RIMS Kôkyûroku Bessatsu
B39 (2013), 035–044

A subgroup of 1-cocycles associated with a group action

By

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Abstract

For a discrete group Γ acting on a compact metric space X, we define a subgroup $\mathcal{W}_{\Gamma,X}$ of the 1-cocycles $Z^1(\Gamma; \mathrm{Map}(X, \mathbb{T}))$ of Γ, where $\mathrm{Map}(X, \mathbb{T})$ is the Γ-module consisting of all continuous maps of X to the 1-dimensional torus $\mathbb{T}$. The group $\mathcal{W}_{\Gamma,X}$ admits a natural homomorphism $W_{\Gamma,X} : \mathcal{W}_{\Gamma,X} \to B^1(\Gamma; \check{\mathrm{H}}^1(X; \mathbb{Z}))$ onto the 1-coboundaries of Γ in the first integral Čech cohomology group $\check{\mathrm{H}}^1(X; \mathbb{Z})$, naturally regarded as a Γ-module. When the fixed point set $\check{\mathrm{H}}^1(X; \mathbb{Z})^\Gamma$ is trivial, the kernel $\mathrm{Ker}\, W_{\Gamma,X}$ contains an image of real-valued 1-coboundaries of Γ in the Γ-module $\mathrm{Map}(X, \mathbb{R})$. Results are applied to canonical actions of torsion-free Gromov hyperbolic group Γ on the boundary $\partial\Gamma$.

§ 1. Introduction and Preliminaries

Throughout the present paper, $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$ denotes the 1-dimensional torus with the standard covering projection

$$e : \mathbb{R} \to \mathbb{T}, \quad \theta \mapsto \exp(i\theta).$$

For a continuous surjection $T : X \to X$ of a compact metric space X and a continuous function $w : X \to \mathbb{T}$, the linear isometry $U_{T,w} : C(X) \to C(X)$ on the Banach space $C(X)$ of all complex-valued continuous functions on X (with the sup norm) defined by

$$(U_{T,w}f)(x) = w(x) \cdot f(T(x)), \quad f \in C(X),\ x \in X$$

is called a weighted composition operator with the unimodular weight w. In [6], the group $\mathcal{W}_T$ of all unimodular weights w such that $U_{T,w}$ has a unimodular eigenfunction

Received September 27, 2011. Revised January 17 2012
2000 Mathematics Subject Classification(s): 37C85, 58E40
Key Words: *group action, dynamical cocycle*
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was studied. It was shown that $\mathcal{W}_T$ carries topological and ergodic aspects of the dynamics when T represents a dynamics of "hyperbolic type." For such $T : X \to X$, the group of all real-valued dynamical coboundaries of T naturally maps onto a subgroup of $\mathcal{W}_T$.

The present paper studies a similar group associated with a discrete group Γ acting on a compact metric space X. It turns out that an appropriate counterpart to $\mathcal{W}_T$ above is a subgroup, denoted by $\mathcal{W}_{\Gamma,X}$, of 1-cocycles $Z^1(\Gamma, \mathrm{Map}(X, \mathbb{T}))$ in the right Γ-module $\mathrm{Map}(X, \mathbb{T})$. The results will be applied to Gromov hyperbolic groups ([4]) acting on their boundaries and the author hopes that the framework presented in this paper, though elementary, might be applied to study dynamical aspect of various group actions.

In the rest of this section, we fix notations and give basic definitions. For a group Γ, its character group $\mathrm{Hom}(\Gamma, \mathbb{T})$ is denoted by $\hat{\Gamma}$. For a left (resp. right) action of a group Γ on a set S, $\gamma \circ x$ (resp. $x \circ \gamma$) denotes the element of S obtained from $x \in S$ by the action of the element $\gamma \in \Gamma$. The fixed point set S^Γ of the left (resp. right) action of Γ on S is defined by

$$S^\Gamma = \{x \in X \mid \gamma \circ x \text{ (resp. } x \circ \gamma \text{)} = x \text{ for each } \gamma \in \Gamma\}.$$

