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The Borsuk-Ulam Theorem and Combinatorics

By

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Abstract

The Borsuk-Ulam antipodal theorem is studied by many mathematicians and generalized in many ways. On the other hand, the Borsuk-Ulam theorem has applications in many mathematical fields. In this paper, we will see some generalization and combinatorial applications of the Borsuk-Ulam theorem.

§ 1. The Borsuk-Ulam Theorem

Let S^n be the unit sphere in $\boldsymbol{R}^{n+1}$. The (Borsuk-Ulam type) antipodal theorem can be stated in several different, but equivalent ways.

1.1. If $f\colon S^m \to S^n$ satisfies $f(-x) = -f(x)$, then $m \leqq n$.

1.2. If $f\colon S^n \to \boldsymbol{R}^n$ satisfies $f(-x) = -f(x)$, then $f^{-1}(0) \neq \emptyset$.

1.3. For every $f\colon S^n \to \boldsymbol{R}^n$ there exists an $x \in S^n$ with $f(-x) = f(x)$.

1.4. For every closed covering $\{M_1, \ldots, M_{n+1}\}$ of S^n, there exists an $i \in \{1, \ldots, n+1\}$ with $M_i \cap (-M_i) \neq \emptyset$.

1.5. $\mathrm{cat}\boldsymbol{R}P^n = n+1$, where $\mathrm{cat}\boldsymbol{R}P^n$ denotes the Ljusternik-Schnirelmann category of the n-dimensional real projective space $\boldsymbol{R}P^n$.

For a topological space X, the Ljusternik-Schnirelmann category $\mathrm{cat}X$ of X is defined by

$$\begin{aligned}\mathrm{cat}X = \min\{k \in \boldsymbol{N} \mid & \text{there esists a closed cover } \{A_1, \ldots A_k\} \text{ of } X \\ & \text{such that all } A_i \text{ are contractible in } X\}.\end{aligned}$$

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There are many way to prove the Borsuk-Ulam theorem. For example, we can prove it by using the cohomology algebra of $\boldsymbol{R}P^n$ (see [22]).

The Borsuk-Ulam theorem is also related with the degree theory of maps between manifolds. Let M and N be closed connected oriented manifolds of dimension n. Then for a continuous map $f\colon M \to N$, we have the induced homomolophism $f_*\colon H_n(M) \to H_n(N)$. Let $[M]$ and $[N]$ be the fundamental homology classes of M and N respectively. We define the degree $\deg f$ of f by $f_*[M] = (\deg f)[N]$. Then the following theorem is known.

1.6. If a continuous map $f\colon S^n \to S^n$ satisfies $f(-x) = -f(x)$, then $\deg f$ is odd.

We can also prove the Borsuk-Ulam theorem by using this theorem. By considering the antipodal $\boldsymbol{Z}_2$-action on S^n, 1.6 is considered as a theorem of equivariant maps. We can prove 1.6 by using the Gysin-Smith (Thom-Gysin) exact sequence([22], [18]).

The Borsuk-Ulam theorem has been generalized and extended in many ways from the view point of transformation group theory (see [5], [7], [12], [23], [24]). Fadell-Husseini and Jaworowski introduced ideal-valued index theory and generalized the Borsuk-Ulam theorem (see [7], [12]). Let G be a compact Lie group and X a G-space. Let $EG \to BG$ be a universal principal G-bundle. We denote by $\bar{H}(-;\boldsymbol{K})$ the Alexander-Spanier cohomology theory with coefficients in a field $\boldsymbol{K}$. We set $\bar{H}^*_G(X;\boldsymbol{K}) = \bar{H}^*(EG \times_G X;\boldsymbol{K})$. The *G-index* $\mathrm{Ind}^G(X;\boldsymbol{K})$ of X is defined to be the kernel of the G-cohomology homomorphism induced by the constant map $c_X : X \to pt$;

$$\mathrm{Ind}^G(X;\boldsymbol{K}) = \mathrm{Ker}(c_X^* : \bar{H}^*_G(pt;\boldsymbol{K}) \to \bar{H}^*_G(X;\boldsymbol{K}))\,,$$

where pt is a one-point space. Since $\bar{H}^*_G(pt;\boldsymbol{K}) = \bar{H}^*(BG;\boldsymbol{K})$, $\mathrm{Ind}^G(X;\boldsymbol{K})$ is an ideal of $\bar{H}^*(BG;\boldsymbol{K})$. The following proposition was proved in [7] and [12].

