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# A topologist's introduction to the motivic homotopy theory for transformation group theorists - 1

By

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**Abstract**

An introductory survey of the motivic homotopy theory for topologisits is given, by focusing upon the algebraic $K$-theory representability and the homotopy purity. The aim is to provide readers with some background to read the Morel-Voevodsky IHES paper. In doing so, some basic properties of algebraic $K$-theory are also reviewed following Schlichting.

## § 1. Introduction

This grew out of a set of slides of my introductory lecture on the (unstable) motovic homotopy theory presented to transformation group theorists. I assumed some familiarity with the *simplicial* model category theory, which plays some vital roles in the motovic homotopy theory, and basic commutative algebra and algebraic geometry. My aim is to convey swiftly the basic ideas of the Morel-Voevodsky IHES paper [28], by focusing upon the $K$-theory representability and the homotopy purity of the $\mathbb{A}^1$-homotopy theory. For both the $K$-theory representability and the homotopy purity, I tried to supply some more backgrounds not touched in the original paper of Morel-Voevodsky.

This is because they together symbolize the clever choice of the Nisnevich topology, which resides between the Zariski topology and the étale topology: The Nisnivech topology is (even after imposing the $\mathbb{A}^1$-equivalence, under the regular base scheme

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assumption) rich enough to represent the $K$-theory, as the Zarisiki topology; the Nisnevich topology, after imposing of the $\mathbb{A}^1$-equivalence, is user-friendly enough to satisfy the homotopy purity, which is a motivic analogue of the excision theorem of the classical homotopy theory, just as the étale topology.

Since, this grew out of slides, some concepts are not defined and some expressions are somewhat umbiguous. However, I hope the brevity and the conciseness of this exposition would allow interested topologists to spend just a day or two on this exposition to be motivated and prepared to read the original paper of Morel-Voevodsky [28]. I am also indebted to the referee for many invaluable comments on the preliminary version of this article, which greatly helped to improve the quality of this article. In fact, the initial version of this paper was stifled with the imposed 20 page limit. However, the referee kindly pretended he does not believe Lemma 3.10, which was briefly explained in just 10 lines in the original Morel-Voevodsky paper [28], and challenged to supply a detailed proof if it were really true. I recognized this as a secret sign which entitles me to break the imposed 20 page limit. At the same time, I took an advantage of this opportunity by supplying more comprehensive information about $K$-theory following the nice paper of Schlichting [38]. I have also supplied some more updated information about algebraic $K$-theory in Remark 2. Here, I would like to express my gratitude to David Gepner for supplying useful information. I hope the detailed proof of Lemma 3.10 and Remark 2 would provide useful information to interested topologists who are not so farmiliar with this kind of mathematics.

Finally, I would like to express my highest gratitude to Professor Mikiya Masuda for patiently waiting for me to write this up.

## § 2. Summary of unstable $\mathbb{A}^1$-homotopy theory

### § 2.1. Nisnevich topology

#### 2.1.1. A "local" preview of the Nisnevich topology

$$\boxed{\text{Zariski topology} \preceq \text{Nisnevich topology} \preceq \text{étale topology}}$$

In fact, the "local ring"at $x \in X$ is:

**<u>in the Zariski topology case</u>** , ordinary local ring $\mathcal{O}_{X,x}$

**<u>in the Nisnevich topology case</u>** , the **<u>henselization</u>** $\mathcal{O}^h_{X,x}$ of $\mathcal{O}_{X,x}$

**<u>in the étale topology case</u>** , (the) **<u>strict henselization</u>** $\mathcal{O}^{sh}_{X,x}$ of $\mathcal{O}_{X,x}$

Here,

- A local ring $(A, \mathfrak{m})$ is called ***Henselian***, if

  For any $P(X) \in A[X],$ monic, such that there exists $a_0 \in A,\ P(a_0) \in \mathfrak{m}, P'(a_0) \notin \mathfrak{m}$, there exists $a \in A,$ such that $P(a) = 0$

- A Henselian local ring $(A, \mathfrak{m})$ is called ***strict Henselian***, if the residue field $A/\mathfrak{m}$ is separably closed.

- **henselization** is determined, unique up to unique isomorphism.

- **strict henselization** is determined, unique, but only up to non-unique isomorphism.

For more on the Henselian rings and henselizations, we refer the reader to Nagata's book [31], Raynaud's book [35], and the fourth volume of EGA IV [15].

### 2.1.2. Definition of the Nisnevich topology

Throughout the rest of this article, we fix a Noetherian scheme $S$ of finite dimension. The full subcategory of $Sch/S$ consisting of smooth schemes of finite type over $S$ is denoted by $Sm/S$.

**Proposition 2.1** ([28, p.95, Proposition 1.1]). *Let $\{U_i\} \to X$ be a finite family of étale morphisms in $Sm/S$. Then the following conditions are equivalent:*

1. *For any $x \in X$, there exist $i$ and $u \in U_i$, such that*

$$\mathcal{O}_{X,x}/\mathfrak{m}_{X,x} \xleftarrow{\cong} \mathcal{O}_{U_i,u}/\mathfrak{m}_{U_i,u}$$

2. *For any $x \in X$, the following morphism of $S$-schemes admits a section:*

$$\coprod_i \left(U_i \times_X Spec\mathcal{O}^h_{X,x}\right) \to Spec\mathcal{O}^h_{X,x}$$

**Definition 2.2** ([28, p.95, Definition 1.2]). Such families of étale morphisms $\{U_i\} \to X$ in $Sm/S$ form a pretopology on the category $Sm/S$. The corresponding topology is called the ***Nisnevich topology***, and the corresponding site is denoted $(Sm/S)_{Nis}$.

### 2.1.3. The elementary distinguished square characterization of the Nisnevich sheaf

**Definition 2.3** ([28, p.96, Definition 1.3]). An ***elementary distinguished square*** in $(Sm/S)_{Nis}$ is a cartesian square of the form

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] & V \arrow[d, "p"', "\text{étale}"] \\ U \arrow[r, "i", "\text{open emb.}"'] & X \end{tikzcd}$$

such that $p^{-1}((X \setminus U)_{red}) \to (X \setminus U)_{red}$ is an isomorphism.

This special Nisnevich cover is of great importance because of the following:

**Proposition 2.4** ([28, p.96, Proposition 1.4]). *A presheaf of sets $F$ on $Sm/S$ is a Nisnevich sheaf if and only if, for any elementary distinguished square, the following commutative diagram is cartesian:*

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d] & F(U) \arrow[d] \\ F(V) \arrow[r] & F(U \times_X V) \end{tikzcd}$$

## § 2.2. Simplicial (Pre)sheaf

### 2.2.1. Simplicial model caregory structures

For the rest of this article, $T$ stands for a site, which we shall soon specialize to the case $T = (Sm/S)_{Nis}$. As usual, let $Preshv(T) := (Sets)^{T^{op}}$ stand for the category of presheaves of sets on $T$, and let $Shv(T)$ stand for the full subcategory of $Preshv(T)$, consisting of sheaves of sets.

**Example 2.5.** We have the following fully faithful embedding:

$$Sm/S \overset{\text{fully faithful}}{\hookrightarrow} Shv(Sm/S)_{Nis}$$

Actually, any scheme is already a sheaf in the etale topolgy [26, p.54, Remark 1.12].

However, we shall mostly work in their simplicial analogues. So, let $\Delta^{op}Preshv(T) \cong (\Delta^{op}Sets)^{T^{op}}$ be the category of simplicial objects of $Preshv(T)$, which can be identified with the category of presheaves of simplicial sets on $T$. Similarly, we let $\Delta^{op}Shv(T)$ be the category of simplicial objects of $Shv(T)$.

**Example 2.6.** For any simplicial sheaf of monoids $M$, its ***classifying space*** $BM$ is also a simplicial sheaf [28, p.123]. In fact, $BM$ is defined to be the diagonal simplicial sheaf of the bisimplicial sheaf:

$$BM(U) : n \mapsto Cat([n], M_n(U)) = (M_n(U))^n \ , \tag{2.1}$$

where we have regarded the ordered set $[n] := \{0 < 1 < \cdots < n-1 < n\}$ as a category and the monoid $M_n(U)$ as a category with a single object, as usual. From this description (2.1) of $BM$, it would be clear to see $BM$ is once again a simplicial sheaf.

Just like the ordinary homotopy theory, we shall eventually (e.g. Theorem 2.17, Theorem 2.19, Theorem 2.26, Theorem 3.3, Theorem 3.4, and Theorem 3.12) work in the ponted analogues $\Delta^{op}Preshv_\bullet(T)$ and $\Delta^{op}Shv(T)_\bullet$ of $\Delta^{op}Preshv(T)$ and $\Delta^{op}Shv(T)$, respectively. In the pointed setting, the most fundamental object is the ***simplicial circle*** $S^1_s$, defined by

$$S^1_s = \Delta^1/\partial\Delta^1, \tag{2.2}$$

which is regarded as a constant simplicial presheaf. we would like to stress the subscript $s$ here; this is because there is another circle $S^1_t = \left(\mathbf{A}^1 \setminus \{0\}, 1\right)$, called the ***Tate circle*** $S^1_t$, in our principal case $T = (Sm/S)_{Nis}$.

Now, let us present a couple of basic constructions of the pointed simplicial sheaves:

**Example 2.7.** For any pointed simplicial presheaf $(\mathcal{P}, p)$, define the pointed simplicial sheaf $\Sigma_s(\mathcal{P}, p)$, called ***suspension***, by applying the degreewise sheafication functor $a$ :

$$\Sigma_s\left(\mathcal{P}, p\right) = a\left(S^1_s \wedge (\mathcal{P}, p)\right) \tag{2.3}$$

Then, since the functor $S^1_s \wedge (-)$ commutes with the direct limit, we easily see the degreewise sheaficatin fuctor $(\mathcal{P}, p) \to a\left(\mathcal{P}, p\right)$ induces an isormorphism of simplicial sheaves:

$$\Sigma_s\left(\mathcal{P}, p\right) = a\left(S^1_s \wedge (\mathcal{P}, p)\right) \xrightarrow{\cong} a\left(S^1_s \wedge a(\mathcal{P}, p)\right) = \Sigma_s a\left(\mathcal{P}, p\right) \tag{2.4}$$

Next, for a simplicial <u>sheaf</u> $(\mathcal{X}, x)$, define the pointed simplicial sheaf $\Omega^1_s(\mathcal{X}, x)$, called ***loop space***, by using the mapping space pointed simplicial presheaf functor $\underline{Hom}_{\Delta^{op}Preshv_\bullet(T)}(-,-) \in \Delta^{op}Preshv_\bullet(T)$ :

$$\Omega^1_s(\mathcal{X}, x) = \underline{Hom}_{\Delta^{op}Preshv_\bullet(T)}\left(S^1_s, (\mathcal{X}, x)\right) \tag{2.5}$$

In fact, $\Omega^1_s(\mathcal{X}, x)$ becomes a pointed simplicial <u>sheaf</u>, because, for any pointed simplicial presheaf $(\mathcal{P}, p)$, the degreewise sheaficatin fuctor $(\mathcal{P}, p) \to a\left(\mathcal{P}, p\right)$ induces an isomorphism

$$
\begin{aligned}
& Hom_{\Delta^{op}Preshv_\bullet(T)}\left(a\left(\mathcal{P},p\right),\Omega_s^1(\mathcal{X},x)\right)\\
\cong & Hom_{\Delta^{op}Preshv_\bullet(T)}\left(a\left(\mathcal{P},p\right),\underline{Hom}_{\Delta^{op}Preshv_\bullet(T)}\left(S_s^1,(\mathcal{X},x)\right)\right)\\
\cong & Hom_{\Delta^{op}Preshv_\bullet(T)}\left(S_s^1\wedge a\left(\mathcal{P},p\right),(\mathcal{X},x)\right) \overset{(\mathcal{X},x)\text{: sheaf}}{\cong} Hom_{\Delta^{op}Preshv_\bullet(T)}\left(\Sigma_s a\left(\mathcal{P},p\right),(\mathcal{X},x)\right)\\
\overset{(2.4)}{\cong} & Hom_{\Delta^{op}Preshv_\bullet(T)}\left(\Sigma_s\left(\mathcal{P},p\right),(\mathcal{X},x)\right) \overset{(\mathcal{X},x)\text{: sheaf}}{\cong} Hom_{\Delta^{op}Preshv_\bullet(T)}\left(S_s^1\wedge\left(\mathcal{P},p\right),(\mathcal{X},x)\right)\\
\cong & Hom_{\Delta^{op}Preshv_\bullet(T)}\left(\left(\mathcal{P},p\right),\underline{Hom}_{\Delta^{op}Preshv_\bullet(T)}\left(S_s^1,(\mathcal{X},x)\right)\right)\\
\cong & Hom_{\Delta^{op}Preshv_\bullet(T)}\left(\left(\mathcal{P},p\right),\Omega_s^1(\mathcal{X},x)\right) \quad \square
\end{aligned}
$$

Just like the case of the classical homotopy theory, the suspension functor $\Sigma_s$ and the loop space functor $\Omega_s^1$ are adjoint to each other:

$$\Sigma_s : \Delta^{op}Shv_\bullet(T) \rightleftarrows \Delta^{op}Shv(T)_\bullet : \Omega_s^1 \tag{2.6}$$

Now, the following special simplicial sheaf will play an important role in the motivic applications to $K$-theory:

**Corollary 2.8.** *The simplicial presheaf $B\left(\coprod_{n\geqq 0} BGL_n\right)$, defined by*

$$B\left(\coprod_{n\geqq 0} BGL_n\right) = \left(U \mapsto B\left(\coprod_{n\geqq 0} BGL_n(\mathcal{O}(U))\right)\right) \tag{2.7}$$

*and its loop space*

$$\Omega_s^1 B\left(\coprod_{n\geqq 0} BGL_n\right) \tag{2.8}$$

*are both simplicial sheaves, where the simplicial presheaf $\coprod_{n\geqq 0} BGL_n$ is regarded as a simplicial monoid by the concatenation (see e.g.* (3.69)*).*

*Proof.* In fact, for each $n \geqq 0$, algebraic group $GL_n$ is a sheaf by Example 2.5, and so, $BGL_n$ is a simplicial sheaf by the constant simplicial monoid case of Example 2.6. So, the simplicial presheaf $\coprod_{n\geqq 0} BGL_n$, regarded as a simplicial monoid by the concatenation, is actully simplicial sheaf of monoid. Then, we see $B\left(\coprod_{n\geqq 0} BGL_n\right)$ is a simplicial sheaf by Example 2.6. Consequently, $\Omega_s^1 B\left(\coprod_{n\geqq 0} BGL_n\right)$ is also a pointed simplicial sheaf, by Example 2.7. $\square$

To deal with simplicial objects, we freely make use of standard techniques of the (simplicial) model categories and their mapping spaces. Results and proofs on these subjects can be found in Hovey's book [18], Hirschhorn's book [16] and papers [8, 9, 10] by Dwyer and Kan.

Now the following theorem was first suggested by Joyal in his letter to Grothendieck:

**Theorem 2.9** ( (Joyal) [19] [28, p.49, Theorem 1.4]). *$\Delta^{op} Shv(T)$ is a proper closed simplicial model category with:*

**Weak equivalences:** *$\pi_0$ equivalence and the stalkwise weak equivalences of simplicial sets, which are characterized by the isomorphism of the $\pi_n$ sheaves for all $n \geqq 1$.*

**Cofibrations:** *monomorphisms*

**Fibrations:** *morphisms having the right lifting property with respect to trivial cofibrations*

**Theorem 2.10** ((Jardine)[19][20, Theorem 11.6]). *$\Delta^{op} Preshv(T)$ is a proper closed simplicial model category with:*

**Weak equivalences:** *$\pi_0$ equivalence and the stalkwise weak equivalences of simplicial sets, which are characterized by the isomorphism of the $\pi_n$ sheaves for all $n \geqq 1$.*

**Cofibrations:** *monomorphisms*

**Fibrations:** *morphisms having the right lifting property with respect to trivial cofibrations*

**Theorem 2.11** ((Jardine) [19][20, Theorem 12.1]). *The above model structures on $\Delta^{op} Shv(T)$ and $\Delta^{op} Preshv(T)$ are Quillen equivalent by the sheafication and the inclusion:*

$$\Delta^{op} Preshv(T) \rightleftarrows \Delta^{op} Shv(T)$$

**Definition 2.12** ([28, p.49]). $\mathcal{H}_s(Sm/S)_{Nis}$ is defined to be the homotopy category of $\Delta^{op} Shv(Sm/S)_{Nis}$ with respect to the Joyal model structure, which is, by Theorem 2.11, equivalent to the homotopy category of $\Delta^{op} PreShv(Sm/S)_{Nis}$ with respect to the Jardine model structure. Here, the subscript $s$ is used in $\mathcal{H}_s(Sm/S)_{Nis}$, because Morel-Voevodsky [28] called the Joyal model structure the ***simplicial model structure***.