For a (discrete) group Γ, a right Γ-module M and an element $\mu \in M$ (referred to as a 0-cochain), we define the coboundary $\delta^0\mu : \Gamma \to M$ in the standard way:

$$\delta^0\mu(\gamma) = \mu - \mu \circ \gamma, \quad \gamma \in \Gamma.$$

A function $\psi : \Gamma \to M$ (referred to as a 1-cochain) is a 1-*coboundary* if there exists a 0-cochain $\mu \in M$ such that $\psi = \delta^0\mu$. A 1-cochain $\varphi : \Gamma \to M$ is called a *1-cocycle* if the equality

$$\varphi(\gamma_1\gamma_2) = \varphi(\gamma_1) \circ \gamma_2 + \varphi(\gamma_2)$$

holds for each $\gamma_1, \gamma_2 \in \Gamma$. The groups of all 1-cocycles and all 1-coboundaries of Γ in M are denoted by $\mathrm{Z}^1(\Gamma; M)$ and $\mathrm{B}^1(\Gamma; M)$ respectively. Then the first cohomology is $\mathrm{H}^1(\Gamma; M) = \mathrm{Z}^1(\Gamma; M)/\mathrm{B}^1(\Gamma; M)$ ([1]). Each Γ-homomorphism $f : M \to N$ induces homomorphisms

$$f_* : \mathrm{Z}^1(\Gamma; M) \to \mathrm{Z}^1(\Gamma; N), \quad f_* : \mathrm{B}^1(\Gamma; M) \to \mathrm{B}^1(\Gamma; N), \text{ and}$$
$$f_* : \mathrm{H}^1(\Gamma; M) \to \mathrm{H}^1(\Gamma; N).$$

Suppose that a discrete group Γ admits a left action on a compact metric space X and let $\mathrm{Map}(X, G)$ be the abelian group of all continuous maps of X to a topological abelian group G, where the group operation of G is multiplicatively denoted by the symbol " $\cdot$ ". The abelian group structure of $\mathrm{Map}(X, G)$ is given by the pointwise multiplication:

$$(f \cdot g)(x) = f(x) \cdot g(x), \quad x \in X, \quad f, g \in \mathrm{Map}(X, G).$$

The group $\mathrm{Map}(X, G)$ is naturally a right Γ-module by the action:

$$(f \circ \gamma)(x) = f(\gamma \circ x), \quad f \in \mathrm{Map}(X, G), \quad \gamma \in \Gamma, \quad x \in X.$$

Also Γ acts on the first integral Čech cohomology group $\check{\mathrm{H}}^1(X; \mathbb{Z})$ by:

$$\varphi \circ \gamma = \gamma^*(\varphi), \quad \varphi \in \check{\mathrm{H}}^1(X; \mathbb{Z}), \quad \gamma \in \Gamma,$$

where γ^* denotes the homomorphism induced by the homeomorphism $\gamma : X \to X$, $x \mapsto \gamma \circ x$.

There exists a natural isomorphism

$$\check{\mathrm{H}}^1(X; \mathbb{Z}) \cong [X, \mathbb{T}] \tag{1.1}$$

between the first integral Čech cohomology group $\check{\mathrm{H}}^1(X; \mathbb{Z})$ of X and the free homotopy classes $[X, \mathbb{T}]$ of the maps $X \to \mathbb{T}$. For a map $f \in \mathrm{Map}(X, \mathbb{T})$, $[f]$ denotes the homotopy class of f. Fixing a generator $\mathbf{e} \in \check{\mathrm{H}}^1(S^1; \mathbb{Z})$, the above isomorphism is given by the correspondence

$$f^*(\mathbf{e}) \leftrightarrow [f], \quad [f] \in [X, \mathbb{T}],$$

where the addition $f^*(\mathbf{e}) + g^*(\mathbf{e})$ corresponds to $[f \cdot g]$ (see [3]). The homotopy classes $[X, \mathbb{T}]$ is naturally a right Γ-module by the action

$$[f] \circ \gamma = [f \circ \gamma], \quad [f] \in [X, \mathbb{T}], \quad \gamma \in \Gamma$$

and it is easy to see that the isomorphism (1.1) is actually an isomorphism as right Γ-modules. This naturally induces a homomorphism