Proposition([7], [12]). *If $f : X \to Y$ is a G-map, then* $\mathrm{Ind}^G(X;\boldsymbol{K}) \supset \mathrm{Ind}^G(Y;\boldsymbol{K})$ *in $\bar{H}^*(BG)$.*

Proposition([7], [12]). *Let X and Y be G-spaces, and W a G -invariant closed subspace of Y. If $f : X \to Y$ is a G -map, then* $\mathrm{Ind}^G f^{-1}(W) \cdot \mathrm{Ind}^G(Y - W;\boldsymbol{K}) \subset \mathrm{Ind}^G(X;\boldsymbol{K})$ *in $\bar{H}^*(BG;\boldsymbol{K})$, where $\cdot$ represents the product of ideals.*

By using these propositions, we have generalized Borsuk-Ulam theorems (see [7], [9], [12], [14]).

§ 2. Applications of the Borsuk-Ulam Theorem in Combinatorics

In this section, we introduce combinatorial applications of the Borsuk-Ulam theorem.

§ 2.1. Tucker's Lemma

Let T be some triangulation of the n-dimensional ball B^n. We call T *antipodally symmetric on the boundary* if the set of simplices of T contained in $S^{n-1} = \partial B^n$ is a triangulation of S^{n-1} and it is antipodally symmetric; that is , if $\sigma \subset S^{n-1}$ is a simplex of T, then $-\sigma$ is also a simplex of T.

Tucker's lemma. *Let T be a triangulation of B^n that is antipodally symmetric on the boundary. Let*

$$\lambda \colon V(T) \to \{+1, -1, +2, -2, \ldots, +n, -n\}$$

be a labelling of the vertices of T such that $\lambda(-v) = -\lambda(v)$ for every vertex $v \in \partial B^n$. Then there exists a 1-simplex $\{v_1, v_2\}$ in T such that $\lambda(v_1) = -\lambda(v_2)$.

Let $\diamondsuit^n = \operatorname{conv}\{\pm e_1, \ldots, \pm e_n\}$ be the n-dimensional cross-polytope and $\partial(\diamondsuit^n) \cong S^{n-1}$ its boundaty with $\boldsymbol{Z}_2$-invariant triangulation.

It is easily seen that the following theorem is equivalent to Tucker's lemma.

Theorem. *Let T be a triangulation of B^n that is antipodally symmetric on the boundary. Then there is no map $\lambda \colon V(T) \to V(\partial(\diamondsuit^n))$ that is a simplitial map of T into $\partial(\diamondsuit^n)$ and is antipodal on the boundary.*

This theorem follows from the fact that the degree of any antipodal map $f \colon S^n \to S^n$ is odd(1.6). Ky Fan generalized this fact from the combinatorial viewpoint.

Ky Fan's theorem([8]). *Let T be a $\boldsymbol{Z}_2$-invariant triangulation of S^n. If $f \colon T \to \partial(\diamondsuit^m)$ is a simplitial $\boldsymbol{Z}_2$-map, then $n < m$ and*

$$\sum_{1 \leqq k_1 < k_2 < \cdots < k_{n+1} \leqq m} \alpha(k_1, -k_2, k_3, -k_4, \ldots, (-1)^n k_{n+1}) \equiv 1 \pmod 2,$$

where $\alpha(j_1, j_2, \ldots, j_{n+1})$ is the number of n-simplices in T mapped to the simplex spanned by vectors $e_{j_1}, e_{j_2}, \ldots, e_{j_{n+1}}$ and by definition $e_{-j} = -e_j$.

§ 2.2. Lovász-Kneser Theorem

First we prepare basic definitions and notations. A graph is a pair (V, E), where V is a set (the *vertex set*) and $E \subset \binom{V}{2}$ is the *edge set*, where $\binom{V}{2}$ denotes the set of all subsets of V of cardinality exactly 2. We denote by $[n]$ the finite set $\{1, 2, \ldots, n\}$. A k-*coloring* of a graph $G = (V, E)$ is a map $c \colon V \to [k]$ such that $c(u) \neq c(v)$ whenever $\{u, v\} \in E$. The *chromatic number* of G, denote by $\chi(G)$, is the smallest k such that G has a k-coloring.