**Definition 2.13** ([28, p.82]). The pointed analogue $\Delta^{op} Shv_\bullet(Sm/S)_{Nis}$ of $\Delta^{op} Shv(Sm/S)_{Nis}$ also posseses the model category structure with respect to the Jardine model structure, by declaring that morphisms in $\Delta^{op} Shv_\bullet(Sm/S)_{Nis}$ are cofibrations, fibrations, or weak equivalences, if they are so after applying the forgetful functor $\Delta^{op} Shv_\bullet(Sm/S)_{Nis} \to \Delta^{op} Shv(Sm/S)_{Nis}$.

Its homotopy category is denoted by $\mathcal{H}_{s,\bullet}(Sm/S)_{Nis}$.

**Proposition 2.14.** *When $T = (Sm/S)_{Nis}$, the adjunction* (2.6)

$$\Sigma_s : \Delta^{op}Shv_\bullet(Sm/S)_{Nis} \rightleftarrows \Delta^{op}Shv_\bullet(Sm/S)_{Nis} : \Omega_s^1$$

*becomes a Quillen adjunction. Consequently, for any fibrant $(\mathcal{X}, x) \in \Delta^{op}Shv_\bullet(Sm/S)_{Nis}$, $\Omega_s^1(\mathcal{X}, x) \in \Delta^{op}Shv_\bullet(Sm/S)_{Nis}$ is a fibrant. However, $\Omega_s^1$ preserves not only fibrations and trivial fibrations, but also weak equivalences.*

*An outline of the proof of Propositio 2.14.* To show the Quillen adjunction property, we check $\Sigma_s$ preserves the cofibrations and trivial cofibrations. To show $\Omega_s^1$ preserves weak equivalences, we observe that the weak equivalences are characterized by the $\pi_n$ sheaves (see Theorem 2.9), and use $\pi_n\Omega_s^1 \cong \pi_{n+1}$. □

### 2.2.2. Fibrant simplicial (pre)sheaf

In both cases $\mathcal{C} = \Delta^{op}Preshv(T), \Delta^{op}Shv(T)$, every object is cofibrant, and fibrant objects, and more generally fibrations, are of particular importance:

**Proposition 2.15** (Fibrations are sectionwise Kan fibrations)**.**
*In both cases $\mathcal{C} = \Delta^{op}Preshv(T), \Delta^{op}Shv(T)$, given $U \in T$,*

- *every fibration $p : \mathcal{X} \to \mathcal{Y}$ induces a Kan fibration*

$$p(U) : \mathcal{X}(U) \to \mathcal{Y}(U)$$

- *every fibrant object $\mathcal{X}$ yields a Kan complex $\mathcal{X}(U)$.*

*Proof.* In fact, since either one of $\mathcal{C} = \Delta^{op}Preshv(T), \Delta^{op}Shv(T)$ is a simplicial model category, we have a bifunctor

$$\mathbf{hom}_{\mathcal{C}} : \mathcal{C}^{op} \times \mathcal{C} \to \Delta^{op}Sets$$

s.t.

$$\begin{aligned}(p(U) : \mathcal{X}(U) \to \mathcal{Y}(U)) &\cong \mathbf{hom}_{\mathcal{C}}(U, p)\\ \in \mathbf{hom}_{\mathcal{C}}(\text{cofibrants}, \text{fibrations}) &\subseteqq \{\text{Kan fibrations}\}\end{aligned}$$

□

**Proposition 2.16** (Stalkwise equiv. between fibrant objects are sectionwise equiv. [20])**.**
*In both cases $\mathcal{C} = \Delta^{op}Preshv(T), \Delta^{op}Shv(T)$, every equivalence*

$$f : \mathcal{X} \to \mathcal{Y}$$

*between fibrant objects is a sectionwise equivalence, i.e. $^\forall U \in T$,*

$$f(U) : \mathcal{X}(U) \to \mathcal{Y}(U)$$

*is a weak equivalence of simplicial sets.*

*Proof.*

- Since every objects in $\mathcal{C}$ is cofibrant, $f$ is a weak equivalence between objects which are simultaneously cofibrant and fibrant.
- Thus, $f$ becomes a homotopy equivalence, defined using the cylinder object constructed by $\times\Delta^1$, by the general theory of simplicial model category.
- This homotopy equivalence induces a weak equivalence of simplicial sets at each section $U \in T$.

□

The following result is implicit in [19, p.72-73]:

**Theorem 2.17** (Fibrants are representable).
*In both cases $\mathcal{C} = \Delta^{op}Preshv(T), \Delta^{op}Shv(T)$, suppose a fibrant $\widetilde{\mathcal{X}}$ is equipped with a global base point $*$. Then, for any $U \in \mathcal{C}$ and $n \in \mathbb{Z}_{\geqq 0}$,*

$$\pi_n\left(\widetilde{\mathcal{X}}(U)\right) \cong \mathrm{Hom}_{\mathcal{H}(\mathcal{C}_\bullet)}\left(S^n \wedge U_+, (\widetilde{\mathcal{X}}, *)\right) \tag{2.9}$$

*Here, $\mathcal{C}_\bullet$ is the pointed model category obtained from $\mathcal{C}$, and $\mathcal{H}(\mathcal{C}_\bullet)$ is the resulting homotopy category.*

*Proof.* First, recall some facts about the set of homotopy classes of maps in a simplicial model category $\mathcal{C}$:

- If $\mathcal{F}$ is fibrant, equipped with a global base point $*$, then $(\mathcal{F}, *)^{(\Delta^n, \partial\Delta^n)}$ is also fibrant.
- Denote by $\pi_{\mathcal{C}}$ the set of homotopy classes quotiented out by the homotopy relation given by the cylinder object $(-) \times \Delta^1$:

$$^\forall\mathcal{X}, {^\forall\mathcal{Y}} \in \mathcal{C}, \quad \pi_{\mathcal{C}}(\mathcal{X}, \mathcal{Y}) := \mathcal{C}(\mathcal{X}, \mathcal{Y}) \Big/ \left((-) \times \Delta^1\text{-homotopy relation}\right)$$

- There is a canonical map to the hom set of the homotopy category $\mathcal{H}(\mathcal{C})$:

$$\pi_{\mathcal{C}}(\mathcal{X}, \mathcal{Y}) \to \mathrm{Hom}_{\mathcal{H}(\mathcal{C})}(\mathcal{X}, \mathcal{Y}),$$

  which is an isomorphism if $\mathcal{Y}$ is fibrant.

Now, the isomorphism (2.9) is obtained by the following composition of isomorphisms, where the above observation is applied to justify the isomorphism ★, whereas the other isomorphisms are more standard consequences of the simplicial model category structure:

$$\pi_n\left(\widetilde{\mathcal{X}}(U),*\right) \cong \pi_n\left(\mathbf{hom}_{\mathcal{C}}\left(U,\widetilde{\mathcal{X}}\right),*\right) \cong \pi_{\Delta^{op}Sets}\left((\Delta^n,\partial\Delta^n),\left(\mathbf{hom}_{\mathcal{C}}\left(U,\widetilde{\mathcal{X}}\right),*\right)\right)$$
$$\cong \pi_{\mathcal{C}}\left((\Delta^n,\partial\Delta^n)\times U,(\widetilde{\mathcal{X}},*)\right) \cong \pi_{\mathcal{C}}\left(U,(\widetilde{\mathcal{X}},*)^{(\Delta^n,\partial\Delta^n)}\right) \overset{\bigstar}{\cong} \mathrm{Hom}_{\mathcal{H}(\mathcal{C})}\left(U,(\widetilde{\mathcal{X}},*)^{(\Delta^n,\partial\Delta^n)}\right)$$
$$\cong \mathrm{Hom}_{\mathcal{H}(\mathcal{C}_\bullet)}\left(U_+,(\widetilde{\mathcal{X}},*)^{(\Delta^n,\partial\Delta^n)}\right) \cong \mathrm{Hom}_{\mathcal{H}(\mathcal{C}_\bullet)}\left((\Delta^n,\partial\Delta^n)\times(U_+,+),(\widetilde{\mathcal{X}},*)\right)$$
$$\cong \mathrm{Hom}_{\mathcal{H}(\mathcal{C}_\bullet)}\left(S^n\wedge U_+,(\widetilde{\mathcal{X}},*)\right)$$

□

### 2.2.3. Descent

In applications, there are many important non-fibrants, which are "almost as nice as" fibrants. So, we slightly enlarge the category of fibrant objects as follows:

**Definition 2.18** ([20, p.24]). In both cases $\mathcal{C} = \Delta^{op}Preshv(T), \Delta^{op}Shv(T)$, $\mathcal{X} \in \mathcal{C}$ is said to ***satisfy descent*** in $\mathcal{C}$, if there is a fibrant replacement

$$j : \mathcal{X} \to \widetilde{\mathcal{X}}$$

which is simultaneously a sectionwise equivalence, i.e. for any $U \in \mathcal{C}$,

$$j(U) : \mathcal{X}(U) \to \widetilde{\mathcal{X}}(U)$$

is a weak equivalence of simplicial sets.

By Prop 2.16, $\mathcal{X} \in \mathcal{C}$ satisfies descent if and only if ANY fibrant replacement is simultaneously a sectionwise equivalence.

Now the following theorem is an immediate consequence of Theorem 2.17:

**Theorem 2.19** (descent implies representability).
*In both cases $\mathcal{C} = \Delta^{op}Preshv(T), \Delta^{op}Shv(T)$, if*

- *$\mathcal{X}$ satisfies descent in $\mathcal{C}$, with a sectionwise equivalent fibrant replacement*

$$j : \mathcal{X} \to \widetilde{\mathcal{X}}$$

- *$\mathcal{X}$ is equipped with a global base point $*$, which also serves as a global base point of $\widetilde{\mathcal{X}}$ via $j : \mathcal{X} \to \widetilde{\mathcal{X}}$.*

*Then, for any $U \in \mathcal{C}$ and $n \in \mathbb{Z}_{\geqq 0}$,*

$$\pi_n\left(\mathcal{X}(U)\right) \cong \mathrm{Hom}_{\mathcal{H}(\mathcal{C}_\bullet)}\left(S^n \wedge U_+, (\widetilde{\mathcal{X}}, *)\right)$$

*Here, $\mathcal{C}_\bullet$ is the pointed model category obtained from $\mathcal{C}$, and $\mathcal{H}(\mathcal{C}_\bullet)$ is the resulting homotopy category.*

### 2.2.4. B.G. property

We now restrict to the special case of $T = (Sm/S)_{Nis}$.

Recalling Proposition 2.4: the characterization of the Nisnevich sheaf in terms of the elementary distinguished square, we may expect the following concept would be important in the simplicial setting:

**Definition 2.20** ([28, p.100, Definition 1.13]). A simplicial presheaf

$$\mathcal{X} : (Sm/S)_{Nis} \to \Delta^{op}Sets$$

is said to ***have the B.G. property*** with respect to $\mathcal{A}$, if and only if, for any elementary distinguished square with $X \in \mathcal{A}$,

(2.10)
$$\begin{array}{ccc} \mathcal{X}(X) & \longrightarrow & \mathcal{X}(U) \\ \downarrow & & \downarrow \\ \mathcal{X}(V) & \longrightarrow & \mathcal{X}(U \times_X V) \end{array}$$

is homotopy cartesian.

As we hoped, we easily obtain the following:

**Proposition 2.21** ([28, p.100, Remark 1.15]). *Any fibrant Nisnevich simplicial sheaf has the B.G. property for all smooth $S$-schemes, i.e. any fibrant object $\mathcal{X}$ of $\Delta^{op}Shv(Sm/S)_{Nis}$ has the B.G. property for all smooth $S$-schemes.*

*Proof.* In fact, from the levelwise Nisnevich sheaf property, (2.10) is cartesian. Moreover, since $\mathcal{X}$ is fibrant and the open embedding $U \times_X V \to V$, which is a monomorphism, is a cofibration, $\mathcal{X}(V) \to \mathcal{X}(U \times_X V)$ in (2.10) is a Kan fibration. Thus, (2.10) is a homotopy cartesian, because the Joyal model category structure is (right) proper by Theorem 2.9. □

Now, the following is of particular importance:

**Theorem 2.22** ([28, p.100, Proposition 1.16]). *Suppose $\mathcal{X} \in \Delta^{op}Shv(Sm/S)_{Nis}$ is sectionwise fibrant, i.e. for any $U \in (Sm/S)_{Nis}$, $\mathcal{X}(U)$ is a Kan complex.*
*Then the following conditions for $\mathcal{X}$ are equivalent:*

- *satisfies descent in* $\Delta^{op}Shv(Sm/S)_{Nis}$*;*
- *has the B.G. property for all smooth* $S$*-schemes.*

We note that the sectionwise fibrant condition does not cause much technical restriction, for we can always apply the sectionwise functorial Kan's $Ex^{\infty}$-fuctor.

*Outline of the proof.*

descent $\implies$ B.G.

This is easy, since any fibrant object $\mathcal{X}$ of $\Delta^{op}Shv(Sm/S)_{Nis}$ has the B.G. property for all smooth $S$-schemes.

B.G. $\implies$ descent

This is more difficult, and Morel-Voevodsky reduced it to showing the following result:

**Lemma 2.23** ([28, p.101, Lemma 1.18]). *In* $\Delta^{op}Preshv(Sm/S)_{Nis}$*, every equivalence*

$$f : \mathcal{X} \to \mathcal{Y}$$

*between objects having the B.G. property for all smooth* $S$*-schemes is a sectionwise equivalence, i.e. for any* $U \in \mathcal{C}$*,*

$$f(U) : \mathcal{X}(U) \to \mathcal{Y}(U)$$

*is a weak equivalence of simplicial sets.*

Though we shall not reproduce the Morel-Voevodsky proof here, in view of Proposition 2.21, we note Lemma 2.23 is a generalization of Proposition 2.16.

□

### § 2.3. Unstable $\mathbb{A}^1$-homotopy theory

While $\mathcal{H}_s(Sm/S)_{Nis}$ contains rich information, it is still difficult to handle...
To make it more accesible, we must invert by the $\mathbb{A}^1$-equivalence, which we now define:

**Definition 2.24** ([28, p.86, Definition 3.1]).

- $\mathcal{Z} \in \Delta^{op}Shv(Sm/S)_{Nis}$ is called ***$\mathbb{A}^1$-local***, if,
  for any $\mathcal{Y} \in \Delta^{op}Shv(Sm/S)_{Nis}$, the projection $\mathcal{Y} \times \mathbb{A}^1 \to \mathcal{Y}$ induces a bijection:

$$\mathrm{Hom}_{\mathcal{H}_s(Sm/S)_{Nis}}(\mathcal{Y}, \mathcal{Z}) \to \mathrm{Hom}_{\mathcal{H}_s(Sm/S)_{Nis}}(\mathcal{Y} \times \mathbb{A}^1, \mathcal{Z})$$

- $(f : \mathcal{X} \to \mathcal{Y}) \in \Delta^{op} Shv(Sm/S)_{Nis}$ is called an ***$\mathbb{A}^1$-weak equivalence***, if for any $\mathbb{A}^1$-local $\mathcal{Z}$, the induced map

$$\mathrm{Hom}_{\mathcal{H}_s(Sm/S)_{Nis}}(\mathcal{Y}, \mathcal{Z}) \to \mathrm{Hom}_{\mathcal{H}_s(Sm/S)_{Nis}}(\mathcal{X}, \mathcal{Z})$$

is a bijection.

