$ -jet of $f(\phi(\hat{\gamma}(v)))$ is transformed into $(0, (p_{2k} - d_{2k})v^{2k+1}, v^2)$. Therefore, the local intersection of g with $\mathcal{P}_{\mathbf{w}_0}$ has an A_{2k} singularity at the origin.

We turn to the case (b). Now we have $x_0 = 0$. It follow form (4.8) that $z_0 = 0$. Hence, $\mathbf{w}_0 = (0, \pm 1, 0)$ and thus the height function h is given by

$$h = v \left(u + \frac{1}{6}b_3v^2 + \frac{1}{24}b_4v^3 + O(v^4) \right).$$

From this and Lemma 4.1, the local component $u = -b_3/6v^2 - b_4/24v^3 + O(v^4)$ of $h^{-1}(0)$ and \mathcal{D} has $(2k + 1)$ -point contact ($k \geq 1$). The rest of the proof can be completed similarly as in the proof of (a).

(2) From assumption and (4.7), we have

$$(4.10) \quad 3a_{02}z_0 - b_3x_0 \neq 0.$$

Firstly, we prove the case that $k = 0$, that is, h is non-singular at $(0, 0)$. From (4.9) and (4.10), it follows that $h^{-1}(0)$ and \mathcal{D} have 2-point contact at $(0, 0)$. Therefor, the proof of (a) of (1) implies that the local intersection of g with $\mathcal{P}_{\mathbf{w}_0}$ has an A_2 singularity at the origin.

Next, we prove the case that $k \geq 1$. Now we note that $\cos \theta_0 \neq 0$. We set a rotation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\cos \theta_0}{\mathcal{A}(\theta_0)} & \frac{a_{11} \cos \theta_0 + a_{02} \sin \theta_0}{\mathcal{A}(\theta_0)} \\ 0 & \frac{-a_{11} \cos \theta_0 - a_{02} \sin \theta_0}{\mathcal{A}(\theta_0)} & \frac{\cos \theta_0}{\mathcal{A}(\theta_0)} \end{pmatrix}.$$

We obtain

$$T \circ g|_{h^{-1}(0)} = \left(u, \mathcal{A}(\theta_0) \sec \theta_0 \left(uv + \frac{b_3}{6} v^3 + O(u, v)^4 \right), 0 \right),$$

which is a plane curve in xy -plane. From the proof of Theorem 3.1, h can be expressed in

$$h = \frac{1}{2} c_{02}^0 \left(v - \tan \theta_0 u + \sum_{i+j \geq 2}^k P_{ij} u^i v^j \right)^2 + Qu^{k+1} + \dots,$$

for some coefficients $P_{ij}, Q (\neq 0) \in \mathbb{R}$, where c_{02}^0 is as in (3.2). We make changes of coordinates

$$(4.11) \quad \hat{v} = v - \tan \theta_0 u + \sum_{i+j \geq 2}^k P_{ij} u^i v^j, \quad \hat{u} = u.$$

In this new system, h is expressed in

$$(4.12) \quad h = \frac{1}{2} c_{02}^0 \hat{v}^2 + Q \hat{u}^{k+1} + \dots.$$

From (4.11), we have $v = \tan \theta_0 \hat{u} + \hat{v} + \sum_{i+j \geq 2}^m \hat{P}_{ij} \hat{u}^i \hat{v}^j + \dots$ for some coefficients $\hat{P}_{ij} \in \mathbb{R}$, where $m = k/2$ (resp. $m = (k+1)/2$) if k is even (resp. odd). So we have

$$T \circ g|_{h^{-1}(0)} = \left(\hat{u}, \mathcal{A}(\theta_0) \sec \theta_0 \left(\tan \theta_0 \hat{u}^2 + \hat{u} \hat{v} + \sum_{i+j \geq 3}^{m+1} \bar{P}_{ij} \hat{u}^i \hat{v}^j + \dots \right), 0 \right) \quad (\bar{P}_{ij} \in \mathbb{R}).$$

Suppose that k is even. It follows from (4.12) that $h^{-1}(0)$ is locally parameterized by $\hat{u} = \alpha t^2, \hat{v} = \beta t^{k+1}$, for some coefficients α and β , not both zero. Hence, we show that

$$T \circ g|_{h^{-1}(0)} = \left(\alpha t^2, \mathcal{A}(\theta_0) \sec \theta_0 \left(\underbrace{\alpha^2 \tan \theta_0 t^4 + \dots + \alpha^{m+1} \bar{P}_{m+1,0} t^{k+2}}_{\text{even order terms}} + \alpha \beta t^{k+3} + \dots \right), 0 \right),$$

which has an A_{k+2} singularity at the origin as a plane curve.