$$\pi : \mathrm{Map}(X, \mathbb{T}) \to \check{\mathrm{H}}^1(X; \mathbb{Z}).$$

Convention: In what follows, the constant map $X \to \mathbb{T}$ taking value $1 \in \mathbb{T}$ is denoted by $1 : X \to \mathbb{T}$:

$$1(x) = 1, \quad x \in X.$$

Its homotopy class $[1] \in [X, \mathbb{T}]$ is denoted by the same symbol 1 for simplicity. The 1-cochain $\mathbf{1} : \Gamma \to \mathrm{Map}(X, \mathbb{T})$ is defined by

$$\mathbf{1}(\gamma) = 1, \quad \gamma \in \Gamma.$$

§ 2. The group $\mathcal{W}_{\Gamma,X}$

For a discrete group Γ which admits a left action on a compact metric space X, we define a subgroup $\mathcal{W}_{\Gamma,X}$ of $\mathrm{Z}^1(\Gamma;\mathrm{Map}(X,\mathbb{T}))$. To simplify notation, for a function $w:\Gamma\to\mathrm{Map}(X,\mathbb{T})$, the map $w(\gamma)\in\mathrm{Map}(X,\mathbb{T})$ is denoted by $w_\gamma : X\to\mathbb{T}$ in the sequel. We define the group $\mathcal{W}_{\Gamma,X}$ by

$$\begin{aligned}\mathcal{W}_{\Gamma,X} = \{\, w\in \mathrm{Z}^1(\Gamma;\mathrm{Map}(X,\mathbb{T}))\mid\ & \\ &\text{there exist } \chi\in\hat{\Gamma} \text{ and } f\in\mathrm{Map}(X,\mathbb{T}) \text{ such that}\\ &w_\gamma\cdot(f\circ\gamma)=\chi(\gamma)\cdot f \text{ for each } \gamma\in\Gamma\}\\ =\ &\hat{\Gamma}\cdot\mathrm{B}^1(\Gamma;\mathrm{Map}(X,\mathbb{T})).\end{aligned}$$

When $\Gamma=\mathbb{Z}$ and the action is generated by a homeomorphism $T:X\to X$, then the group $\mathcal{W}_{\Gamma,X}$ coincides with the group $\mathcal{W}_T$ of [6].

The homomorphism $\pi:\mathrm{Map}(X;\mathbb{T})\to\check{\mathrm{H}}^1(X;\mathbb{Z})\cong[X,\mathbb{T}]$ induces a homomorphism

$$\pi_*:\mathrm{Z}^1(\Gamma;\mathrm{Map}(X,\mathbb{T}))\to\mathrm{Z}^1(\Gamma;\check{\mathrm{H}}^1(X;\mathbb{Z}))$$

which is explicitly written as

$$\pi(w)(\gamma)=[w_\gamma],\quad \gamma\in\Gamma.$$

The restriction of π_* above to $\mathcal{W}_{\Gamma,X}$ is denoted by $W_{\Gamma,X}$:

$$W_{\Gamma,X}:\mathcal{W}_{\Gamma,X}\to\mathrm{Z}^1(\Gamma;\check{\mathrm{H}}^1(X;\mathbb{Z})),\quad W_{\Gamma,X}(w)_\gamma=[w_\gamma].$$

In what follows, we examine the image $\mathrm{Im}\,W_{\Gamma,X}$ and the kernel $\mathrm{Ker}\,W_{\Gamma,X}$.

Example 2.1. *Let $\mathbb{F}_n$ be the free group of rank n and let $\partial\mathbb{F}_n$ be the boundary of $\mathbb{F}_n$ as a hyperbolic group [4]. Since $\partial\mathbb{F}_n$ is homeomorphic to the Cantor set, we see that $\check{\mathrm{H}}^1(\partial\mathbb{F}_n;\mathbb{Z})=0$. Hence $\mathcal{W}_{\mathbb{F}_n,\partial\mathbb{F}_n}=\mathrm{Ker}\,W_{\mathbb{F}_n,\partial\mathbb{F}_n}$.*