*What we really want* is the following:

**Theorem 2.25** ([28, p.86, Theorem 3.2; p.87, Example 4]). *$\Delta^{op} Shv(Sm/S)_{Nis}$ is a proper model category with:*

**Weak equivalences:** *$\mathbb{A}^1$-weak equivalence*

**Cofibrations:** *monomorphisms*

**Fibrations:** *morphisms having the right lifting property with respect to trivial cofibrations*

Accordingly, let us fix some notations:

**$\mathcal{H}(S)$:** the homotopy category of $\Delta^{op} Shv(Sm/S)_{Nis}$ w.r.t. the above model structure

**$*$:** the simplicial sheaf (associated to) $\triangle^0$, which is the final object in $\Delta^{op} Shv(T)$ and is called ***the point***

**$\mathcal{H}_\bullet(S)$:** the pointed analogue of $\mathcal{H}(S)$.

**Theorem 2.26** ([17, p.671, Theorem 3.1]). *Given a simplicial preshaef*

$$P : (Sm/S)_{Nis} \to \Delta^{op} Sets$$

(i) *Suppose $P$ has the B.G. property with respect to all smooth schemes of finite type, then $X \in (Sm/S)_{Nis}$,*

$$\pi_n (P(X)) \cong Hom_{\mathcal{H}_{s,\bullet}(Sm/S)_{Nis}} (S^n \wedge X_+, (aP)_f) \tag{2.11}$$

*Here, $X_+ = X \coprod S$ and $(aP)_f$ is the fibrant replacement in the Joyal model structure of the levelwise sheafication $aP$ of $P$ with respect to the Nisnevich topology.*
(ii) *Suppose further that $P$ is $\mathbb{A}^1$-homotopy invariant, then*

$$\pi_n (P(X)) \cong Hom_{\mathcal{H}_\bullet(S)} (S^n \wedge X_+, (aP)_f) \tag{2.12}$$

*Proof.* (i) By the assumption and Proposition 2.21, the canonical map $P \to (aP)_f$ is an equivalence between objects with the B.G. property with respect to all smooth schemes of finite type. Thus, it is a sectionwise weak equivalence by Lemma 2.23. Now the claim follows from Theorem 2.19.
(ii) When $P$ is $\mathbb{A}^1$-invariant, $(aP)_f$ is $\mathbb{A}^1$-fibrant in the sense of the model category structure in Theorem 2.25 by [28, p.80. Proposition 2.28]. Since every object is cofibrant in the model category structure in Theorem 2.25, by the standard result of the model category theory, every object in $Hom_{\mathcal{H}_\bullet(S)}(S^n \wedge X_+, (aP)_f)$ is represented by an honest morphism, and the equivalence relation is given by a cylinder object

$$S^n \wedge X_+ \xrightarrow[\mathbb{A}^1\text{-weak equivalence}]{i_0} Cyl\,(S^n \wedge X_+) \xleftarrow[\mathbb{A}^1\text{-weak equivalence}]{i_1} S^n \wedge X_+$$

However, as $(aP)_f$ is $\mathbb{A}^1$-local, this equivalence relation is already valied in $Hom_{\mathcal{H}_{s,\bullet}(Sm/S)_{Nis}}(S^n \wedge X_+, (aP)_f)$, and so, the canonical epimorphism

$$Hom_{\mathcal{H}_{s,\bullet}(Sm/S)_{Nis}}(S^n \wedge X_+, (aP)_f) \to Hom_{\mathcal{H}_\bullet(S)}(S^n \wedge X_+, (aP)_f)$$

turns out to be an isomorphism in this case. Thus, the claim follows from (i). □

*Remark* 1. Although we have attributed Theorem 2.26 to [17], it was certainly well-understood by the authors of [28]. Historically, Brown-Gersten [5] first considered the Zariski analogues of the B. G. property and Theorem 2.22, where the Zariski analogue of the elementary distinguished square, defined in Definition 2.3, is nothing but its special case when $p : V \to X$ is also an open embedding. With respect to such Zariski analogues, the Zariski analogue of Theorem 2.26 can be proven by essentially the same line as in the Nisnevich case presented above.

## § 3. Two advantages of unstable $\mathbb{A}^1$-homotopy theory

### § 3.1. $K$-theory representability

Before we explain the Morel-Voevodsky $K$-theory representability, we must prepare some basic facts about the algebraic $K$-theory, from "the pre-Voevodsky era." The original references here are Quillen [34], Waldhausen [46], and espcially Thomason-Trobaugh [44], but we mostly follow the "modern" streamlined presentation by Schlichting [37]. To quickly provide readers with a bird's-eye view of what is going on, we first summarize these basic facts, differing their (rough ideas of) proofs and definitions of some terminologies:

- For an exact category $\mathcal{E}$, we can canstruct the following three kinds of categories:

- we may apply the Quillen construction to obtain the category

$$Q\mathcal{E} \tag{3.1}$$

- we may associate the Waldhausen category (also known as the category with cofibrations and weak equivalences)

$$(\mathcal{E}, i) \tag{3.2}$$

with admissible monomorphisms as cofibrations and isomorphisms as weak equivalences

- we may associate the complicial exact category (i.e. an exact category equipped with a bi-exact action of the symmetric monoidal category $\mathrm{Ch}^b(\mathbb{Z})$ ) with weak equivalences

$$(\mathrm{Ch}^b\,\mathcal{E}, quis) \tag{3.3}$$

- For a complicial exact category with weak equivalences $(\mathcal{C}, w)$, we may also associate the Waldhausen category

$$(\mathcal{C}, w) \tag{3.4}$$

with admissible monomorphisms as cofibrations and morphisms in $w$ as weak equivalences. Note that this is in general different from another Waldhausen category (3.2)

$$(\mathcal{C}, i),$$

obtained by forgetting its complicial structure and weak equivalences,

- Especially, if we specialize to the case $(\mathcal{C}, w) = (\mathrm{Ch}^b\,\mathcal{E}, quis)$, we obtain the Waldhausen category

$$(\mathrm{Ch}^b\,\mathcal{E}, quis) \tag{3.5}$$

with levlelwise split dmissible monomorphisms as cofibrations and quasi-isomorphisms as weak equivalences.

- Corresponding to the various categories shown up above, we may define respective $K$-theory spaces:
  - the ***Quillen $K$-theory space $K^Q(\mathcal{E})$ of an exact category*** $\mathcal{E}$ and the ***Quillen $K$-group $K_i^Q(\mathcal{E})$ $(i \in \mathbb{Z}_{\geqq 0})$ of an exact category*** $\mathcal{E}$ are defined from (3.1) by

$$K^Q(\mathcal{E}) := \Omega B(Q\mathcal{E}) = \Omega\big|N_\bullet(Q\mathcal{E})\big| \tag{3.6a}$$

$$K_i^Q(\mathcal{E}) := \pi_i \Omega B(Q\mathcal{E}) = \pi_{i+1} B(Q\mathcal{E}) \ (i \in \mathbb{Z}_{\geqq 0}) \tag{3.6b}$$

However, when we wish to work in the category of simplicial sets, we may simply think of the classifying space functur $B$ in (3.6a) as the nerve functor $N_\bullet$ by omitting the geometric realization functor $\big| - \big|$.

– the **Waldhausen $K$-theory space $K^W(\mathcal{W}, w)$ of a Waldhausen category** $(\mathcal{W}, w)$ and the ***Waldhausen $K$-group $K_i^W(\mathcal{W}, w)$ $(i \in \mathbb{Z}_{\geqq 0})$ of a Waldhausen category*** are defined by

$$K^W(\mathcal{W}, w) := \Omega\big|N_\bullet\,(wS_\bullet\mathcal{W})\big| \tag{3.7a}$$

$$K_i^W(\mathcal{W}, w) := \pi_i\Omega\big|N_\bullet\,(wS_\bullet\mathcal{W})\big| = \pi_{i+1}\big|N_\bullet\,(wS_\bullet\mathcal{W})\big| \;\; (i \in \mathbb{Z}_{\geqq 0}) \tag{3.7b}$$

where $wS_\bullet\mathcal{W}$ is the simplicial category with moriphisms levelwise weak equivalences in $w$, obtained by the Waldhausen construction $S_\bullet$.

$\big| - \big|$ is the geometric realization of a bisimplicial set, which is defined to be the usual geometric realization of the diagonal simplicial set. However, when we work in the category of simplicial sets, we omit the geometric realization functor $\big| - \big|$ in (3.7a).

– the ***Thomason-Trobaugh $K$-theory space $K^{TT}(\mathcal{C}, w)$ of a complicial exact category with weak equivalences*** $(\mathcal{C}, w)$ and the ***Thomason-Trobaugh $K$-group $K_i^{TT}(\mathcal{C}, w)$ of a complicial exact category with weak equivalences*** $(\mathcal{C}, w)$ $(i \in \mathbb{Z}_{\geqq 0})$ are defined by the Waldhausen $K$-theory space (3.7a) applied to the associated Waldhausen category (3.4);

$$K^{TT}(\mathcal{C}, w) := K^W(\mathcal{C}, w) \tag{3.8a}$$

$$K_i^{TT}(\mathcal{C}, w) := K^W(\mathcal{C}, w) \;\; (i \in \mathbb{Z}_{\geqq 0}) \tag{3.8b}$$

Starting with an exact category $\mathcal{E}$, we have three kinds of categories and respective algebraic $K$-theories:

$$\begin{cases} Q\mathcal{E} & \mapsto & K^Q(\mathcal{E}) & (3.1)(3.6\text{a}) \\ (\mathcal{E}, i) & \mapsto & K^W(\mathcal{E}, i) & (3.2)(3.7\text{a}) \\ (\mathrm{Ch}^b\,\mathcal{E}, quis) & \mapsto & K^{TT}(\mathrm{Ch}^b\,\mathcal{E}, quis) & (3.5)(3.8\text{b}) \end{cases} \tag{3.9}$$

Equivalences of the $K$-theory spaces originated in a fixed exact category $\mathcal{E}$

The $K$-theory spaces in (3.9) are homotopy equivalenct, natural w.r.t. $\mathcal{E}$:

$$K^Q(\mathcal{E}) \simeq K^W(\mathcal{E}, i) \simeq K^{TT}(\operatorname{Ch}^b \mathcal{E}, quis) \tag{3.10}$$

Consequently, their $K$-groups are equivalent

$$K_i^Q(\mathcal{E}) = K_i^W(\mathcal{E}, i) = K_i^{TT}(\operatorname{Ch}^b \mathcal{E}, quis) \ (i \in \mathbb{Z}_{\geqq 0}) \tag{3.11}$$

In fact, the first homotopy equivalence in (3.10) $K^Q(\mathcal{E}) \simeq K^W(\mathcal{E}, i)$ (obtained by the Segal subdivision) is shown by Waldhausen [46, 1.9.], and the second (zig-zag) homotopy equivalence in (3.10) is shown by Thomason-Trobaugh [44, p.279, 1.11.7.].

The reason why we still wish to consider the most complicated looking Thomason-Trobaugh $K$-theory $K_i^{TT}(\operatorname{Ch}^b \mathcal{E}, quis)$ of the complicial exact category with weak equivalences $(\operatorname{Ch}^b \mathcal{E}, quis)$ is because we may associate a triangulated category for each complicial exact category with weak equivalences, which allows us to apply the powerful triangulated category technique [21, 32, 33] to study the Thomason-Trobaugh $K$-theory $K^{TT}$. We shall see such applications soon.

On the other hand, because of (3.10), we shall mostly regard $K^Q$ as a part of $K^W$ in this review.

- To study $K^Q$ (and $K^W$) for exact categories, probably the most powerful tool had been the associated $K$-theory (space) fibration sequence for certain class of exact sequences of exact categories.
  - ***Quillen localization theorem*** [34, §5] Let $\mathcal{B}$ be a ***Serre subcategory*** of a small abelian category $\mathcal{A}$, i.e.

    $$\begin{gathered} \forall\ M_0 \rightarrowtail M_1 \twoheadrightarrow M_2, \text{short exact sequence in } \mathcal{A}\ , \\ M_1 \in \mathcal{B} \iff M_0 \text{ and } M_2 \in \mathcal{B} \end{gathered} \tag{3.12}$$

    Then there is a homotopy fibration sequence of $K$-theory spaces

    $$K^Q(\mathcal{B}) \to K^Q(\mathcal{A}) \to K^Q(\mathcal{A}/\mathcal{B}) \tag{3.13}$$

  - [36, p.1097, Theorem 2.1.] Let $\mathcal{A}$ be an idempotent complete right $s$-filtering subcategory (see [36, p.1097, Theorem 2.1.] for the definition of "$s$-filtering") of an exact category $\mathcal{U}$. Then there is a homotopy fibration sequence of $K$-theory spaces

    $$K^W(\mathcal{A}) \to K^W(\mathcal{U}) \to K^W(\mathcal{U}/\mathcal{A}) \tag{3.14}$$

However, (3.14) is sometimes not so applicable, because the assumption is not so easy to handle. Fortunately, exploiting the triangulated category techniques, user-friendly $K$-theory (space) fibration sequences are obtained in the context of the Thomason-Trobaugh $K$-theory, as shall see now.

- For each complicial exact category $\mathcal{C}$ (i.e. an exact category equipped with a bi-exact action

$$\otimes : \mathrm{Ch}^b(\mathbb{Z}) \times \mathcal{C} \to \mathcal{C} \tag{3.15}$$

of the symmetric monoidal category $\mathrm{Ch}^b(\mathbb{Z})$ ), we may associate a triangulated category $\underline{\mathcal{C}}$:

  - the exact category of bounded chain complexes of finitely generated free $\mathbb{Z}$-modules $\mathrm{Ch}^b(\mathbb{Z})$ is a symmetric monoidal category with the monoidal unit

$$\mathbb{I} := \left( \cdots 0 \xrightarrow{d} 0 \xrightarrow{d} 0 \xrightarrow{d} \mathbb{Z}\langle 1_{\mathbb{Z}} \rangle \xrightarrow{d} 0 \xrightarrow{d} 0 \cdots \;\middle|\; |1_{\mathbb{Z}}| = 0, d = 0 \right) \tag{3.16}$$

  which admits an exact sequence, obtained by an embedding in an acyclic complex $C$ and the resulting quotient on to a complex $T$

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{I} & \to & C & \to & T & \to & 0 \qquad (3.17) \\ & & 1_{\mathbb{Z}} & \mapsto & 1_C, (\eta, 1_C) & \mapsto & (\eta_T, 0) & & \qquad (3.18) \end{array}$$

  Here, $C, T \in \mathrm{Ch}^b(\mathbb{Z})$ are defined as follows:

  (3.19a)
$$C := \left( \cdots 0 \xrightarrow{d} 0 \xrightarrow{d} Z\langle \eta \rangle \xrightarrow{d} \mathbb{Z}\langle 1_C \rangle \xrightarrow{d} 0 \xrightarrow{d} 0 \cdots \;\middle|\; |\eta| = -1, |1_C| = 0, d\eta = 1_C \right)$$

$$T := \left( \cdots 0 \xrightarrow{d} 0 \xrightarrow{d} Z\langle \eta_T \rangle \xrightarrow{d} 0 \xrightarrow{d} 0 \xrightarrow{d} 0 \cdots \;\middle|\; |\eta_T| = -1, d = 0 \right) \tag{3.19b}$$

  - for each object $U$ in a complicial exact category $\mathcal{C}$, abbreviate the resulting functorial conflation of the bi-exact action (3.15) of (3.17) on $U$

$$\mathbb{I} \otimes U \quad \rightarrowtail \quad C \otimes U \quad \twoheadrightarrow \quad T \otimes U \tag{3.20}$$

  as

$$U \quad \rightarrowtail \quad CU \quad \twoheadrightarrow \quad TU, \tag{3.21}$$

  by setting

$$CU := C \otimes U \in \mathcal{C}, \; TU := T \otimes U \quad \in \quad \mathcal{C}. \tag{3.22}$$

– given a morphism $f : X \to Y$ in $\mathcal{C}$, define the ***cone of*** $f$ $C(f)$ and the conflation

$$Y \rightarrowtail C(f) \twoheadrightarrow TX, \tag{3.23}$$

from the following commutative diagram

$$\begin{tikzcd} X \arrow[r, tail] \arrow[d, "f"'] & CX \arrow[r, two heads] \arrow[d] & TX \arrow[d, equal] \\ Y \arrow[r, tail] & C(f) \arrow[r, two heads] & TX \end{tikzcd} \tag{3.24}$$

where the upper row is the conflation (3.21) applied to the case $U = X$, and the left square is a pushout diagram.