Suppose that k is odd. It follows from (4.12) that $h^{-1}(0)$ is locally parameterized by $\hat{u} = t, \hat{v} = \pm\sqrt{-2Q/c_{02}^0} t^{(k+1)/2}$. Hence, we show that

$$T \circ g|_{h^{-1}(0)} = \left(t, \mathcal{A}(\theta_0) \sec \theta_0 \left(\tan \theta_0 t^2 + \dots + \bar{P}_{m,0} t^{\frac{k+1}{2}} \pm \sqrt{-\frac{2Q}{c_{02}^0}} t^{\frac{k+3}{2}} + \dots \right), 0 \right)$$

which has an A_{k+2}^\pm singularity at the origin as a plane curve.

□

In the case (1)(b), the local intersection contains a regular curve which is tangent to x -axis at the origin. The curvature of the regular curve is considered in [9].

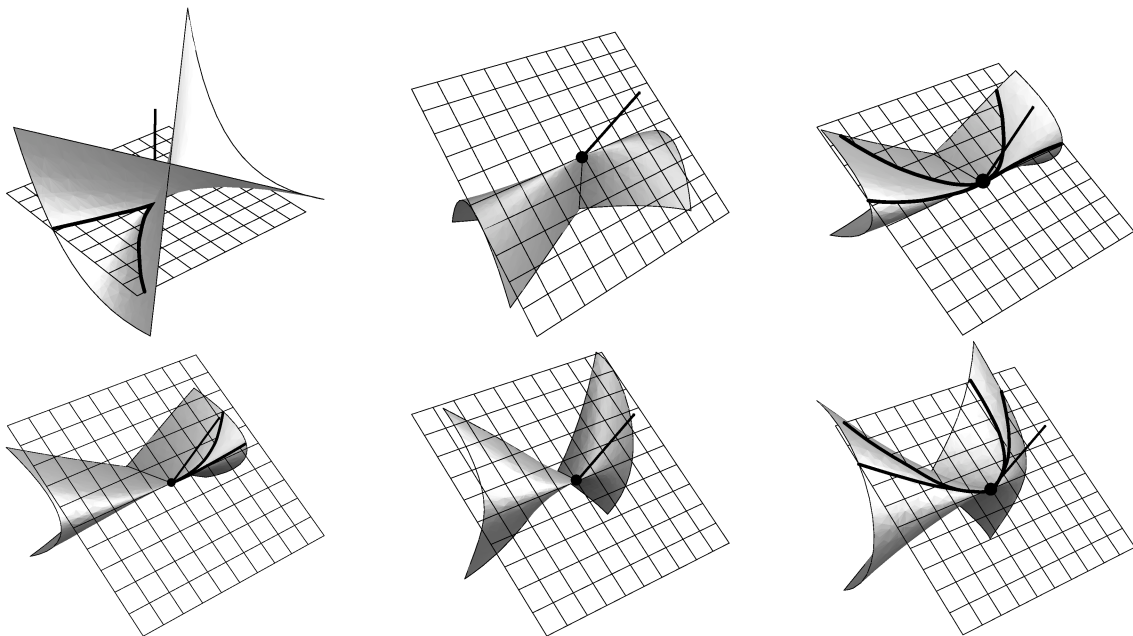


Figure 3. Intersection of the Whitney umbrella with the plane $\mathcal{P}_{\mathbf{w}_0}$ so that $h_{\mathbf{w}_0}$ has a singularity of type A_0 (top left), A_1^+ (top center), A_1^- (top right), A_2 (bottom left), A_3^+ (bottom center), and A_3^- (bottom right) when $\mathcal{P}_{\mathbf{w}_0}$ does not contain the tangent line to the double point locus on the Whitney umbrella.

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