Example 2.2. *As is shown in [2] and [5], there are many hyperbolic groups Γ such that $\partial\Gamma$ is homeomorphic to the Menger curve or the Sierpinski carpet. For such a group Γ, the homomorphism $W_{\Gamma,\partial\Gamma}$ maps $\mathcal{W}_{\Gamma,\partial\Gamma}$ to $\mathrm{Z}^1(\Gamma;\oplus_\infty\mathbb{Z})$, where $\oplus_\infty\mathbb{Z}$ denotes the direct sum of countably many copies of $\mathbb{Z}$.*

First we examine the image $\mathrm{Im}\,W_{\Gamma,X}$.

Theorem 2.3. *We have the following equality.*

$$\mathrm{Im}\,W_{\Gamma,X}=\mathrm{B}^1(\Gamma;\check{\mathrm{H}}^1(X;\mathbb{Z})).$$

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{Map}(X,\mathbb{T}) & \xrightarrow{\delta^0} & \mathrm{B}^1(\Gamma;\mathrm{Map}(X,\mathbb{T})) & \xrightarrow{\subset} & \mathrm{Z}^1(\Gamma;\mathrm{Map}(X,\mathbb{T})) \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\pi_*}\downarrow & & {\scriptstyle\pi_*}\downarrow \\ \check{\mathrm{H}}^1(X;\mathbb{Z}) & \xrightarrow{\delta^0} & \mathrm{B}^1(\Gamma;\check{\mathrm{H}}^1(X;\mathbb{Z})) & \xrightarrow{\subset} & \mathrm{Z}^1(\Gamma;\check{\mathrm{H}}^1(X;\mathbb{Z})) \end{array}$$

By definition, each δ^0 and π are surjective. It follows from this that

$$\pi_*(\mathrm{B}^1(\Gamma;\mathrm{Map}(X,\mathbb{T}))) = \mathrm{B}^1(\Gamma;\check{\mathrm{H}}^1(X;\mathbb{Z})).$$

Also it is easy to see that $\pi_*(\hat{\Gamma}) = \{\mathbf{1}\} \subset \mathrm{B}^1(\Gamma;\check{\mathrm{H}}^1(X;\mathbb{Z}))$. Then the desired equality is obtained as follows:

$$\begin{aligned} \mathrm{Im}\, W_{\Gamma,X} &= \pi_*(\mathcal{W}_{\Gamma,X}) \\ &= \pi_*(\hat{\Gamma}\cdot \mathrm{B}^1(\Gamma;\mathrm{Map}(X,\mathbb{T})) \\ &= \pi_*(\mathrm{B}^1(\Gamma;\mathrm{Map}(X,\mathbb{T}))) \\ &= \mathrm{B}^1(\Gamma;\check{\mathrm{H}}^1(X;\mathbb{Z})). \end{aligned}$$

□

Corollary 2.4. *A 1-cocycle $v \in \mathrm{Z}^1(\Gamma;\check{\mathrm{H}}^1(X;\mathbb{Z}))$ belongs to $\mathrm{Im}(W_{\Gamma,X})$ if and only if v represents the trivial element in the cohomology $\mathrm{H}^1(\Gamma;\check{\mathrm{H}}^1(X;\mathbb{Z}))$.*

In order to examine the kernel $\mathrm{Ker}\, W_{\Gamma,X}$, it is convenient to introduce the following two subgroups of $\mathrm{Map}(X,\mathbb{T})$:

$$\begin{aligned} \mathrm{Map}(X,\mathbb{T})_\Gamma &= \{f \in \mathrm{Map}(X,\mathbb{T}) \mid f\circ\gamma \simeq f \text{ for each } \gamma\in\Gamma\} \quad \text{and} \\ \mathrm{Map}(X,\mathbb{T})_1 &= \{f \in \mathrm{Map}(X,\mathbb{T}) \mid f \simeq 1\}. \end{aligned}$$

Notice that

$$\mathrm{Map}(X,\mathbb{T})_1 = e_*(\mathrm{Map}(X,\mathbb{R})) \subset \mathrm{Map}(X,\mathbb{T})_\Gamma = \mathrm{Ker}(\pi_*\delta^0) \tag{2.1}$$

Theorem 2.5. *Under the above notation, we have the following.*

(1) $\mathrm{Ker}(W_{\Gamma,X}) = \hat{\Gamma}\cdot\delta^0(\mathrm{Map}(X,\mathbb{T})_\Gamma)$.