– a conflation $X \rightarrowtail Y \twoheadrightarrow Z$ in $\mathcal{C}$ is called a ***Frobenius conflation***, if for every $U \in \mathcal{C}$, the following dotted arrows always exist, i.e. the corresponding extension problem and the lifting problem are always solvable:

$$\begin{tikzcd} X \arrow[r] \arrow[d, tail] & CU \\ Y \arrow[ur, dotted] & \end{tikzcd} \qquad \begin{tikzcd} & Y \arrow[d, two heads] \\ CU \arrow[r] \arrow[ur, dotted] & Z \end{tikzcd} \tag{3.25}$$

– then, as is shown in [38, p.225, Lemma A.2.16], the complicial exact category $\mathcal{C}$ together with the Frobenius conflations becomes a ***Frobenius exact category***, i.e. an exact category with enough injectives and enough projectives, and where injectives and projectives coincide to be the direct factors of objects of the form $CU$ for some $U \in \mathcal{C}$ [38, p.225, Lemma A.2.16].

– for a Frobenius exact category $\mathcal{F}$, its ***stable category*** $\underline{\mathcal{F}}$ is defined by

$$\mathrm{Ob}\,\underline{\mathcal{F}} = \mathrm{Ob}\,\mathcal{F}; \qquad \mathrm{Hom}_{\underline{\mathcal{F}}}(X, Y) = \mathrm{Hom}_{\mathcal{F}}(X, Y)/\sim, \tag{3.26}$$

where $f, g : X \to Y$ are $f \sim g$ if and only if their difference factors through a projective-injective object.

– a stable category $\underline{\mathcal{F}}$ becomes a triangulated category.

  * if a Frobenius category $\mathcal{F}$ is a complicial exact category $\mathcal{C}$ together with the Frobenius conflations, then the distinguished triangles of the triangulated category $\underline{\mathcal{C}}$ are of the form

$$X \xrightarrow{f} Y \to C(f) \to TX, \tag{3.27}$$

  where the unnamed maps are constructed in (3.24).

* for a general Frobenius category $\mathcal{F}$, the distinguished triangles of the triangulated category $\underline{\mathcal{F}}$ are of the form

$$X \xrightarrow{f} Y \to I(X) \coprod_X Y \to I(X)/X, \tag{3.28}$$

where the unnamed maps are constructed in the following commutative diagram

$$\begin{tikzcd} X \arrow[r, tail] \arrow[d, "f"'] & I(X) \arrow[r, two heads] \arrow[d] & I(X)/X \arrow[d, equal] \\ Y \arrow[r, tail] & I(X)\coprod_X Y \arrow[r, two heads] & I(X)/X, \end{tikzcd} \tag{3.29}$$

which is constructed just like (3.24), beginning with an inflation $X \rightarrowtail I(X)$ into an injective object.

- For each complicial exact category with weak equivalence $(\mathcal{C}, w)$, we may associate a triangulated category $\mathcal{T}(\mathcal{C}, w)$:
  - set $\mathcal{C}^w \subsetneqq \mathcal{C}$ be the full exact subcategory, consisting of $X \in \mathcal{C}$ such that $(0 \to X) \in w$. Then, $\mathcal{C}^w$ is still a complicial exact category, whose resulting Frobenius exact category structure has the same injective-projective objects just as $\mathcal{C}$, i.e. objects which are the direct factors of objects of the form $CU$ for some $U \in \mathcal{C}$ [38, p.191, 3.2.15.; p.225, Lemma A.2.16]. Consequently, we obtain a full embedding of triangulated stable categories:

$$\underline{\mathcal{C}^w} \subsetneqq \underline{\mathcal{C}} \tag{3.30}$$

  - when we have a full triangulated emebedding $\mathcal{B} \subsetneqq \mathcal{A}$, consider the class $b$ of morphisms whose cones (see [33] for the general construction, but, when the triangulated category is the stable category of a Frobenius category, they are given by (3.27)) are isomorphic to objects of $\mathcal{B}$. Now the ***Verdier quotient*** [45] [33, p.74, Theorem 2.1.8.] $\mathcal{A}/\mathcal{B}$ is defined by the localization with respect to $b$:

$$\mathcal{A}/\mathcal{B} = \mathcal{A}\left[b^{-1}\right] \tag{3.31}$$

  - let $\mathcal{B}' \subsetneqq \mathcal{A}$ be the full subtriangulated category consisting of those objects sent to zero in the Verdier quotient $\mathcal{A}/\mathcal{B}$. Then $\mathcal{B}'$ is the idempotent completion of $\mathcal{B}$ in $\mathcal{A}$, i.e. we have full embeddings of triangulated categories

$$\mathcal{B} \subsetneqq \mathcal{B}' \subsetneqq \mathcal{A} \tag{3.32}$$

where objects of $\mathcal{B}'$ consist of those objects of $\mathcal{A}$, which are direct summands of objects in $\mathcal{B}$ [33, p.91, Lemma 2.1.33.] [38, p.222, A.7.].

– for each complicial exact category with weak equivalence $(\mathcal{C}, w)$, its associated triangulated category $\mathcal{T}(\mathcal{C}, w)$ is defined by the Verdier quotient of (3.30):

$$\mathcal{T}(\mathcal{C}, w) = \underline{\mathcal{C}}/\underline{\mathcal{C}^w} \tag{3.33}$$

- There is a user-friendly fiber sequences of the Thomason-Trobaugh $K$-theory (space) $K^{TT}$, which exploits the triangulated category technology.

  – [38, p.184, Definition 3.1.5.] a sequence of triangulated categories

  $$\mathcal{A} \to \mathcal{B} \to \mathcal{C}$$

  is called ***exact***, if the following conditions are satisfied:

  * the composition sends $\mathcal{A}$ to 0,
  * $\mathcal{A} \to \mathcal{B}$ is fully faithful and identifies $\mathcal{A}$, up tp equivalences, with the subcategory consisting of those objects in $\mathcal{B}$ sent to 0 in $\mathcal{C}$,
  * the induced functor from the Verdier quotient (3.31) $\mathcal{B}/\mathcal{A}$ to $\mathcal{C}$ is an equivalence.

  – ***Thomason-Waldhausen Localization, Connective Version***
  [38, p.193, Theorem 3.2.23.] Given a sequence $\mathcal{C}_0 \to \mathcal{C}_1 \to \mathcal{C}_2$ of complicial exact categories with weak equivalences. Assume that the associated sequence $\mathcal{T}\mathcal{C}_0 \to \mathcal{T}\mathcal{C}_1 \to \mathcal{T}\mathcal{C}_2$ of triangulated categories is exact. Then there is a homotopy fibration sequence of K-theory spaces

  $$K^{TT}(\mathcal{C}_0) \to K^{TT}(\mathcal{C}_1) \to K^{TT}(\mathcal{C}_2) \tag{3.34}$$

- Both $K^W$ and $K^{TT}$ are parts of appriori non-connected spectra $\mathbb{K}^W$ and $\mathbb{K}^{TT}$ (denoted by $K^B$ in [44], but we shall follow more conceptually transparent treatments of Schlichting [36, 37, 38]):

  – [36] for an exact category $\mathcal{E}$, there is a left $s$-filtering embedding

  $$\mathcal{E} \subseteqq \mathcal{F}\mathcal{E}$$

  into an exact category $\mathcal{F}\mathcal{E}$ whose $K$-theory space $K^W(\mathcal{F}\mathcal{E})$ is contractible. Then setting

  $$S\mathcal{E} = \mathcal{F}\mathcal{E}/\mathcal{E},$$

  the ***Schlichting $K$-theory spectrum*** $\mathbb{K}^W(\mathcal{E})$ is defined so that its $n$-th space of the spectrum is given by $K^W(S^n\mathcal{E})$.

  - [37] for a complicial exact category with weak equivalence $(\mathcal{C}, w)$, there is a fully exact functor of complicial exact categories with weak equivalences

$$(\mathcal{C}, w) \to \mathcal{F}(\mathcal{C}, w),$$

whose associated functor of triangulated categories $\mathcal{T}(\mathcal{C}, w) \to \mathcal{T}(\mathcal{F}(\mathcal{C}, w))$ is fully faithful, and the $K$-theory space $K^{TT}(\mathcal{F}(\mathcal{C}, w))$ is contractible. Then setting

$$S(\mathcal{C}, w)$$

so that its underlying complicial exact category is $\mathcal{F}(\mathcal{C}, w)$ and and its weak equivalences are those which become isomorphisms in the Verdier quotient $\mathcal{T}(\mathcal{F}(\mathcal{C}, w))/\mathcal{T}(\mathcal{C}, w)$, the ***Schlichting $K$-theory spectrum*** $\mathbb{K}^{TT}(\mathcal{C}, w)$ is defined so that its $n$-th space of the spectrum is given by $K^{TT}(S^n(\mathcal{C}, w))$.

- Then the associated $K$-theory space fibration sequences (3.14) (3.34) can be upgraded to the level of spectra, under the weaker "up to factors" conditions:

  - [36, p.1101, Theorem 2.10.] Let $\mathcal{A}$ be an idempotent complete right $s$-filtering subcategory (see [36, p.1097, Theorem 2.1.] for the definition of "$s$-filtering") of an exact category $\mathcal{U}$. Then there is a homotopy fibration sequence of $K$-theory spectra

$$\mathbb{K}^W(\mathcal{A}) \to \mathbb{K}^W(\mathcal{U}) \to \mathbb{K}^W(\mathcal{U}/\mathcal{A}) \tag{3.35}$$

However, just like (3.14), (3.35) is sometimes not so applicable, because the assumption is not so easy to handle. Fortunately, exploiting the triangulated category techniques, user-friendly $K$-theory (spectra) fibration sequences (to be recalled in (3.36)) are obtained in the context of the Thomason-Trobaugh $K$-theory, just as before.

- [37, p.125, Theorem 9.] [38, p.195, Theorem 3.2.27.].

  - [38, p.180, 2.4.1.] An inclusion $\mathcal{A} \subset \mathcal{B}$ of exact categories is called ***cofinal***, or ***equivalence up to factors***, if the following conditions are satisfied:
    * every object of $\mathcal{A}$ is a direct factor of an object of $\mathcal{B}$,
    * the inclusion is extension closed,
    * preserves and detects conflations.
  - [38, p.186, Definition 3.1.10.] a sequence of triangulated categories

$$\mathcal{A} \to \mathcal{B} \to \mathcal{C}$$

is called ***exact up to factors***, if the following conditions are satisfied:

  * the composition sends $\mathcal{A}$ to 0,
  * $\mathcal{A} \to \mathcal{B}$ is fully faithful and identifies $\mathcal{A}$, up tp equivalences, with the subcategory consisting of those objects in $\mathcal{B}$ sent to 0 in $\mathcal{C}$,
  * the induced functor from the Verdier quotient (3.31) $\mathcal{B}/\mathcal{A}$ to $\mathcal{C}$ is an equivalence up to factors.

- ***Thomason-Waldhausen Localization, Non-Connective Version*** [38, p.195, Theorem 3.2.27.] Given a sequence $\mathcal{C}_0 \to \mathcal{C}_1 \to \mathcal{C}_2$ of complicial exact categories with weak equivalences. Assume that the associated sequence $\mathcal{T}\mathcal{C}_0 \to \mathcal{T}\mathcal{C}_1 \to \mathcal{T}\mathcal{C}_2$ of triangulated categories is exact up to factors. Then there is a homotopy fibration sequence of K-theory spectra

$$\mathbb{K}^{TT}(\mathcal{C}_0) \to \mathbb{K}^{TT}(\mathcal{C}_1) \to \mathbb{K}^{TT}(\mathcal{C}_2) \tag{3.36}$$

- [33, p.14, Theorem 1.14., p.143, Theorem 4.4.9.] [37, p.105, Theorem 2, p.106, Corollary 3, Lemma 3.] [38, p.203, Theorem 3.4.5.] There is a useful way to creat an exact sequence up to factors of triangulated categories:
  - [33, p.130.] [38, p.203.] An object $A$ of a triangulated category $\mathcal{A}$ with all small coproducts is called ***compact*** , if the canonical map

$$\oplus_{i\in I} \operatorname{Hom}(A, E_i) \to \operatorname{Hom}(A, \oplus_{i\in I} E_i) \tag{3.37}$$

    is always an isomorphism.
    Denote the full subcategory od compact objects of $\mathcal{A}$ by $\mathcal{A}^c$, which becomes an idempotent complete triangulated subcategory of $\mathcal{A}$.
  - [33, p.140, p.274.] [38, p.203.] A set $S$ of compact objects is said to ***generate*** $\mathcal{A}$ , or $\mathcal{A}$ is compactly generated by $S$, if for every object $E \in \mathcal{A}$ we have

$$\operatorname{Hom}(A, E) = 0, \forall A \in S \quad \Longrightarrow \quad E = 0. \tag{3.38}$$

  - [33, p.14, Theorem 1.14., p.143, Theorem 4.4.9.] [38, p.203, Theorem 3.4.5.] Given a set $S_0$ of compact objects in a compactly generated triangulated category $\mathcal{R}$, which is closed under taking shifts, let $\mathcal{S} \subsetneqq \mathcal{R}$ be the smallest full triangulated subcategory containing the set $S_0$, which is closed under formation of coproducts in $\mathcal{R}$. Then the sequence

$$\mathcal{S} \to \mathcal{R} \to \mathcal{R}/\mathcal{S} \tag{3.39}$$

    induces a sequence of triangulated categories of compact objects

$$\mathcal{S}^c \to \mathcal{R}^c \to (\mathcal{R}/\mathcal{S})^c \tag{3.40}$$

    which is exact up to factors.

- Specializing to the case where the $K$-theory spaces and spectra are originated in a fixed exact category $\mathcal{E}$, we may summarize as follows:

**Equivalences of the $K$-theory spectra originated in a fixed exact category $\mathcal{E}$**

The $K$-theory spectra are homotopy equivalenct, natural w.r.t. $\mathcal{E}$:

$$\mathbb{K}^W(\mathcal{E}, i) \simeq \mathbb{K}^{TT}(\mathrm{Ch}^b\,\mathcal{E}, quis) \tag{3.41}$$

Consequently, their $K$-groups are equivalent

$$\mathbb{K}^W_i(\mathcal{E}, i) = \mathbb{K}^{TT}_i(\mathrm{Ch}^b\,\mathcal{E}, quis)\ (i \in \mathbb{Z}) \tag{3.42}$$

For $i \in \mathbb{Z}_{\geqq 0}$, there is a natural map

$$\begin{tikzcd} K^W_i(\mathcal{E}, i) \arrow[rr] \arrow[d, Rightarrow, "\simeq"] & & \mathbb{K}^W_i(\mathcal{E}, i) \arrow[d, equal, "\simeq"] \\ K^{TT}_i(\mathrm{Ch}^b\,\mathcal{E}, quis) \arrow[rr] & & \mathbb{K}^{TT}_i(\mathrm{Ch}^b\,\mathcal{E}, quis) \end{tikzcd} \tag{3.43}$$

which is

$$\begin{cases} \text{an isomorphism} & \text{if } i \geqq 1 \\ K^W_0(\mathcal{E}, i) \to K^W_0(\widetilde{\mathcal{E}}, i) & \text{if } i = 0, \end{cases} \tag{3.44}$$

where $\widetilde{\mathcal{E}}$ is an idempotent completion of $\mathcal{E}$ [38, p.181, 2.4.3.].

We now apply the preceeding general theory to study algebraic geometry.