Let X be a finite set and let $\mathcal{F} \subset 2^X$ be a set system. The *Kneser graph* of $\mathcal{F}$, denoted by $\mathrm{KG}(\mathcal{F})$, has $\mathcal{F}$ as the vertex set, and two sets $F_1, F_2 \in \mathcal{F}$ are adjacent if and only if $F_1 \cap F_2 = \emptyset$. Let $\mathrm{KG}_{n,k}$ denote the Kneser graph of the system $\mathcal{F} = \binom{[n]}{k}$ (all k-element subsets of $[n]$). The following theorem was expected by Kneser and proved by Lovász ([15]).

Lovász-Kneser theorem. *For all $k > 0$ and $n \geqq 2k - 1$, the chromatic number of the Kneser graph $\mathrm{KG}_{n,k}$ is $n - 2k + 2$.*

It is easy to prove $\chi(\mathrm{KG}_{n,k}) \leqq n-2k+2$. We define a coloring $c\colon \binom{[n]}{k} \to [n-2k+2]$ of the Kneser graph $\mathrm{KG}_{n,k} = \mathrm{KG}(\binom{[n]}{k})$ by

$$c(F) = \min\{\min(F), n - 2k + 2\}\,.$$

If two sets F_1, F_2 get the same color $c(F_1) = c(F_2) = i < n - 2k + 2$, then they cannnot be disjoint, since they both contain the element i. If the two sets both get color $n - 2k + 2$, then they are both contained in the set $\{n - 2k + 2, n - 2k + 3, \ldots, n\}$. Since $|\{n - 2k + 2, n - 2k + 3, \ldots, n\}| = 2k - 1$, they can not be disjoint either.

Lovász used the neighborhood complex of a graph to prove the Lovász-Kneser theorem. The neighborhood complex $\mathcal{N}(G)$ of a graph G is the simplicial complex whose vertices are the vertices of G and whose simplices are those subsets of $V(G)$ which have a common neighbor. Denote by $\bar{\mathcal{N}}(G)$ the polyhedron determined by $\mathcal{N}(G)$. Lovász proved that if $\bar{\mathcal{N}}(G)$ is i-connected, then $\chi(G) > i+2$ by using the Borsuk-Ulam theorem (1.4). Moreover he proved $\bar{\mathcal{N}}(\mathrm{KG}_{n,k})$ is $(n - 2k - 1)$-connected and therefore he proved the Lovász-Kneser theorem. The Lovász-Kneser theorem was proved in other ways after Lovász proved it (see [3], [17], [20]).

A *hypergraph* is a pair $(X, \mathcal{F})$, where X is a finite set and $\mathcal{F} \subset 2^X$ is a system of subsets of X. The element of $\mathcal{F}$ are called the *edges* or *hyperedges*. A hypergraph H is m-colorable if its vertices can be colored by m colors such that no hyperedge becomes monochromatic. We define the m-colorability defect $\mathrm{cd}_m(\mathcal{F})$ of a set system $\emptyset \notin \mathcal{F}$ by

$$\mathrm{cd}_m(\mathcal{F}) = \min\bigl\{|Y| \,:\, (X - Y, \{F \in \mathcal{F} \,|\, F \cap Y = \emptyset\}) \text{ is } m\text{-colorable}\bigr\}\,.$$

Dol'nikov prove the following theorem in [6].

Dol'nikov's theorem. *For any set system $\emptyset \notin \mathcal{F}$, the inequality*

$$\mathrm{cd}_2(\mathcal{F}) \leqq \chi(\mathrm{KG}(\mathcal{F}))$$

holds.

This theorem generalizes the Lovász-Kneser theorem. Because it is easy to prove that if $\mathcal{F}$ consists of all the k-subsets of an n-set with $k \leqq n/2$, then $\mathrm{cd}_2(\mathcal{F}) = n-2k+2$.

We say that a graph is *completely multicolored* in a coloring if all its vertices receive different colors. For $x \in \boldsymbol{R}$, let $\lfloor x \rfloor = \max\{n \in \boldsymbol{Z} \mid n \leqq x\}$ and $\lceil x \rceil = \min\{n \in \boldsymbol{Z} \mid n \geqq x\}$. Symonyi and Tardos prove the following theorem.

Theorem([21]). *Let $\mathcal{F}$ be a finite family of sets, $\emptyset \notin \mathcal{F}$ and $\mathrm{KG}(\mathcal{F})$ its general Kneser graph. Let $r = \mathrm{cd}_2(\mathcal{F})$. Then any proper coloring of $\mathrm{KG}(\mathcal{F})$ with colors $1, \dots, m$ (m arbitary) must contain a completely multicolored complete bipartite graph $K_{\lceil r/2 \rceil, \lfloor r/2 \rfloor}$ such that the r different colors occur alternating on the two sides of the bipartite graph with respect to their natural order.*

This theorem generalizes Dol'nikov's theorem, because it implies that any proper coloring must use at least $\mathrm{cd}_2(\mathcal{F})$ different colors. In the proof of the above theorem, Ky Fan's theorem is used.