(2) Assume that for each $\gamma\in\Gamma$, the induced homeomorphism $\gamma : X \to X$ has a fixed point. Then we have

$$\mathrm{Ker}\, W_{\Gamma,X} = \hat{\Gamma}\oplus\delta^0(\mathrm{Map}(X,\mathbb{T})_\Gamma) \text{ (a direct sum decomposition).}$$

Proof. (1) For each $w \in \mathcal{W}_{\Gamma,X}$, there exist $\chi \in \hat{\Gamma}$ and $f \in \mathrm{Map}(X,\mathbb{T})$ such that $w = \chi \cdot \delta^0(f)$. Noticing that $\pi_*(\chi) = \mathbf{1}$, we see that $\pi_*(w) = \pi_*\delta^0(f)$. Under this notation, the equality (1) is a consequence of the following sequence of equivalences.

$$\begin{aligned} w \in \mathrm{Ker}\, W_{\Gamma,X} &\Leftrightarrow \pi_*\delta^0(f) = 0 \in \mathrm{Z}^1(\Gamma; \check{\mathrm{H}}^1(X;\mathbb{Z})) \\ &\Leftrightarrow f \in \mathrm{Map}(X,\mathbb{T})_\Gamma \text{ (by (2.1))} \\ &\Leftrightarrow w \in \hat{\Gamma} \cdot \delta^0(\mathrm{Map}(X,\mathbb{T})_\Gamma). \end{aligned}$$

In order to show (2), it suffices to prove

$$\hat{\Gamma} \cap \delta^0(\mathrm{Map}(X,\mathbb{T})_\Gamma) = \{\mathbf{1}\}.$$

If $\chi \in \hat{\Gamma}$ belongs to $\delta^0(\mathrm{Map}(X,\mathbb{T})_\Gamma)$, then there exists $f \in \mathrm{Map}(X,\mathbb{T})_\Gamma$ such that $\chi = \delta^0(f)$. This means that, for each $\gamma \in \Gamma$ and for each $x \in X$, we have

$$(f \cdot (f \circ \gamma)^{-1})(x) = \chi(\gamma).$$

Evaluating the left hand side of the above at the fixed point x_γ of $\gamma : X \to X$, we see that $\chi(\gamma) = 1$ and hence $\chi = \mathbf{1} \in \delta^0(\mathrm{Map}(X,\mathbb{T})_\Gamma)$. □

Theorem 2.6. *Assume that the fixed point set of the action of* Γ *on the first Čech cohomology is trivial:* $\check{\mathrm{H}}^1(X;\mathbb{Z})^\Gamma = \{0\}$. *Then we have the following.*

(1) $\delta^0(\mathrm{Map}(X,\mathbb{T})_\Gamma) = e_*(\mathrm{B}^1(\Gamma, \mathrm{Map}(X,\mathbb{R})))$.

(2) Assume further that X *is connected. Then*

$$\mathrm{Ker}\, W_{\Gamma,X} = \hat{\Gamma} \oplus e_*(\mathrm{B}^1(\Gamma, \mathrm{Map}(X,\mathbb{R}))) \text{ (a direct sum decomposition).}$$

Proof. First we observe the following equivalences:

$$\begin{aligned} \check{\mathrm{H}}^1(X;\mathbb{Z})^\Gamma = \{0\} &\Leftrightarrow [X,\mathbb{T}]^\Gamma = \{1\} \\ &\Leftrightarrow \mathrm{Map}(X,\mathbb{T})_\Gamma = \mathrm{Map}(X,\mathbb{T})_1 = e_*(\mathrm{Map}(X,\mathbb{R})). \end{aligned}$$