- [38, p.175, Definition 2.2.6.] For a scheme $X$, let $\mathrm{Vect}(\mathbf{X})$ be the category of ***vector bundles***, i.e. locally free sheaves of finite rank on $X$. Then, $\mathrm{Vect}(\mathbf{X})$ becomes extension closed in the category of quasi-coherent $\mathcal{O}_X$ -modules $\mathrm{Qcoh}(X)$

$$\mathrm{Vect}(\mathbf{X}) \subset \mathrm{Qcoh}(\mathbf{X}), \tag{3.45}$$

through which we may regard $\mathrm{Vect}(\mathbf{X})$ as an exact category. Now, we define the **Quillen $K$-theory space of a scheme $X$** by

$$K^Q(X) := K^Q(\mathrm{Vect}(\mathbf{X})). \tag{3.46}$$

- [38, p.202, 3.4.1.] The preceding discussion using the vector bundles in the framework of exact categories can be upgraded to the framework of complicial exact category with weak equivalences using the ***perfect complexes***:

- for a quasi-compact and separated scheme $X$, a complex $(A, d)$ of quasi-coherent $\mathcal{O}_X$ -modules is called ***perfect*** if there is a covering

  (3.47) $$X = \cup_{i \in I} U_i$$

  by affine open subschemes $U_i \subset X$ such that, for any $i \in I$,

  (3.48) $$(A, d)\big|_{U_i} \overset{\text{quis}}{\cong} \text{ a bounded complex of vector bundles}$$

- For a closed subset $Z \subset X$ with quasi-compact open complement $X \setminus Z$, let $\operatorname{Perf}_Z(X)$ be the full subcategory in $\operatorname{Ch}\operatorname{Qcoh}(X)$

  (3.49) $$\operatorname{Perf}_Z(X) \subset \operatorname{Ch}\operatorname{Qcoh}(X)$$

  of perfect complexes on $X$ which are acyclic over $X \setminus Z$. The inclusion (3.49) is extension closed, and makes $(\operatorname{Perf}_Z(X), quis)$ a complicial exact category with weak equivalences. Let us compare this situation (3.49) with (3.45), but notice that we are now free to relativize the situation by taking into accout a closed subset $Z \subset X$ with quasi-compact open complement $X \setminus Z$. For a notational consistency between (3.45) and (3.49), it is set

  (3.50) $$\operatorname{Perf}(X) := \operatorname{Perf}_X(X).$$

- [38, p.202, Definition 3.4.2.] Now the ***Thomason-Trobaugh $K$-theory space of $X$ with support in $Z$*** $K^{TT}(XonZ)$ and the ***Schlichting Thomason-Trobaugh $K$-theory spectrum of $X$ with support in $Z$*** $\mathbb{K}^{TT}(XonZ)$ are respectively defined by

  (3.51a)
  $$K^{TT}(X \text{ on } Z) := K^{TT}(\operatorname{Perf}_Z(X), quis) \qquad \big(K^{TT}(X) := K^{TT}(X \text{ on } X)\big)$$

  (3.51b)
  $$\mathbb{K}^{TT}(X \text{ on } Z) := \mathbb{K}^{TT}(\operatorname{Perf}_Z(X), quis) \qquad \big(\mathbb{K}^{TT}(X) := \mathbb{K}^{TT}(X \text{ on } X)\big)$$

- With the precious complicial exact category with weak equivalence $(\operatorname{Perf}_Z(X), quis)$ at hand, in view of (3.34) (3.36), it is natural for us to pay a great attention to its associated triangulated category (3.33) $\mathcal{T}(\operatorname{Perf}_Z(X), quis)$, which is usually denoted by

  (3.52) $$D\operatorname{Perf}_Z(X) := \mathcal{T}\left(\operatorname{Perf}_Z(X), quis\right).$$

  This becomes idempotent complete.

- [38, p.204, Proposition 3.4.8.] ***Suppose $X$ is a quasi-compact and separated scheme which has an ample family of line bundles.*** Then the inclusion of bounded complexes of vector bundles into perfect complexes

$$\mathrm{Ch}^b \,\mathrm{Vect}(X) \subset \mathrm{Perf}(X) \tag{3.53}$$

induces an equivalence of triangulated categories

$$D^b \,\mathrm{Vect}(X) \cong D \,\mathrm{Perf}(X). \tag{3.54}$$

By the Thomason-Waldhausen Localization, (3.36) (3.36), this equivalence (3.54) implies (3.53) induces a homotopy equivalence of (Schlichting) Thomason-Trobaugh $K$-theories:

$$K^{TT}\left(\mathrm{Ch}^b \,\mathrm{Vect}(X), quis\right) \xrightarrow{\simeq} K^{TT}\left(\mathrm{Perf}(X), quis\right) \tag{3.55a}$$

$$\mathbb{K}^{TT}\left(\mathrm{Ch}^b \,\mathrm{Vect}(X), quis\right) \xrightarrow{\simeq} \mathbb{K}^{TT}\left(\mathrm{Perf}(X), quis\right) \tag{3.55b}$$

- Combining (3.55) (3.51b) (3.10) (3.41) (3.46), we may summarize as follows:

quasi-compact separated scheme with an ample family of line bundles

***Suppose $X$ is a quasi-compact and separated scheme which has an ample family of line bundles.*** Then, we have the following homotopy equivalence of $K$-theory spaces and $K$-theory spectra:

$$\begin{aligned} K^Q(X) \xrightarrow[\simeq]{(3.46)} K^W(\mathrm{Vect}(X), i) \xrightarrow[\simeq]{(3.10)} K^{TT}(\mathrm{Ch}^b \,\mathrm{Vect}(X), quis) \\ \xrightarrow[\simeq]{(3.55)} K^{TT}\left(\mathrm{Perf}(X), quis\right) \overset{(3.51a)}{=:} K^{TT}(X) \end{aligned} \tag{3.56}$$

$$\begin{aligned} \mathbb{K}^W(\mathrm{Vect}(X), i) \xrightarrow[\simeq]{(3.41)} \mathbb{K}^{TT}(\mathrm{Ch}^b \,\mathrm{Vect}(X), quis) \\ \xrightarrow[\simeq]{(3.55)} \mathbb{K}^{TT}\left(\mathrm{Perf}(X), quis\right) \overset{(3.51b)}{=:} \mathbb{K}^{TT}(X) \end{aligned} \tag{3.57}$$

Furthermore, since $\mathrm{Vect}(\mathbf{X})$ is idempotent complete, the natural map (3.43) induces an isomorphism

$$K_i^Q(X) \xrightarrow{\simeq} \mathbb{K}_i^{TT}(X) \qquad \left(\forall i \in \mathbb{Z}_{\geqq 0}\right) \tag{3.58}$$

by (3.44).

- [38, p.203, Proposition 3.4.6.] Denote by $D \,\mathrm{Qcoh}(X)$ the unbounded derived category of $\mathrm{Qcoh}(X)$, and for a closed subset $Z \subset X$, denote by $D_Z \,\mathrm{Qcoh}(X)$ the full subcategory of $D\,\mathrm{Qcoh}(X)$

$$D_Z\,\mathrm{Qcoh}(X) \subsetneqq D\,\mathrm{Qcoh}(X) \tag{3.59}$$

of those complexes which are acyclic when restricted to $X \setminus Z$.

Then the triangulated category $D_Z\,\mathrm{Qcoh}(X)$ is compactly generated with category of compact objects the derived category of perfect complexes $D\,\mathrm{Perf}_Z(X)$:

$$(D_Z\,\mathrm{Qcoh}(X))^c \cong D\,\mathrm{Perf}_Z(X) \tag{3.60}$$

- We would like to apply (3.60) to study the Schlichting Thomason-Trobaugh $K$-theory $\mathbb{K}^{TT}$ (3.51b), using (3.40) (3.36). We now list a couple of results which are used for this purpose:
  - [44, p.307, Theorem 2.6.3.] Let $f : X' \to X$ be a quasi-separated map of quasi-compact schemes. Let $i : Y \to X$ be a finitely presented closed immersion. Suppose that $f$ is an isomorphism infinitely near $Y$ (2.6.2.1). Set $Y' = f^{-1}(Y) = Y \times_X X'$. Then, we have an equivalence of drived categories

    $$Lf^* : D_{Y'}\,\mathrm{Qcoh}(X') \cong D_Y\,\mathrm{Qcoh}(X) : Rf_* \tag{3.61}$$

  - [38, p.202, Lemma 3.4.3.] Let $Z \subset X$ be a closed subset of a quasi-compact and separated scheme $X$ with quasicompact open complement $X \setminus Z \subset X$. Then the following sequence of triangulated categories is exact:

    $$D_Z\,\mathrm{Qcoh}(X) \to D\,\mathrm{Qcoh}(X) \to D\,\mathrm{Qcoh}(X \setminus Z). \tag{3.62}$$

Now, we are ready to provide an outline to prove the following important theorems of Thomason-Trobaugh [44]:

**Theorem 3.1** ([44, p.322, Proposition 3.19; p.364, Theorem 7.1]). *In the commutative diagram:*

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] & V \arrow[d, "p"'] \\ U \arrow[r, "i", "\textit{open emb.}"'] & X \end{tikzcd} \tag{3.63}$$

*suppose the following conditions (whose precise definitions are not reviewed here, but are satisfied if* (3.63) *is a distinguished square diagram) are satisfied:*

- *$p$ is a map of quasi-compact and quasi-separated schemes.*

- *$U$ is quasi-compact.*
- *$p$ is an isomorphism infinitely near $X \setminus U$.*

*Then, there are homotopy equivalences of spectra:*

$$p^* : \mathbb{K}^{TT}(X \text{ on } X \setminus U) \xrightarrow{\simeq} \mathbb{K}^{TT}(V \text{ on } V \setminus (U \times_X V))$$

**Theorem 3.2** ([44, p.365, Theorem 7.4])**.** *Suppose*

- *$X$ is a quasi-compact and quasi-separated scheme.*
- *$i : U \to X$ is an open immersion with $U$ quasi-compact.*

*Then, there is a homotopy fibre sequence of spectra*

$$\mathbb{K}^{TT}(X \text{ on } X \setminus U) \to \mathbb{K}^{TT}(X) \to \mathbb{K}^{TT}(U)$$

*Proof.* In fact, Theorem 3.1 follows from (3.61) (3.60) (3.40) (3.36). Similarly, Theorem 3.2 follows from (3.62) (3.60) (3.40) (3.36).

□

We have now reviewed necessary "after Quillen" techniques to prove the following $K$-theory representability theorems:

**Theorem 3.3** ([28, p.139, Proposition 3.9])**.** *For any $X \in (Sm/S)$ and for any $m \in \mathbb{Z}_{\geqq 0}$, we have the following representability:*

$$K_m^{TT}(X) \cong \mathrm{Hom}_{\mathcal{H}_{s,\bullet}(Sm/S)_{Nis}}\left(\Sigma_s^m(X_+), (\mathbf{R}\Omega_s^1)B(\coprod_{n\geqq 0} BGL_n)\right) \tag{3.65}$$

*Here the $m$ fold simplicial suspension is defined by $\Sigma_s^m(X_+) = (X_+) \wedge S^m$, and the derived simplicial loop space $\mathbf{R}\Omega_s^1(\ )$ is the right adjoint to the simplicial suspension*

$$\Sigma_s : \mathcal{H}_{s,\bullet}(Sm/S)_{Nis} \to \mathcal{H}_{s,\bullet}(Sm/S)_{Nis}$$

***Suppose further $X$ is a quasi-compact and separated scheme which has an ample family of line bundles.*** *Then, we have from* (3.56) (3.65) *isomorphisms for any $n \in \mathbb{Z}_{\geqq}$:*

(3.66)

$$K_m^Q(X) \xrightarrow[\simeq]{(3.56)} K_m^{TT}(X) \xrightarrow[\simeq]{(3.65)} \mathrm{Hom}_{\mathcal{H}_{s,\bullet}(Sm/S)_{Nis}}\left(\Sigma_s^m(X_+), (\mathbf{R}\Omega_s^1)B(\coprod_{n\geqq 0} BGL_n)\right)$$

**Theorem 3.4** (a special case of [28, p.140. Theorem 3.13]). *Suppose further $S$ is regular. Then, for any $m \in \mathbb{Z}_{\geqq 0}$, and for any $X \in Sm/S$, we have the following representability:*

$$K_m^{TT}(X) \xrightarrow{\cong} Hom_{\mathcal{H}_\bullet(S)}\left(\Sigma_s^m(X_+), BGL_\infty \times \mathbb{Z}\right) \tag{3.67}$$

***Suppose further $X$ is a quasi-compact and separated scheme which has an ample family of line bundles.*** *Then, we have from* (3.56) (3.67) *isomorphisms for any $m \in \mathbb{Z}_{\geqq 0}$:*

$$K_m^Q(X) \xrightarrow[\simeq]{(3.56)} K_m^{TT}(X) \xrightarrow[\simeq]{(3.67)} Hom_{\mathcal{H}_\bullet(S)}\left(\Sigma_s^m(X_+), BGL_\infty \times \mathbb{Z}\right) \tag{3.68}$$

We note that the Zariski analogue of Theorem 3.3 was shown by Gillet-Soulé [13, Proposition 5], as may be expected from Remark 1.

The core of the proof of Theorem 3.3 is the Morel-Voevodsky observation that a "friendly" model of $\left(a_{Nis}K^{TT}\right)_f$ is provided by $(\mathbf{R}\Omega_s^1)B(\coprod_{n\geqq 0} BGL_n)$. Since the referee asked us to supply some details of how this observation of Morel-Voevodsky is proven, we shall isolate it as Lemma 3.10, and present a complete proof. For this purpose, we have already reviewed necessary "after Quillen" techniques, and we now start reviewing more necessary techniques from "Quillen era":

- [47, I. Definition 1.1] A ring $R$ is said to satisfy the (right) invariant basis property if the based free (right) $R$-modules $R^m$ and $R^n$ are not isomorphic for $m \neq n$. Any commutative ring satisfies the invariant basis property.
- [47, IV. Example 4.1.1] For a ring $R$ satisfying the invariant basis property, let $\mathbf{bF}(R)$ be the cagtegory of finitely based free (right) $R$-modules, whose objects and morphisms are respectively the based free $R$-modules $\{0, R, R^2, \cdots, R^n, \cdots\}$ and the (right) $R$-module homomorphisms.
- $\mathbf{bF}(R)$ becomes a symmetric monoidal category by the concatenation of basis:

$$R^m \oplus R^n := R^{m+n}, \quad \begin{cases} GL_m(R) \times GL_n(R) & \to GL_{m+n}(R) \\ (a, b) & \mapsto \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \end{cases} \tag{3.69}$$

- The symmetric monoidal category structure (3.69) on $\mathbf{bF}(R)$, when restricted to the subcategory $i\mathbf{bF}(R)$ of isomorphisms, not only endow $Ob(i\mathbf{bF}(R))$ with the honest monoid structure, but also endow the nerve $N_\bullet i\mathbf{bF}(R)$ with a monoid object structure in the category of simplicial sets. (We shall call such a symmetric monoidal category with an honest monoid structure a *symmetric strict monoidal category*, though the terminology "permutative category" [24] might be more familiar.)