$S \in \binom{[n]}{k}$ is said to be *stable* if it does not contain any two adjacent elemens modulo n. In other words, S corresponds to an independent set in the cycle C_n. We denote by $\binom{[n]}{k}_{\mathrm{stab}}$ the family of stable k-subsets of $[n]$. We define the *Schrijver graph* by

$$\mathrm{SG}_{n,k} = \mathrm{KG}\left(\binom{[n]}{k}_{\mathrm{stab}}\right).$$

It is an induced subgraph of the Kneser graph $\mathrm{KG}_{n,k}$. In [19], Schrijver defined the Schrijver graph and proved $\chi(SG_{n,k}) = \chi(KG_{n,k}) = n - 2k + 2$ for all $n \geqq 2k \geqq 0$.

§ 2.3. Necklace Theorem

Two thieves have stolen a precious necklace, which has n beads. These beads belong to t different types. Assume that there is an even number of beads of each type, say $2a_i$ beads of type i, for each $i \in \{1, 2, \dots, t\}$, where a_i is a nonzero integer. Remark that we have $2\sum_{i=1}^{t} a_i = n$. The beads are fixed on an open chain made of gold.

As we do not know the exact value of each type of bead, a fair division of the necklace consists of giving the same number of beads of each type to each thief. The number of beads of each type is even, hence such a division is always possible: cut the chain at the $n-1$ possible positions. But we want to do the division with fewer cuts. The following theorem was proved by Goldberg and West.

Theorem. *A fair division of the necklace with t types of beads between two thieves can be done with no more than t cuts.*

Alon and West gave a simpler proof by using the Borsuk-Ulam theorem in [2].

In 1987, Alon proved the following generalization, for a necklace having qa_i beads for each type i, a_i integer, using a generalized Borsuk-Ulam theorem:

Theorem([1]). *A fair division of the necklace with t types of beads between q thieves can be done with no more than $t(q-1)$ cuts.*

The following theorem is considered as a continuous version of the necklace theorem.

Hobby-Rice theorem([11]). *Let $\mu_1, \mu_2, \ldots, \mu_d$ be continuous probability measures on the unit interval. Then there is a partition of $[0,1]$ into $d+1$ intervals $I_0, I_1, \ldots, I_d$ and signs $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_d \in \{-1, +1\}$ with*

$$\sum_{j=0}^{d} \varepsilon_j \mu_i(I_j) = 0 \ \text{for} \ i = 1, 2, \ldots, d.$$

Alon generalized Hobby-Rice theorem as follows.

Theorem([1]). Let $\mu_1, \mu_2, \ldots, \mu_t$ be t continuous probability measures on the unit interval. Then it is possible to cut the interval in $(k-1) \cdot t$ places and partition the $(k-1) \cdot t + 1$ resulting intervals into k families $F_1, F_2, \ldots, F_k$ such that $\mu_i(\cup F_j) = l/k$ for all $l \leqq i \leqq t$, $l \leqq j \leqq k$. The number $(k-l) \cdot t$ is best possible.

Recently de Longueville and Živaljeić generalize the theorem above and got a higher-dimensional necklace theorem in [16].

§ 2.4. Tverberg's Theorem

H. Tverberg showed the following theorem in [25]

Theorem([25]). *Consider a finite set $X \subset \boldsymbol{R}^d$ with $|X| = (d+1)(r-1)+1$. Then X can be partitioned into r subset $X_1, \ldots, X_r$ so that*

$$\bigcap_{i=1}^{r} \mathrm{conv} X_i \neq \emptyset\,.$$

The following theorem is a topological generalization of Tverberg's theorem.

Theorem([4], [26]). *Let $q = p^r$ be a prime power and $d \geqq 1$. Put $N = (d+1)(q-1)$. For every continuous map $f \colon \|\sigma^N\| \to \boldsymbol{R}^d$ there are q disjyoint faces $F_1, F_2, \ldots, F_q$ of the standard N-simplex σ^N whose images under f intersect: $\bigcap_{i=1}^{q} f(\|F_i\|) \neq \emptyset$.*

This theorem was proved by using a Borsuk-Ulam type theorem. It is still unknown whether such a theorem holds for q not equal to a prime power.