The above implies the following equalities which prove (1):

$$\begin{aligned} \delta^0(\mathrm{Map}(X,\mathbb{T})_\Gamma) &= \delta^0(\mathrm{Map}(X,\mathbb{T})_1) \\ &= \delta^0 e_*(\mathrm{Map}(X,\mathbb{R})) \\ &= e_*\delta^0(\mathrm{Map}(X,\mathbb{R})) \\ &= e_*(\mathrm{B}^1(\Gamma, \mathrm{Map}(X,\mathbb{R}))). \end{aligned}$$

In order to show (2), first we notice that

$$\mathrm{Ker}\, W_{\Gamma,X} = \hat{\Gamma} \cdot e_*(\mathrm{B}^1(\Gamma, \mathrm{Map}(X,\mathbb{R})))$$

which follows from (1) and Theorem 2.5, (1). It remains to be shown that

$$\hat{\Gamma} \cap e_*(\mathrm{B}^1(\Gamma, \mathrm{Map}(X, \mathbb{R}))) = \{\mathbf{1}\}. \tag{2.2}$$

If $\chi \in \hat{\Gamma}$ belongs to $e_*(\mathrm{B}^1(\Gamma; \mathrm{Map}(X, \mathbb{R}))$, then there exists a map $\varphi : \mathrm{Map}(X, \mathbb{R})$ such that $\chi(\gamma) = \exp(i(\varphi - \varphi \circ \gamma))$ for each $\gamma \in \Gamma$. Since X is connected, we see that, for each $\gamma \in \Gamma$, the function $\varphi - \varphi \circ \gamma$ takes a constant value θ_γ:

$$\varphi - \varphi \circ \gamma \equiv \theta_\gamma.$$

For each nonnegative integer n, we have

$$\begin{aligned} \varphi \circ \gamma^n - \varphi &= \sum_{i=0}^{n-1} (\varphi \circ \gamma^{i+1} - \varphi \circ \gamma^i) \\ &= \sum_{i=0}^{n-1} (\varphi \circ \gamma - \varphi) \circ \gamma^i \\ &= \sum_{i=0}^{n-1} \theta_\gamma = n\theta_\gamma. \end{aligned} \tag{2.3}$$

Let $\| \varphi \| := \max_{x \in X} |\varphi(x)|$. Then we have $|\varphi \circ \gamma^n - \varphi| \leq 2 \| \varphi \|$. By (2.3), we see that

$$n|\theta_\gamma| \leq 2 \| \varphi \| .$$

for each non-negative integer n. The above holds only when $\theta_\gamma = 0$ and thus, only when $\varphi \circ \gamma = \varphi$. This implies that $\chi(\gamma) = 1$ for each $\gamma \in \Gamma$. This proves (2.2) and hence completes the proof of (2). □

Example 2.7. *Let $\mathbb{F}_n$ be the free group of rank n of Example 2.1. Recalling $\check{\mathrm{H}}^1(\partial \mathbb{F}_n; \mathbb{Z}) = 0$ and noticing that every $\gamma : \partial \mathbb{F}_n \to \partial \mathbb{F}_n$ has two fixed points in $\partial \mathbb{F}_n$, we obtain, by Theorem 2.5 and Theorem 2.6, that*

$$\begin{aligned} \mathcal{W}_{\mathbb{F}_n, \partial \mathbb{F}_n} &= \mathrm{Ker}\, W_{\mathbb{F}_n, \partial \mathbb{F}_n} \\ &\cong \mathbb{T}^n \oplus e_*(\mathrm{B}^1(\mathbb{F}_n; \mathrm{Map}(\partial \mathbb{F}_n, \mathbb{R}))). \end{aligned}$$