Thus, with respect to its monoid structure $\oplus$, we may further take its nerve to form a bisimplicial set

$$N_\bullet\left(N_\bullet i\mathbf{bF}(R), \oplus\right) \tag{3.70}$$

- Define $i_m^{m+n} : R^m \to R^{m+n}$ and $p_n^{m+n} : R^{m+n} \to R^n$ as follows:

$$0 \to R^m \xrightarrow{i_m^{m+n}} \quad R^{m+n} \xrightarrow{p_n^{m+n}} R^n \to 0$$
$$v \mapsto v \oplus 0, \ v \oplus w \mapsto w$$

Then, $\mathbf{bF}(R)$ becomes an exact category in the sense of Quillen [34, p.92] (see also [47, II Definition 7.0.]), whose admissible monomorphisms and admissible epimorphisms are respectively of the form

$$\begin{cases} g i_m^{m+n} : R^m \xrightarrow{i_m^{m+n}} R^{m+n} \xrightarrow{g \in GL_{m+n}R} R^{m+n} \\ p_n^{m+n} h : R^{m+n} \xrightarrow{h \in GL_{m+n}R} R^{m+n} \xrightarrow{p_n^{m+n}} R^n \end{cases} \tag{3.72}$$

- [47, II. Example 7.1.1., 7.3., Example 7.3.1.] For a ring $R$ satisfying the invariant basis property, we define the category $\mathbf{P}(R)$ of finitely generated projective (right) modules by the idempotent completion of $\mathbf{bF}(R)$. Thus an object of $\mathbf{P}(R)$ consists of elements of the form $(\mathbb{R}^m, e)$ with $e \in \mathrm{End}_{\mathbf{bF}(R)}(\mathbb{R}^n)$ an idempotent $e^2 = e$, and a morphism from $(R^m, e)$ to $(R^n, e')$ is a morphism $f \in \mathrm{Hom}_{\mathbf{bF}(R)}(R^m, R^n)$ such that $f = e'fe$. (We define $\mathbf{P}(R)$ in this way not to worry about set theoretical problems.)
- The symmetric monoidal category structure (3.69) on $\mathbf{bF}(R)$ induces a strict monoidal category structure on the subcategory $i\mathbf{P}(R)$ of isomorphisms of $\mathbf{P}(R)$. Just like the case of $\mathbf{bF}(R)$ (see (3.70)), it endows the nerve $N_\bullet i\mathbf{P}(R)$ with a monoid object structure in the category of simplicial sets, by which, we may further take its nerve to form a bisimplicial set

$$N_\bullet\left(N_\bullet i\mathbf{P}(R), \oplus\right) \tag{3.73}$$

- Then, by the general theory [47, II. Exercise 7.6., Lemma 7.2], $\mathbf{P}(R)$ also becomes an exact category, and the canonical embedding

$$c : \mathbf{bF}(R) \to \mathbf{P}(R) \tag{3.74a}$$
$$R^n \mapsto (R^n, \mathrm{Id}_{R^n}) \tag{3.74b}$$

makes $\mathbf{bF}(R)$ an exact subcategory of $\mathbf{P}(R)$ [47, II.7.0.1.]. Also, $c$ is a morphism of strict monoidal category, and induces a morphism of bisimplicial sets from (3.70) to (3.73):

$$\tilde{c} : N_\bullet\left(N_\bullet i\mathbf{bF}(R), \oplus\right) \to N_\bullet\left(N_\bullet i\mathbf{P}(R), \oplus\right),$$

which, upon applying the diagonalization fuctor, which intuitively regard as the geometric realization functor, further induces

$$B\tilde{c} : B\left(\coprod_{n \geqq 0} BGL_n(R)\right) \to B\left(B(i\mathbf{P}(R))\right) \tag{3.75}$$

- [14] [41, p.128, p.133] [46, IV Definition 4.2., Definition 4.3.] In general, for a symmetric monoidal category $(S, \oplus)$, its symmetric monoidal $K$-theory space $K^{\oplus}(S)$ is given by

$$K^{\oplus}(S) = B\left(S^{-1}S\right), \tag{3.76}$$

where $S^{-1}S$ is the category such that

(3.77a)
$$\mathrm{Ob}\, S^{-1}S = \mathrm{Ob}\, S \times \mathrm{Ob}\, S$$

(3.77b)
$$\mathrm{Mor}_{S^{-1}S}\left((m_1, m_2), (n_1, n_2)\right) = \left\{(s \in \mathrm{Ob}\, S, f \in \mathrm{Mor}_S(s \oplus m_1, n_1), g \in \mathrm{Mor}_S(s \oplus m_2, n_2)\right\} \big/ \simeq,$$

where $(s \in \mathrm{Ob}\, S, f \in \mathrm{Mor}_S(s \oplus m_1, n_1), g \in \mathrm{Mor}_S(s \oplus m_2, n_2)$ is interpreted as the composite

$$(m_1, m_2) \xrightarrow{s\Box} (s\Box m_1, s\Box m_2) \xrightarrow{(f,g)} (n_1, n_2)$$

To understand (3.75), we start with the following three observations concerning the symmetric monoidal $K$-theory space; first, its delooping in the strict case, second, a delooped cofinality theorem, and third, its relevance with the Quillen $K$-theory :

**Theorem 3.5** (delooped symmetric monoidal $K$-theory space). *When $S$ is a symmetric strict monoidal such that every morphism is an isomorphism and the translation is faithful, i.e. for any $s, t \in S$, $\mathrm{Aut}_S(s) \to \mathrm{Aut}_S(s \oplus t)$ is injection.*

*Then we have a natural zig-zag homotopy equivalence*

$$\Omega B\left(BS\right) \simeq B\left(S^{-1}S\right) = K^{\oplus}(S) \tag{3.78}$$

*Proof.* Actually, (3.78) is the composite of the following homotopy equivalences

$$\Omega B\left(BS\right) \xleftarrow[{[25]}]{\simeq} (BS)^{+} \xrightarrow[{[14][47, IV Theorem 4.8.]}]{\simeq} B\left(S^{-1}S\right),$$

where, $(BS)^{+}$ stands for the group completion of $BS$. $\square$

**Theorem 3.6** (delooped cofinality theorem for symmetric monoidal K-theory). *Suppose $S$ and $T$ are ymmetric strict monoidal categories such that every morphism is an isomorphism and the translation is faithful (in the sense of Theorem 3.5.*

*Suppose a morphism of symmetric strict monoidal categories $f : S \to T$ satisfies the following conditions:*

**cofinality** *For any $t \in T$, there exist $t' \in T, s \in S$ such that $t \square t' \cong f(s)$.*

**fully faithfulness** *For any $s \in S$, $\mathrm{Aut}_S(s) \cong \mathrm{Aut}_T(f(s))$.*

*Then, we have the following natural fibration-up-to-homotopy*

$$B\left(BS\right) \xrightarrow{B(Bf)} B\left(BT\right) \to B\left(K_0^{\oplus}(T)/K_0^{\oplus}(S)\right), \tag{3.79}$$

*where $K_0^{\oplus}$ stands for the $K_0$ group for a symmetric monoidal category (see [47, II Definition 5.1.2.] for instance), and the second map is the composite*

$$B\left(BT\right) \to K\left(\pi_1 B\left(BT\right), 1\right) = B\left(K_0^{\oplus}(T)\right) \to B\left(K_0^{\oplus}(T)/K_0^{\oplus}(S)\right)$$

*Proof.* In view of Theorem 3.5, the claim follows from the usual cofinality theorem concerning the map $B\left(S^{-1}S\right) \to B\left(T^{-1}T\right)$ [14] [47, IV Cofinality Theorem 4.11.]. □

**Theorem 3.7** ("$Q = \oplus$"theorem [14][41, Theorem 7.7.][47, IV Theorem 7.1.]). *For any split exact category $\mathcal{A}$, there is a natural homotopy equivalence between the Quillen $K$-theory space $K^Q(\mathcal{A})$ of the exact category $\mathcal{A}$ and the symmetric monoidal category $K$-theory space $K^{\oplus}(i\mathcal{A})$ for the symmetric monoidal category $i\mathcal{A}$:*

$$K^Q(\mathcal{A}) = \Omega BQ\mathcal{A} \xrightarrow{\simeq} B\left((i\mathcal{A})^{-1}(i\mathcal{A})\right) = K^{\oplus}(i\mathcal{A}) \qquad \square \tag{3.80}$$

Now the importance of $B\tilde{c}$ (3.75) is revealed by the following delooped "$+ = \oplus$ "theorem, which is an immediate consequence of the delooped cofinality theorem for symmetric monoidal K-theory Threorem 3.6:

**Theorem 3.8** (delooped "$+ = \oplus$ "theorem). *For a ring $R$ with the invariant basis property, there is a fibration-sequence-up-to-homotopy*

$$B\left(\coprod_{n \geqq 0} BGL_n(R)\right) \xrightarrow{B\tilde{c}} B\left(B(i\mathbf{P}(R))\right) \to B\left(K_0(R)/\mathbb{Z}\right), \tag{3.81}$$

*which is natural with respect to ring homomorphisms between rings with invariant basis property.* □

Applying $\Omega$ to (3.82), together with Theorem 3.5 and Theorem 3.7, we have the following variant of the famous Quillen "$+ = Q$"theorem:

**Theorem 3.9** (variant of "$+ = Q$ "theorem). *For a ring $R$ with the invariant basis property, we shall write $K^Q(\mathbf{P}(R))$, the Quillen $K$-theory space of the exact category $\mathbf{P}(R)$ (see Theorem 3.7) by $K^Q_{ring}(R)$. Then, there is a fibration-sequence-up-to-homotopy*

$$\Omega B\left(\coprod_{n\geqq 0} BGL_n(R)\right) \xrightarrow{\overline{c}} K^Q_{ring}(R) \to K_0(R)/\mathbb{Z} \tag{3.82}$$

*which is natural with respect to ring homomorphisms between rings with invariant basis property.* □

Now, we are ready to answer the referee's request, by finishing our proof, whose first part is a reminiscence of the proof of [13, Lemma 18.]:

**Lemma 3.10.** *We can take $(\mathbf{R}\Omega^1_s)B(\coprod_{n\geqq 0} BGL_n)$ as a "friendly" model of $\left(a_{Nis}K^Q\right)_f \xrightarrow{\simeq} \left(a_{Nis}K^{TT}\right)_f$ in $\mathcal{H}_{s,\bullet}(Sm/S)_{Nis}$.*

*Proof of Lemma 3.10.*

We first note the natural equivalence of simplicial sheaves

$$a_{Nis}K^Q \xrightarrow{\simeq} a_{Nis}K^{TT} \tag{3.83}$$

This is because, when we stufy the behavior of the natural map of simplicial presheaves

$$K^Q \to K^{TT} \tag{3.84}$$

at stalks, we may restrict our attention to the affine schemes, which have an ample family of line bundles. Thus, we may apply the homotopy equivalence (3.56) to conclude (3.83).

Thus, it suffices to show that we can take $(\mathbf{R}\Omega^1_s)B(\coprod_{n\geqq 0} BGL_n)$ as a "friendly" model of $\left(a_{Nis}K^Q\right)_f$.

For this purpose, consider the following diagram in $\triangle^{op}Preshv(Sm/S)_{Nis}$ :

(3.85)

$$\begin{array}{ccc} & & K^Q := (U \mapsto \Omega BQ\,(\mathrm{Vect}(\mathbf{U}))) \\ & & \downarrow \simeq \\ \Omega^1_s B(\coprod_{n\geqq 0} BGL_n) := \left(U \mapsto \left(\Omega B(\coprod_{n\geqq 0} BGL_n\,(\mathcal{O}(U)))\right)\right) & \xrightarrow[\simeq]{\overline{c}} & K^Q_{\mathrm{ring}} := (U \mapsto \Omega BQ\,(\mathbf{P}\,(\mathcal{O}(U)))) \end{array}$$

Here, the right vertical map is induced by a morphism of exact categories from the category of finite rank vector bundles on $U$ to the category of finitely generated projective $\mathcal{O}(U)$ modules

$$\mathrm{Vect}(\mathbf{U}) \to \mathbf{P}(\mathbf{U}), \tag{3.86}$$

which is an equivalence of categories when $U$ is affine. Thus, the right vertical map is a weak equivalence.

Now, the bottom horizontal map is also a weak equivalence, for each stalk in the Nisnevich site is a (Hensel) local ring, and any finitely generated module over a local ring is free, in which case, the last term in (3.82) degenerates to a single point to force $\overline{c}$ to be a homotopy equivalence.

Then, let us apply the sheafication functor $a$ to (3.85), which preserves the weak equivalences because the sheafication functor $a$ induces stalk isomorphisms:

(3.87)
$$\begin{tikzcd} & & a\left(K^{Q}\right) \arrow[d, "\simeq"] \\ \Omega_s^1 B(\coprod_{n\geqq 0} BGL_n) \arrow[r, "Corollary\ 2.8", "\cong"'] & a\left(\Omega_s^1 B(\coprod_{n\geqq 0} BGL_n)\right) \arrow[r, "a(\overline{c})", "\simeq"'] & a\left(K^{Q}_{\mathrm{ring}}\right) \end{tikzcd}$$

Here, we used Corollary 2.8, which claims $\Omega_s^1 B(\coprod_{n\geqq 0} BGL_n)$ is a pointed simplicial sheaf.

Next, we apply the fibrant replacement functor $f$ to (3.88):

(3.88)
$$\begin{tikzcd} \Omega_s^1\left(B(\coprod_{n\geqq 0} BGL_n)_f\right) \arrow[rr, dotted, "\simeq"'] & & \left(a\left(K^{Q}\right)\right)_f \arrow[d, "\simeq"] \\ \left(\Omega_s^1 B(\coprod_{n\geqq 0} BGL_n)\right)_f \arrow[u, "\simeq"] \arrow[r, "\cong"'] & \left(a\left(\Omega_s^1 B(\coprod_{n\geqq 0} BGL_n)\right)\right)_f \arrow[r, "\simeq"'] & \left(a\left(K^{Q}_{\mathrm{ring}}\right)\right)_f \end{tikzcd}$$

Here, the left upper map is defined because $\Omega_s^1$ sends a fibrant to a fibrant by Proposition 2.14. Next, this left upper map is a weak equivalence becuase $\Omega_s^1$ preserves weak equivalences. From this, we see cofibrant and fibrant objects $\left(a\left(K^{Q}\right)\right)_f$ and $\Omega_s^1\left(B(\coprod_{n\geqq 0} BGL_n)_f\right)$ are connected by weak equivalnces between cofibrant and fibrant objects.

Thus, we see a model of $\left(a\left(K^{Q}\right)\right)_f$ is given by $\Omega_s^1\left(B(\coprod_{n\geqq 0} BGL_n)_f\right)$, which is nothing but the right derived functor $(\mathbf{R}\Omega_s^1)B(\coprod_{n\geqq 0} BGL_n)$, in the sense of Quillen model category. This completes the proof. □

*Proof of Theorem 3.3.*

Now the claim immediately follows from Theorem 2.26 (i), Theorem 3.2 and Theorem 3.1.

□

*Proof of Theorem 3.4.*

By Theorem 2.26 (ii) and the above Proof of Theorem 3.3, we see

$$K_n^{TT}(X) \cong \mathrm{Hom}_{\mathcal{H}_\bullet(S)}\left(\Sigma_s^n(X_+), (\mathbf{R}\Omega_s^1)B(\coprod_{n\geqq 0} BGL_n)\right)$$

Now the claim follows because the natural map

$$BGL_\infty \times \mathbb{Z} \to (\mathbf{R}\Omega_s^1)B(\coprod_{n\geqq 0} BGL_n)$$

is an $\mathbb{A}^1$-equivalence [28, p.139, Proposition 3.10]. □

*Remark* 2. (i) The reader might had been sick and tired of the complexity of the proof of the $K$-theory representability Theorem 3.3 presented here. In fact, the essence of the $K$-theoretical input in the proof of Theorem 3.3 was the Thomason-Trobaugh Excision Theorem 3.1 and Localization Theorem 3.2, both of which are shown in the framework of (Bass like) Waldhausen $K$-theory of perfect complexes, we had to resort to the original Quillen $K$-theory and the symmetric monoidal $K$-theory to prove Lemma 3.10, following the original approach of Morel-Voevodsky [28].

However, we can completely eliminate the Quillen $K$-theory and the symmetric monoidal $K$-theory, and can avoid the delooped "$+ = \oplus$" theorem. In fact, we can work entirely in the framework of the Waldhausen $K$-theory, by using the delooped "$+ = S$" theorem (see e.g. [27]), instead.

Although the delooped "$+ = S$" theorem is conceptually very simple and can be proven in a straightforward fashion, we opted to follow the (more complicated) original approach of Morel-Voevodsky [28] to prove Theorem 3.3 here. This is because we found some topologists are used to the Quillen $K$-theory much more than the Waldhausen $K$-theory. So, we thought the orignal Morel-Voevodsky [28] of proving Theorem 3.3 would provide such readers with a smooth transition from the Quillen $K$-theory to the Waldhausen $K$-theory.