§ 3. A Generalization of Tucker's Lemma

Tucker's lemma is equivalent to the Borsuk-Ulam theorem (see [17]) We shall consider a generalization of Tucker's lemma as analogous as generalizations of the Borsuk-Ulam theorem.

Let K be a simplicial complex and A a subcomplex of K. Suppose that A has a simplicial $\boldsymbol{Z}_2$-action and $\boldsymbol{Z}_2$-action on $|A|$ is free, where $|A|$ denotes the polyhedron determined by A. Then we can consider the first Stiefel-Whitney class $w_1(|A|/\boldsymbol{Z}_2)$ of the double covering $\pi\colon |A| \to |A|/\boldsymbol{Z}_2$. Let $\pi^!\colon H^*(|A|;\boldsymbol{Z}_2) \to H^*(|A|/\boldsymbol{Z}_2;\boldsymbol{Z}_2)$ be the transfer. We denotes by $i : |A| \to |K|$ the inculusion. Then we have the following.

Theorem. *Suppose that* $w_1(|A|/\boldsymbol{Z}_2)^{n-1} \notin \pi^! \circ i^*(H^{n-1}(|K|;\boldsymbol{Z}_2))$. *If*

$$\lambda\colon V(K) \to \{\pm 1, \pm 2, \dots, \pm n\}$$

satisfies $\lambda(-v) = -\lambda(v)$ *for* $v \in V(A)$*, then there exisits a* 1*-simplex* $\{v_1, v_2\}$ *in* K *such that* $\lambda(v_1) = -\lambda(v_2)$.

Proof. Suppose that there is no 1-simplex $\{v_1, v_2\}$ in K such that $\lambda(v_1) = -\lambda(v_2)$. Then we have a simplicial map $f_\lambda\colon K \to \partial(\diamondsuit^n)$, where $\partial(\diamondsuit^n)$ denotes a triangulation of S^{n-1} in section 2.1. Since $f_\lambda \circ i\colon A \to \partial(\diamondsuit^n)$ is a $\boldsymbol{Z}_2$-map, we have a continuous map $f\colon |A|/\boldsymbol{Z}_2 \to |\partial(\diamondsuit^n)|/\boldsymbol{Z}_2$ such that $f \circ \pi = \pi_S \circ f_\lambda \circ i$, where $\pi_S\colon |\partial(\diamondsuit^n)| \to |\partial(\diamondsuit^n)|/\boldsymbol{Z}_2$ is a projection. Let α be the generator of $H^{n-1}(\partial(\diamondsuit^n);\boldsymbol{Z}_2)$. Since $\pi^! \circ (f_\lambda \circ i)^*(\alpha) = f^* \circ \pi_S^!(\alpha) = f^*(w_1(S^{n-1}/\boldsymbol{Z}_2)^{n-1}) = w_1(|A|/\boldsymbol{Z}_2)^{n-1}$, $\pi^! \circ i^* \circ f_\lambda^*(\alpha) = w_1(|A|/\boldsymbol{Z}_2)^{n-1}$. This contradicts $w_1(|A|/\boldsymbol{Z}_2)^{n-1} \notin \pi^! \circ i^*(H^{n-1}(|K|;\boldsymbol{Z}_2))$. $\square$

Remark. In the above theorem, if $w_1(|A|/\boldsymbol{Z}_2)^n \neq 0$, then there exisits a 1-simplex $\{v_1, v_2\}$ in A such that $\lambda(v_1) = -\lambda(v_2)$. Because if there is no 1-simplex $\{v_1, v_2\}$ in A such that $\lambda(v_1) = -\lambda(v_2)$, then we have an equivariant map $f_\lambda\colon |A| \to |\partial(\diamondsuit^n)|$ from λ. Since $|\partial(\diamondsuit^n)|/\boldsymbol{Z}_2 \cong \boldsymbol{R}P^{n-1}$, $w_1(|A|/\boldsymbol{Z}_2)^n = \bar{f}_\lambda^*(w_1(\boldsymbol{R}P^{n-1})^n) = 0$, where $\bar{f}_\lambda\colon |A|/\boldsymbol{Z}_2 \to \boldsymbol{R}P^{n-1}$ is a map determined by f_λ.

In the above theorem we consider S^{n-1} and its triangulation $\partial(\diamondsuit^n)$. In [7], [12] and [14], we see Borsuk-Ulam type theorems on Stiefel manilolds.

Problem. Consider a generalization of Tucker's lemma on Stiefel manifolds.

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