Example 2.8. *Let Γ be a torsion-free hyperbolic group such that $\partial\Gamma$ is homeomorphic to the Sierpinski carpet S. Every non-identity element γ of Γ acts on $\partial\Gamma \approx S$ as a hyperbolic homeomorphism: there exist exactly two fixed points $x_{+\infty}$ and $x_{-\infty}$ such that for each point $x \in \partial\Gamma \setminus \{x_{-\infty}\}$ (resp. $x \in \partial\Gamma \setminus \{x_{+\infty}\}$), $\lim_{n\to\infty} \gamma^n x = x_{+\infty}$ (resp. $\lim_{n\to-\infty} \gamma^n x = x_{-\infty}$). Noticing that generators of $\check{\mathrm{H}}^1(S; \mathbb{Z}) \cong \oplus_\infty \mathbb{Z}$ are represented by the "peripheral circles" of S ([5, p.651]), we see that every hyperbolic homeomorphism of S does not fix any peripheral circle of S. Thus we have $\mathrm{Fix}(\gamma^*) = \{0\} \subset \check{\mathrm{H}}^1(S; \mathbb{Z})$, and hence $\check{\mathrm{H}}^1(S; \mathbb{Z})^\Gamma = \{0\}$. By Theorem 2.6, we obtain the following:*

$$\mathrm{Ker}\, W_{\Gamma, \partial\Gamma} = \hat{\Gamma} \oplus e_*(\mathrm{B}^1(\Gamma; \mathrm{Map}(\partial\Gamma, \mathbb{R}))).$$

Now assume that $\check{\mathrm{H}}^1(X;\mathbb{Z})^\Gamma = \{0\}$ and suppose further that, either

(1) the induced homeomorphism $\gamma : X \to X$ has a fixed point for each $\gamma \in \Gamma$, or

(2) X is connected.

Then Theorem 2.5 and Theorem 2.6 reduce the detection of the kernel $\mathrm{Ker}\, W_{\Gamma,X}$ to the detection of $\mathrm{B}^1(\Gamma; \mathrm{Map}(X,\mathbb{R}))$. When the action is minimal in the sense that the orbit $\Gamma \circ x$ is dense in X for each $x \in X$, we may appeal to the following generalization of the classical Gottschalk-Hedlund theorem. For a map $f \in \mathrm{Map}(X,\mathbb{R})$, let $\| f \| = \max_{x\in X} |f(x)|$. We say that a 1-cochain $\varphi : \Gamma \to \mathrm{Map}(X,\mathbb{R})$ is *uniformly bounded* if there exists a constant $M > 0$ such that $\| \varphi_\gamma \| < M$ for each $\gamma \in \Gamma$.

Theorem 2.9 ([7], Theorem 2.1). *Let Γ be a discrete group acting minimally on a compact Hausdorff space X. Then a 1-cocycle $\varphi \in \mathrm{Z}^1(\Gamma; \mathrm{Map}(X,\mathbb{R}))$ is a 1-coboundary if and only if φ is uniformly bounded.*

The action of an arbitrary torsion-free hyperbolic group Γ on its boundary is minimal ([4]) and each element $\gamma \in \Gamma$ has two fixed points on $\partial\Gamma$. Combining these, our results are summarized as follows.

Corollary 2.10. *Let Γ be a torsion-free hyperbolic group acting on its boundary $\partial\Gamma$. Assume that $\check{\mathrm{H}}^1(\partial\Gamma,\mathbb{Z})^\Gamma = \{0\}$. Then we have the following*

(1) $\mathrm{Im}\, W_{\Gamma,\partial\Gamma} = \mathrm{B}^1(\Gamma; \check{\mathrm{H}}^1(\partial\Gamma;\mathbb{Z}))$,

(2) $\mathrm{Ker}\, W_{\Gamma,\partial\Gamma} = \hat{\Gamma} \oplus e_*(\mathrm{B}^1(\Gamma; \mathrm{Map}(\partial\Gamma,\mathbb{R})))$*, and*

(3) $\mathrm{B}^1(\Gamma; \mathrm{Map}(\partial\Gamma,\mathbb{R}))$ *is the group of uniformly bounded 1-cocycles.*

These results suggest that dynamical information of the action shall be lost in the cohomology level. The following result makes this statement explicit. The canonical projection $\mathrm{Z}^1(\Gamma; M) \to \mathrm{H}^1(\Gamma; M)$ is denoted by $q_M : \mathrm{Z}^1(\Gamma; M) \to \mathrm{H}^1(\Gamma; M)$. For simplicity, let $q := q_{\mathrm{Map}(X,\mathbb{T})}$ and $q_1 := q_{\mathrm{Map}(X,\mathbb{T})_1}$.