(ii) However, it is fair to say that Theorem 3.3, which is a statement before inverting the $\mathbf{A}^1$-equivalence, essentially belongs to the "B.V." (= before Voevodsky) era, and might be well expected by many experts around the time. This is probably the reason why Theorem 3.3 was merely a proposition in the Morel-Voevodsky paper [28, p.139, Proposition 3.8.].

The deepest part of the $K$-theory representability in the Morel-Voevodsky paper [28] is Theorem 3.4, and especially their [28, p.139, Proposition 3.10], which claims he natural map

$$BGL_\infty \times \mathbb{Z} \to (\mathbf{R}\Omega_s^1)B(\coprod_{n\geqq 0} BGL_n)$$

is an $\mathbb{A}^1$-equivalence. It is very unfortunate that, in this exposition, we failed to say even a word about the proof of this $\mathbb{A}^1$-equivalence, though many topologists would find this $\mathbb{A}^1$-equivalence claim very convincing...

(iii) Nowadays, the Thomason-Trobaugh Excision Theorem 3.1 and Localization Theorem 3.2, both of which were the core of the proof of Theorem 3.3, can be shown in much shorter and conceptual ways. In fact, this development was already foreseen by Thomason and Trobaugh by themselves. Actually, in [44, p.302, 2.4.4.][1], Thomason-Trobaugh writes as follows:

> *To summarize, 2.4.3 roughly characterizes perfect complexes on schemes with ample families of line bundles as the finitely presented objects (in the sense of Grothendieck [EGA] IV 8.14 that* Mor *out of them preserves direct colimits) in the derived category $D(\mathcal{O}_X - Mod)_{qc}$ of complexes with quasi-coherent cohomology. On a general scheme, the prefect complexes are the locally finitely presented objects in the "***homotopy stack****" of derived categories. (We must say "roughly characterizes" as we always take our direct systems in the category $C(\mathcal{O}_X - Mod)$ of chain complexes, and have not examined the question of lifting a direct system if $D(\mathcal{O}_X - Mod)$ to $C(\mathcal{O}_X - Mod)$ up to cofinality.)*

What was not availabe at the time [44] was written was an appropriate theoretical foundation which makes their above point of view rigorous. Now, the first breakthrough for achieving this goal was provided by Neeman [32], who used the Bousfield localization technique. Then, Schilichting [37] gave a more general conceptual definition of the negative $K$-theory, which generalizes the Thomason-Trobaugh Bass $K$-theory $K^B$. In [38, p.205, Theorem 3.4.12.], Schlichting oultlined a proof of Mayer-Vietoris for open covers. Finally, in [3], necessary theoretical foundation was provided in the framework of Lurie's (stable) infinite category theory [22, 23].

## § 3.2. Homotopy Purity

**Definition 3.11** ([28, p.111, Definition 2.16])**.** Let $X$ be a smooth scheme over $S$ and $\mathcal{E}$ be a vector bundle over $X$. The ***Thom space*** of $\mathcal{E}$ is the pointed sheaf

$$Th(\mathcal{E}) = Th(\mathcal{E}/X) := \mathcal{E}/\left(\mathcal{E} \setminus i(X)\right)$$

where $i : X \to \mathcal{E}$ is the zero section of $\mathcal{E}$.

Now the following theorem is the homotopy purity:

[1] We would like to thank David Gepner for this information.

**Theorem 3.12** ([28, p.115, Theorem 2.23]). *Let $i : Z \to X$ be a closed embedding of smooth schemes over $S$. Denote by $\mathcal{N}_{X,Z} \to Z$ the normal vector bundle to $i$. Then there is a canonical isomorphism in $\mathcal{H}_\bullet(S)$ of the form*

$$X\big/ (X \setminus i(Z)) \cong Th(\mathcal{N}_{X,Z}).$$

*Proof.*
This is proven as follows:

- If the embedding of $Z$ in $X$ is a ***regular embedding***, i.e. local equations for the ideal of $Z$ in $X$ form a regular sequence in local rings of $X$, then the sheaf theoretically defined ***normal cone*** $C_Z X$ becomes a vector bundle, called the ***normal bundle*** to $Z$ in $X$, denoted $\mathcal{N}_{X,Z}$.
- [15, §17] For a closed embedding $i : Z \to X$ of smooth schemes over $S$, there exists a finite affine Zariski open covering $X = \cup U_\alpha$ such that, for all $i$, there exists an étale morphism $q_\alpha : U_\alpha \to \mathbb{A}^{n_\alpha}$ such that $q^{-1}\left(\mathbb{A}^{n_\alpha - c_\alpha} \times \{\underbrace{0,\ldots,0}_{c_\alpha}\}\right) = i(Z \cap U_\alpha)$ for some $n_\alpha$ and $c_\alpha$. In this case, we have a distinguished diagram

$$\begin{tikzcd} U_\alpha \setminus Z \arrow[r, hook, "\text{open}"'] \arrow[d] & U_\alpha \arrow[d, "\text{étale}"] \\ \mathbb{A}^{n_\alpha} \setminus \mathbb{A}^{n_\alpha - c_\alpha} \times \{\underbrace{0,\ldots,0}_{c_\alpha}\} \arrow[r, hook, "\text{open}"'] & \mathbb{A}^{n_\alpha} \end{tikzcd}$$

which presents

$$\left\{\mathbb{A}^{n_\alpha} \setminus \mathbb{A}^{n_\alpha - c_\alpha} \times \{\underbrace{0,\ldots,0}_{c_\alpha}\}, U_\alpha\right\}$$

as a Nisnevich covering of $\mathbb{A}^{n_\alpha}$. Thus, we may interpret the induced closed embedding $i_\alpha : Z \cup U_\alpha \hookrightarrow U_\alpha$ as a "***Nisnevich neighborhood***" of the closed embedding $\overline{i_\alpha} : \mathbb{A}^{n_\alpha - c_\alpha} \times \{\underbrace{0,\ldots,0}_{c_\alpha}\} \hookrightarrow \mathbb{A}^{n_\alpha}$ :

$$\begin{tikzcd} Z \cap U_\alpha \arrow[r, hook, "i_\alpha", "\text{closed}"'] \arrow[d, "\cong"'] & U_\alpha \arrow[d, "\text{étale}"] \\ \mathbb{A}^{n_\alpha - c_\alpha} \times \{\underbrace{0,\ldots,0}_{c_\alpha}\} \arrow[r, hook, "\overline{i_\alpha}", "\text{closed}"'] & \mathbb{A}^{n_\alpha} \end{tikzcd} \tag{3.89}$$

This fact conversely allows us to interpret that any closed embedding $i : Z \hookrightarrow X$ of smooth schemes over $S$ is Nisnevich locally isomorphic to the closed embedding

$$\mathbb{A}^{n-c} \times \{\underbrace{0, \ldots, 0}_{c}\} \hookrightarrow \mathbb{A}^n$$

In particular, when we consider a closed embedding $i : Z \hookrightarrow X$ of smooth schemes over $S$ in Nisnevich topology, we may regard the normal cone $C_Z X$ as the normal bundle $\mathcal{N}_{X,Z}$.

- As is slightly more evident in the exposition of [29] than in [28] (though their model structures are different), the proof essentially makes use of ***the deformation to the normal cone*** (which was discovered by MacPherson, and used extensively in [11] for instance) of a closed embedding of smooth schemes $i : Z \hookrightarrow X$ over $S$:

$$\begin{tikzcd}
Z \times \mathbb{A}^1 = (Z \times \mathbb{A}^1)_Z \arrow[rr, "j(i)"] \arrow[dr, "p"'] & & (X \times \mathbb{A}^1)_Z \setminus X_Z \arrow[dl, "p"] \\
& \mathbb{A}^1 &
\end{tikzcd}$$

Here $(Z \times \mathbb{A}^1)_Z$, $(X \times \mathbb{A}^1)_Z$ and $X_Z$ are respectively blow-ups of the closed embeddings $Z \times \{0\} \hookrightarrow Z \times \mathbb{A}^1$, $Z \times \{0\} \hookrightarrow X \times \mathbb{A}^1$ and $Z \hookrightarrow X$, respectively, and $j(i) : Z \times \mathbb{A}^1 = (Z \times \mathbb{A}^1)_Z \to (X \times \mathbb{A}^1)_Z \setminus X_Z$ is the resulting morphism of schemes over $\mathbb{A}^1$ such that

$$j(i)\big|_{p^{-1}(t)} \cong \begin{cases} \text{the given embedding } Z \hookrightarrow X & \text{if } t \neq 0 \\ \text{the zero section of the normal cone } C_Z X & \text{if } t = 0 \end{cases} \tag{3.90}$$

Since we work in the Nisnevich topology, we may interpret $C_Z X$ as the normal bundle $\mathcal{N}_{X,Z}$, as above.

- To prove the homotopy purity, we restrict (3.90) to cartesian diagrams which correspond to the cases $t = 0, 1$:

$$\begin{tikzcd}
Z \arrow[r, hook, "i", "\text{closed}"'] \arrow[d, "s_1"] & X \arrow[d, "s_1"] \\
Z \times \mathbb{A}^1 \arrow[r, hook, "j(i)", "\text{closed}"'] & (X \times \mathbb{A}^1)_Z \setminus X_Z
\end{tikzcd}
\qquad
\begin{tikzcd}
Z \arrow[r, hook, "\text{zero section}", "\text{closed}"'] \arrow[d, "s_0"] & \mathcal{N}_{X,Z} \arrow[d, "s_0"] \\
Z \times \mathbb{A}^1 \arrow[r, hook, "j(i)", "\text{closed}"'] & (X \times \mathbb{A}^1)_Z \setminus X_Z
\end{tikzcd}$$

These diagrams respectively induce

(3.91)
$$\eta(i) : T(i) := X/\left(X \setminus i(Z)\right) \;\to\; T\left(j(i)\right) := \left((X\times\mathbb{A}^1)_Z \setminus X_Z\right)/\left(\left((X\times\mathbb{A}^1)_Z \setminus X_Z\right) \setminus j(i)\left(Z\times\mathbb{A}^1\right)\right)$$
(3.92)
$$\kappa(i) : \; Th(\mathcal{N}_{X,Z}) \;\to\; T\left(j(i)\right) := \left((X\times\mathbb{A}^1)_Z \setminus X_Z\right)/\left(\left((X\times\mathbb{A}^1)_Z \setminus X_Z\right) \setminus j(i)\left(Z\times\mathbb{A}^1\right)\right)$$

Thus, it suffices to show both $\eta(i)$ and $\kappa(i)$ are $\mathbb{A}^1$-weak equivalences.

- To show both $\eta(i)$ and $\kappa(i)$ are $\mathbb{A}^1$-weak equivalences, we cover $X = \cup U_\alpha$ with finitely many affine Zariski open $U_\alpha$ with a " ***Nisnevich neighborhood*** " (3.89) of the closed embedding $\mathbb{A}^{n_\alpha - c_\alpha} \times \{\underbrace{0,\dots,0}_{c_\alpha}\} \hookrightarrow \mathbb{A}^{n_\alpha}$ :

(3.93)
$$\begin{tikzcd} Z\cap U_\alpha \arrow[r, hook, "i_\alpha", "\text{closed}"'] \arrow[d, "\cong"'] & U_\alpha \arrow[d, "\text{étale}"] \\ \mathbb{A}^{n_\alpha - c_\alpha} \times \{\underbrace{0,\dots,0}_{c_\alpha}\} \arrow[r, hook, "\overline{i_\alpha}", "\text{closed}"'] & \mathbb{A}^{n_\alpha} \end{tikzcd}$$

as above [15, §17].

- By some Meyer-Vietoris property, it is reduced to showing that, for each $\alpha$, both

(3.94)
$$\eta(i_\alpha) : T\left(i_\alpha\right) \to T\left(j(i_\alpha)\right)$$

(cf. (3.91)) and

(3.95)
$$\kappa(i_\alpha) : Th(\mathcal{N}_{U_\alpha, Z\cap U_\alpha}) \to T\left(j(i_\alpha)\right)$$

(cf. (3.92)) are $\mathbb{A}^1$-weak equivalences.

- From (3.93), we see

(3.96)
$$Th(\mathcal{N}_{U_\alpha, Z\cap U_\alpha}) \xrightarrow[\mathbb{A}^1-w.e.]{\simeq} Th\left(\mathcal{N}_{\mathbb{A}^{n_\alpha}, \mathbb{A}^{n_\alpha - c_\alpha}\times\{\underbrace{0,\dots,0}_{c_\alpha}\}}\right)$$

By some Nisnevich property applied to (3.93), we see

(3.97)
$$T\left(i_\alpha\right) \xrightarrow[\mathbb{A}^1-w.e.]{\simeq} T\left(\overline{i_\alpha}\right)$$

The Nisnevich neighborhood diagram (3.93) also induces another Nisnevich neighborhood diagram

$$\begin{tikzcd}
(Z\cap U_\alpha)\times\mathbb{A}^1 \arrow[r, hook, "j(i_\alpha)", "\text{closed}"'] \arrow[d, "\cong"'] & \left(U_\alpha\times\mathbb{A}^1\right)_{Z\cap U_\alpha}\setminus (U_\alpha)_{Z\cap U_\alpha} \arrow[d, "\text{étale}"] \\
\mathbb{A}^{n_\alpha-c_\alpha}\times\{\underbrace{0,\dots,0}_{c_\alpha}\}\times\mathbb{A}^1 \arrow[r, hook, "j(\overline{i_\alpha})", "\text{closed}"'] & \left(\mathbb{A}^{n_\alpha}\times\mathbb{A}^1\right)_{\mathbb{A}^{n_\alpha-c_\alpha}\times\{0,\dots,0\}}\setminus(\mathbb{A}^{n_\alpha})_{\mathbb{A}^{n_\alpha-c_\alpha}\times\{\underbrace{0,\dots,0}_{c_\alpha}\}}
\end{tikzcd} \tag{3.98}$$

Applying some Nisnevich property again, this time to (3.98), we see

$$T\left(j(i_\alpha)\right)\xrightarrow[\mathbb{A}^1-w.e.]{\simeq} T\left(j(\overline{i_\alpha})\right) \tag{3.99}$$

- From (3.96), (3.97), (3.99), the $\mathbb{A}^1$-equivalence properties of $\eta(i_\alpha)$ (3.94) and $\kappa(i_\alpha)$ (3.95) are equivalent to the $\mathbb{A}^1$-equivalence properties of

$$\eta(\overline{i_\alpha}) : T\left(\overline{i_\alpha}\right)\to T\left(j(\overline{i_\alpha})\right) \tag{3.100}$$

and

$$\kappa(\overline{i_\alpha}) : Th\left(\mathcal{N}_{\mathbb{A}^{n_\alpha},\mathbb{A}^{n_\alpha-c_\alpha}\times\{\underbrace{0,\dots,0}_{c_\alpha}\}}\right)\to T\left(j(\overline{i_\alpha})\right) \tag{3.101}$$

respectively, corresponding to the canonical closed embedding

$$\overline{i_\alpha} : \mathbb{A}^{n_\alpha-c_\alpha}\times\{\underbrace{0,\dots,0}_{c_\alpha}\}\hookrightarrow\mathbb{A}^{n_\alpha} \tag{3.102}$$

However, in this case, we have a canonical equivalence

$$\mathcal{N}_{\mathbb{A}^{n_\alpha},\mathbb{A}^{n_\alpha-c_\alpha}\times\{\underbrace{0,\dots,0}_{c_\alpha}\}}\cong\mathbb{A}^{n_\alpha-c_\alpha}\times\{\underbrace{0,\dots,0}_{c_\alpha}\}\times\mathbb{A}^{c_\alpha}\cong\mathbb{A}^{c_\alpha}$$

Thus, the proof of the homotopy purity is now reduced to showing the $\mathbb{A}^1$-equivalence of

$$\lambda_t(\overline{i_\alpha}) : T\left(\overline{i_\alpha}\right)\to T\left(j(\overline{i_\alpha})\right)\qquad (t=0,1) \tag{3.103}$$

induced by the cartesian diagram

$$\begin{tikzcd}
\mathbb{A}^{n_\alpha-c_\alpha}\times\{\underbrace{0,\dots,0}_{c_\alpha}\} \arrow[r, hook, "\overline{i_\alpha}", "\text{closed}"'] \arrow[d, "s_t"] & \mathbb{A}^{n_\alpha} \arrow[d, "s_t"] \\
\mathbb{A}^{n_\alpha-c_\alpha}\times\{\underbrace{0,\dots,0}_{c_\alpha}\}\times\mathbb{A}^1 \arrow[r, hook, "j(\overline{i_\alpha})", "\text{closed}"'] & \left(\mathbb{A}^{n_\alpha}\times\mathbb{A}^1\right)_{\mathbb{A}^{n_\alpha-c_\alpha}\times\{0,\dots,0\}}\setminus(\mathbb{A}^{n_\alpha})_{\mathbb{A}^{n_\alpha-c_\alpha}\times\{\underbrace{0,\dots,0}_{c_\alpha}\}}
\end{tikzcd} \tag{3.104}$$

where $t=0,1$ and $\lambda_0(\overline{i_\alpha})=\kappa(\overline{i_\alpha}),\ \lambda_1(\overline{i_\alpha})=\eta(\overline{i_\alpha})$.

- To prove the $\mathbb{A}^1$-equivalence of (3.103), we set

$$Z := \mathbb{A}^{n_\alpha - c_\alpha} \times \{\underbrace{0,\dots,0}_{c_\alpha}\}, \quad d := c_\alpha, \quad s_{\vec{0}} := \overline{i_\alpha} \tag{3.105}$$

and observe that

$$\lambda_t(s_{\vec{0}}) : T\left(s_{\vec{0}}\right) \to T\left(j(s_{\vec{0}})\right) \qquad (t = 0, 1) \tag{3.106}$$

is induced by the cartesian diagram

$$\begin{tikzcd}
Z \arrow[r, hook, "s_{\vec{0}}", "\text{closed}"'] \arrow[d, "s_t"] & Z \times \mathbb{A}^d \arrow[d, "s_t"] \\
Z \times \mathbb{A}^1 \arrow[r, hook, "j(s_{\vec{0}})", "\text{closed}"'] & \left(Z \times \mathbb{A}^d \times \mathbb{A}^1\right)_{Z \times \{\underbrace{0,\dots,0}_{d+1}\}} \setminus \left(Z \times \mathbb{A}^d\right)_{Z \times \{\underbrace{0,\dots,0}_{d}\}}
\end{tikzcd}$$

$$= Z \times \left( \begin{tikzcd}
pt \arrow[r, hook, "s_{\vec{0}}", "\text{closed}"'] \arrow[d, "s_t"] & \mathbb{A}^d \arrow[d, "s_t"] \\
\mathbb{A}^1 \arrow[r, hook, "j(s_{\vec{0}})", "\text{closed}"'] & \left(\mathbb{A}^d \times \mathbb{A}^1\right)_{\{\underbrace{0,\dots,0}_{d+1}\}} \setminus \left(\mathbb{A}^d\right)_{\{\underbrace{0,\dots,0}_{d}\}}
\end{tikzcd} \right) \tag{3.107}$$

Thus, to prove the $\mathbb{A}^1$-equivalence of (3.106) to finish the proof of the homotopy purity, we may suppose $Z = pt$ .

- The critical obervation is the following identifications of the relevant spaces as the total spaces of appropritate line bundles:

$$\begin{tikzcd}
\mathbb{A}^1 \arrow[r] \arrow[d, equal] & \left(\mathbb{A}^d \times \mathbb{A}^1\right)_{\{\underbrace{0,\dots,0}_{d+1}\}} \arrow[d, equal] & \left(\mathbb{A}^d\right)_{\{\underbrace{0,\dots,0}_{d}\}} \arrow[l] \arrow[d] \\
\left( (\{\underbrace{0,\dots,0}_{d}\} \times \mathbb{A}^1) \setminus \{\underbrace{0,\dots,0}_{d+1}\} \right) \times_{(\mathbb{A}^1 \setminus \{0\})} \mathbb{A}^1 \arrow[r] \arrow[d] & \left( (\mathbb{A}^d \times \mathbb{A}^1) \setminus \{\underbrace{0,\dots,0}_{d+1}\} \right) \times_{(\mathbb{A}^1 \setminus \{0\})} \mathbb{A}^1 \arrow[d] & \left( \mathbb{A}^d \times \{0\} \setminus \{\underbrace{0,\dots,0}_{d+1}\} \right) \times_{(\mathbb{A}^1 \setminus \{0\})} \mathbb{A}^1 \arrow[l] \arrow[d] \\
pt \arrow[r] & \mathbb{P}^d & \mathbb{P}^{d-1} \arrow[l]
\end{tikzcd} \tag{3.108}$$

$$pt \mapsto \Big[\underbrace{0,\dots,0}_{d},1\Big] \qquad \Big[\underbrace{v_1,\dots,v_d}_{d},0\Big] \leftarrow\!\shortmid \Big[\underbrace{v_1,\dots,v_d}_{d}\Big]$$

This restricts to the following identification:

$$\begin{tikzcd}
 & \mathbb{A}^1 \arrow[rr, hook, "j(s_{\vec{0}})", "\text{closed}"'] \arrow[d, equal, "t \mapsto \left(\underbrace{(0,\ldots,0,1)}_{d},t\right)"'] & & \left(\mathbb{A}^d \times \mathbb{A}^1\right)_{\underbrace{\{0,\ldots,0\}}_{d+1}} \setminus \left(\mathbb{A}^d\right)_{\underbrace{\{0,\ldots,0\}}_{d}} \arrow[d, equal] & \\
 & \left(\underbrace{\{0,\ldots,0\}}_{d} \times (\mathbb{A}^1 \setminus \{0\})\right) \times_{(\mathbb{A}^1 \setminus \{0\})} \mathbb{A}^1 \arrow[rr, hook, "\text{closed}"'] \arrow[d] & & \left(\mathbb{A}^d \times (\mathbb{A}^1 \setminus \{0\})\right) \times_{(\mathbb{A}^1 \setminus \{0\})} \mathbb{A}^1 \arrow[r, equal, "((\vec{v},\lambda),t) \mapsto (\lambda^{-1}\vec{v},\lambda t)"'] \arrow[d] & \mathbb{A}^d \times \mathbb{A}^1 \arrow[d] \\
 & pt \arrow[rr] & & \mathbb{P}^d \setminus \mathbb{P}^{d-1} \arrow[r, equal] & \mathbb{A}^d
\end{tikzcd} \tag{3.109}$$

$$pt \mapsto \left[\underbrace{0,\ldots,0}_{d},1\right] \qquad \left[\underbrace{v_1,\ldots,v_d}_{d},1\right] \leftarrow\!\shortmid \left(\underbrace{v_1,\ldots,v_d}_{d}\right)$$

From this, we may identify

$$\begin{tikzcd}
pt \arrow[r, hook, "s_{\vec{0}}", "\text{closed}"'] \arrow[d, "s_t"] & \mathbb{A}^d \arrow[d, "s_t"] \\
\mathbb{A}^1 \arrow[r, hook, "j(s_{\vec{0}})", "\text{closed}"'] & \left(\mathbb{A}^d \times \mathbb{A}^1\right)_{\underbrace{\{0,\ldots,0\}}_{d+1}} \setminus \left(\mathbb{A}^d\right)_{\underbrace{\{0,\ldots,0\}}_{d}}
\end{tikzcd} \qquad (t = 0,1) \tag{3.110}$$

with

$$\begin{tikzcd}
pt \arrow[r, hook, "s_{\vec{0}}", "\text{closed}"'] \arrow[d, "s_t"] & \mathbb{A}^d \arrow[d, "s_t"] \\
\mathbb{A}^1 \arrow[r, hook, "s_{\vec{0}}", "\text{closed}"'] & \mathbb{A}^d \times \mathbb{A}^1
\end{tikzcd} \qquad (t = 0,1)$$

Thus, we see the induced map

$$\lambda_t(s_{\vec{0}}) : T\left(s_{\vec{0}}\right) \to T\left(j(s_{\vec{0}})\right) \qquad (t = 0,1) \tag{3.111}$$

of (3.110) is induced by the following relative morphism

$$s_t : \left(\mathbb{A}^d, \mathbb{A}^d \setminus \{\vec{0}\}\right) \xrightarrow{\vec{v} \mapsto (\vec{v},t)} \left(\mathbb{A}^d, \mathbb{A}^d \setminus \{\vec{0}\}\right) \times \mathbb{A}^1 \qquad (t = 0,1)$$

from which, we see easily that $\lambda_t(s_{\vec{0}})$ (3.111) is an $\mathbb{A}^1$-weak equivalence. This completes the proof of the homotopy purity.

$\square$

## References


[1] Anderson, D. W., *Chain functors and homology theories*, Lecture Notes in Mathematicsl, Springer-Verlag, **249** (1971).

[2] Berthelot, P., Grothendieck, A., Illusie, L., *Théorie des intersections et theoreme de Riemann-Roeh (SGA6)*, Leeture Notes in Math., Springer-Verlag, **225** (1971).

[3] Blumberg, A. J., Gepner D., and Tabuada, G., "A universal characterization of higher algebraic K-theory," **arXiv10012282**,

[4] Bousfield, A. K., and Friedlander, E., M., *Homotopy theory of Gamma-spaces, spectra, and bisimplicial sets,* Geometric applications of homotopy theory, II, pp 80-130, Lecture Notes in Math, 658.

[5] Brown, K.S. and Gersten, S.M., *Algebraic K-theory and generalizied sheaf cohomology,* Lecture Notes in Math., Springer-Verlag, **341** (1973), 266–292.

[6] Bjorn Ian Dundas, B. I., Goodwillie, G., and McCarthy, R., *The local structure of algebraic K-theory,* 424 pages, May 7, 2012

[7] Dundas, B. I., Levine, M., 0stvaer, P.A., Röndigs, R., and Voevodsky, V., *Motivic Homotopy Theory,* Lectures at a Summer School in Nordfjordeid, Norway, August 2002, Springer, 2007.

[8] Dwyer, W. G. and Kan, D. W., "Calculating simplicial localizations," J. Pure Appl. Algebra 18.1 (1980), pp. 17-35.

[9] Dwyer, W. G. and Kan, D. W., "Function complexes in homotopical algebra," Topology 19.4 (1980), pp. 42-440.

[10] Dwyer, W. G. and Kan, D. W., "Simplicial localizations of categories," J. Pure Appl. Algebra 17.3 (1980), pp. 267-284.

[11] Fulton, W., *Intersection theory,* Ergeb. der Math. und ihrer Grenzgebiete, vol. 3, Springer-Verlag, 1984.

[12] Gersten, S. M., "The localization theorem for projective modules," Comm. Algebra 2 (1974), 317-350.

[13] Gille, H. and Soulé, C., *Filtrations in higher algebraic K-theory,* Algebraic K-theory (Seattle, WA, 1997), Proc. Symp. Pure Math., vol. 67, AMS, 1999, pp. 89-148.

[14] Grayson, D. R., "Higher algebraic K-theory II [after Daniel Quillen]," Algebraic K-theory, Evanston 1976, Lecture Notes in Math. 551, Springer, Berlin, Heidelberg, New York, 1976, pp. 217-240.

[15] Grothendieck, A. and Dieudonne, J., *Etude Locale des Schemas et des Morphismes de Schemas (EGA 4)*, Publ. Math. IHES,**20,24,28,32,** 1964-67.

[16] Hirschhorn, P. S., *Model categories and their localizations,* Vol. 99. Mathematical Surveys and Monographs. Providence, RI: American Mathematical Society, 2003, pp. xvi+457.

[17] Hornbostel, J., "$A^1$-representability of hermitian $K$-theory and Witt groups," *Topology,* **44** (2005), 661–687.

[18] Hovey, M., *Model Categories,* Vol. 63. Mathematical Surveys and Monographs. Providence, RI: American Mathematical Society, 1998, pp. xvi+209.

[19] Jardine, J. F., "Simplicial presheaves," *J. Pure Appl. Appl. Algebra,* **47** (1987), 35–87.

[20] Jardine, J. F., *Simplicial presheaves, Fields Lectures*, January 31, 2007.

[21] Keller, B., *Derived categories and their uses*, In: Handbook of algebra, Vol. 1, pp. 671-701. North-Holland, Amsterdam, 1996.

[22] Lurie, J., *Higher topos theory*, Annals of Mathematics Studies, 170, Princeton University Press, 2009.

[23] Lurie, J., *Derived algebraic geometry I: Stable 1-categories*, Preprint, available at **www.math.harvard.edu/lurie**.

[24] May, J. P, "The spectra associated to permutative categories,"Topology 17 (1978), no 3, 225-228.

[25] McDuff, D. and Segal, G., "Homology fibrations and the " groupcompletion " theorem," Invent. Math., 31(3), 279-284, 1975/76.

[26] Milne, J. S., *Etale Cohomology,* PMS-33, Princeton University Press, 1980.

[27] Minami, N., "A delooped "$+ = S$ theorem" of algebraic $K$-theory, preprint.

[28] Morel, F. and Voevodsky, V., "$\mathbb{A}^1$-homotopy theory of schemes," *Inst. Hautes Etudes Sci. Publ. Math.* **90** (1999), 45–143 (2001).

[29] Morel, F., *Homotopy Theory of Schemes,* Asterisque 1999, American Mathematical Society, 2006.

[30] Morel, F., $\mathbb{A}^1$*-algebraic topology,* In: International Congress of Mathematicians. Vol. II. Eur. Math. Soc., Zrich, 2006, pp. 1035-1059.

[31] Nagata, M., *Local rings, Corrected reprint,* Robert E. Krieger Publishing Co., Huntington, N.Y., 1975, pp. xiii+234.

[32] Neeman, A., "The connection between the K-theory localization theorem of Thomason, Trobaugh, and Yao and the smashing subcategories of Bousfield and Ravenel," Ann. Sci. Ecole. Norm. Sup. 25(4)(1992), no. 5, 547-566.

[33] Neeman, A., *Triangulated categories,* vol. 148 of Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2001.

[34] Quillen, D., *Higher algebraic K-theory. I.* In Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), pages 85-147. Lecture Notes in Math., Vol. 341. Springer, Berlin, 1973.

[35] Raynaud, M., *Anneaux locaux hensliens,* Lecture Notes in Mathematics, Vol. 169. Berlin: Springer-Verlag, 1970, pp. v+129. Macro

[36] Schlichting, M., "Delooping the K-theory of exact categories," Topology 43(5) (2004), 1089-1103.

[37] Schlichting, M., "Negative K-theory of derived categories," Math. Z. 253 (2006), no. 1, 97-134.

[38] Schlichting, M., *Higher algebraicK-theory*, Topics in algebraic and topologicalK-theory, 167-241, Lecture Notes in Math., 2008, Springer, Berlin, 2011.

[39] Segal, G., "Configuration-spaces and iterated loop-spaces," Invent. Math. 21 (1973), 213-221.

[40] Segal, G., "Categories and cohomology theories," Topology 13 (1974), 293-312.

[41] Srinivas, V., *Algebraic K-theory,* volume 90 of Progress in Mathematics. Birkhauser Boston Inc., Boston, MA, second edition, 1996.

[42] Staffeldt, R. E., "On fundamental theorems of algebraic K-theory," K-Theory, 2(4) 511-532, 1989.

[43] Thomason, R. W., "Algebraic K-theory and etale cohomology," Annales scientifiques de l'É. N. S. **18-3** (1985), 435-552; Erratum: **22-4** (1989), 675-677.

[44] Thomsason, R. W. and Trobaugh. T., *Higher algebraic K-theory of schemes and derived categories,* In *The Grothendieck festchrift*, volume 3, 247-436. Birkhauser, Boston, 1990.

[45] Verdier, J.-L.: *Des catégories dérivées des catégories abéliennes,* Astérisque 239, pp.xii+253 (1996) With a preface by Illusie, L., edited and with a note by Maltsiniotis, G.

[46] Waldhausen, F., *Algebraic K-theory of spaces,* Lecture Notes in Mathematics, Vol. 126,

Springer, New York, (1985), pp. 318-419.
[47] Charles Weibel, Weibel, C., *The K-book - An introduction to algebraic K-theory,* a graduate textbook in progress, 566 pages, summer 2012.