Proposition 2.11. *Under the notation above, we have the following.*

(1) $q(\mathcal{W}_{\Gamma,X}) = q(\mathrm{Ker}\, W_{\Gamma,X}) = q(\hat{\Gamma})$.

(2) Assume that $\check{\mathrm{H}}^1(X,\mathbb{Z})^\Gamma = \{0\}$. *If, either*

(2a) the induced homeomorphism $\gamma : X \to X$ has a fixed point for each $\gamma \in \Gamma$, or

(2b) X is connected,

then the restriction $q|\hat{\Gamma} : \hat{\Gamma} \to q(\hat{\Gamma})$ is an isomorphism.

Proof. (1) is an immediate consequence of Theorem 2.5 and the equality

$$\mathcal{W}_{\Gamma,X} = \hat{\Gamma}\cdot \mathrm{B}^1(\Gamma, \mathrm{Map}(X;\mathbb{T})).$$

For the proof of (2), consider the following diagram:

$$\begin{array}{ccccc}
\hat{\Gamma} & \xrightarrow{\subset} & \mathrm{Z}^1(\Gamma;\mathrm{Map}(X,\mathbb{T})_1) & \longrightarrow & \mathrm{Z}^1(\Gamma;\mathrm{Map}(X,\mathbb{T})) \\
 & & {\scriptstyle q_1}\downarrow & & {\scriptstyle q}\downarrow \\
\mathrm{H}^0(\Gamma;\check{\mathrm{H}}^1(X,\mathbb{Z})) & \longrightarrow & \mathrm{H}^1(\Gamma;\mathrm{Map}(X,\mathbb{T})_1) & \xrightarrow{i_*} & \mathrm{H}^1(\Gamma;\mathrm{Map}(X;\mathbb{T})),
\end{array}$$

where the bottom row is a part of the Bockstein exact sequence induced by

$$0 \longrightarrow \mathrm{Map}(X,\mathbb{T})_1 \xrightarrow{i} \mathrm{Map}(X,\mathbb{T}) \xrightarrow{j} \check{\mathrm{H}}^1(X;\mathbb{Z}) \longrightarrow 0$$

By the hypothesis, we obtain that $\mathrm{H}^0(\Gamma;\check{\mathrm{H}}^1(X;\mathbb{Z})) = \check{\mathrm{H}}^1(X;\mathbb{Z})^{\Gamma} = 0$ and hence i_* is a monomorphism.

In order to prove that $q|\hat{\Gamma}$ is a monomorphism, take $\chi \in \hat{\Gamma}$ such that $q(\chi) = 0 \in \mathrm{H}^1(\Gamma, \mathrm{Map}(X,\mathbb{T}))$. Since i_* is a monomorphism, we see $q_1(\chi) = 0$ and hence $\chi \in \mathrm{B}^1(\Gamma, \mathrm{Map}(X,\mathbb{T})_1)$. On the other hand, by (2.1), the assumption $\check{\mathrm{H}}^1(X,\mathbb{Z})^{\Gamma} = \{0\}$ and Theorem 2.6, we obtain

$$\begin{aligned}
\mathrm{B}^1(\Gamma, \mathrm{Map}(X,\mathbb{T})_1) &= \delta^0 e_*(\mathrm{Map}(X,\mathbb{R})) = \delta^0(\mathrm{Map}(X,\mathbb{T})_{\Gamma}) \\
&= e_*(\mathrm{B}^1(\Gamma, \mathrm{Map}(X,\mathbb{R})).
\end{aligned}$$

By Theorem 2.5 and Theorem 2.6, we see that, under the hypothesis (2a) or (2b), the equality

$$\hat{\Gamma} \cap e_*(\mathrm{B}^1(\Gamma, \mathrm{Map}(X,\mathbb{R}))) = \hat{\Gamma} \cap \delta^0(\mathrm{Map}(X,\mathbb{T})_{\Gamma}) = \{\mathbf{1}\}$$

holds. Hence $\chi = 1$ and we obtain the desired conclusion. □

Acknowledgment. The author expresses his sincere and deep gratitude to the referee whose comments considerably improved the results with simpler proofs.

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