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New examples of the Borsuk-Ulam groups

By

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Abstract

A Borsuk-Ulam group (BUG) G is a group which satisfies the inequality $\dim V - \dim V^G \leqq \dim W - \dim W^G$ if whenever we have an isovariant map $\varphi : V \to W$, where V and W are G-representations. Except some cases, it is still unknown what kind of groups are Borsuk-Ulam groups. In this paper, we introduce a new sufficient condition for Borsuk-Ulam groups called the Möbius condition. It yields that $PSL(2,p^r)$,$SL(2,p^r)$ and $GL(2,p^r)$,which are unknown to be Borsuk-Ulam groups, are Borsuk-Ulam groups for any prime p.

§ 1. Introduction

Let G be a group. Suppose X and Y are G-spaces. A G-equivariant map $\varphi : X \to Y$ is called a *G-isovariant map*, if it preserves the isotropy groups, that is, $G_x = G_{\varphi(x)}$ holds for all $x \in X$. As is well known, the Borsuk-Ulam theorem ([1]) is stated as follows:

Proposition 1.1. *Let C_2 be a cyclic group of order 2. Assume that C_2 acts on both S^m and S^n antipodally. If there exists a continuous C_2-map $f : S^m \to S^n$, then $m \leqq n$ holds.*

The Borsuk-Ulam theorem is regarded to be a theorem for isovariant maps, since the actions on both spheres are free. Using the concept of isovariant maps, Wasserman

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defined the Borsuk-Ulam groups ([4]). Let G be a compact Lie group. Let V and W be G-representations with the G-fixed point sets V^G and W^G respectively. We say that G is a Borsuk-Ulam group (BUG) if whenever we have a G-isovariant map $\varphi : V \to W$, $\dim V/V^G \leqq \dim W/W^G$, that is,

$$\dim V - \dim V^G \leqq \dim W - \dim W^G$$

holds. As we will see in the next section, Wasserman gave some sufficient conditions that a group is a BUG in [4], that is, he proved that a group G is a BUG if G is solvable or satisfies the prime condition. However, it has been unknown whether his criterions are necessary or not. The purpose of this paper is to present a new sufficient condition for being a BUG and construct new examples of BUGs which does not satisfy Wasserman's conditions. Our new condition is called the Möbius condition because it is expressed with the Möbius function. Our main result is as follows:

Main Theorem. *Let $\mathbb{F}_q$ be a finite field with $q = p^r$ elements, where p is a prime number, and r is a positive integer. Then,*

$$\begin{aligned} GL(2,q) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \; a,b,c,d \in \mathbb{F}_q, \; ad - bc \neq 0 \right\}, \\ SL(2,q) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \; a,b,c,d \in \mathbb{F}_q, \; ad - bc = 1 \right\}, \textit{and} \\ PSL(2,q) &= SL(2,q)/\{\pm I\} \end{aligned}$$

are BUGs.

Remark. The groups in our Main Theorem are not solvable if $q > 3$. Moreover, there are infinitely many such kinds of finite groups which do not satisfy the prime condition.

This paper is organized as follows. In section 2, we review some properties of BUGs from [4]. In section 3, we present our new sufficient condition that a finite group becomes a BUG. Section 4 is devoted to the proof of our main result. In the last section, by constructing examples which do not satisfy the prime condition, we show that our results truly give new examples of BUGs.

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§ 2. The Borsuk-Ulam groups

In this section, we review the Borsuk-Ulam groups and the prime conditions from [4]. Let G be a compact Lie group. Let V and W be G-representations with the G-fixed point sets V^G and W^G respectively. Then, there exists a G-isovariant map $\varphi : V \to W$ if and only if there exists a G-isovariant map $\varphi' : V/V^G \to W/W^G$. As mentioned in the first section, the Borsuk-Ulam group (BUG) is defined as follows.

Definition 2.1. Let G, V and W be as above. We say that G is a *Borsuk-Ulam group (BUG)* if whenever we have a G-isovariant map $\varphi : V \to W$, $\dim V/V^G \leqq \dim W/W^G$, that is,

$$\dim V - \dim V^G \leqq \dim W - \dim W^G$$

holds.

For example, any cyclic group of prime order is a BUG. In fact, let C_p be a finite cyclic group of prime order p. Then, V/V^{C_p} and W/W^{C_p} are free C_p-representations. Hence, if $p = 2$, $\dim V/V^{C_2} \leqq \dim W/W^{C_2}$ holds by the Borsuk-Ulam theorem. Since the Borsuk-Ulam theorem also holds between the spheres with free C_p-actions for any odd prime p ([2]), the inequality $\dim V/V^{C_p} \leqq \dim W/W^{C_p}$ also holds.

Wasserman conjectured that all compact Lie groups are BUGs. However, only few things are known about BUGs. For example, it is unknown whether a subgroup of a BUG is also a BUG or not. For the study of the BUGs, the following two properties are useful.

Lemma 2.2 ([4]). *Let G be a BUG. If H is a closed normal subgroup of G, then G/H is a BUG.*

Lemma 2.3 ([4]). *Let H and K be BUGs. If $1 \to H \to G \to K \to 1$ is an exact sequence of compact Lie groups, then G is a BUG.*

By Lemma 2.3, we see that if there exists a finite group which is not a BUG, then there exists a finite simple group which is not a BUG. Since any cyclic group of prime order is a BUG, Lemma 2.3 yields the following proposition.

Proposition 2.4 ([4]). *Any solvable compact Lie group is a BUG.*

Wasserman introduced the prime condition for positive integers and finite groups as follows:

Definition 2.5 ([4]).

(1) An integer n is said to satisfy the prime condition if $\sum_{i=1}^{s} \frac{1}{p_i} \leqq 1$ holds, where $n = p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s}$ is the prime factorization of n.

(2) A finite simple group G is said to satisfy the prime condition if, for each $g \in G$, $|g|$ satisfies the prime condition.

(3) Let G be a finite group, and $\{e\} = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_r = G$ a composition series of G. A finite group G is said to satisfy the prime condition if each component factor G_{i+1}/G_i of G satisfies the prime condition.

This condition gives a sufficient condition for being a BUG. In fact the following holds.

Proposition 2.6 ([4]). *If a finite group G satisfies the prime condition, then G is a BUG.*

§ 3. New criterion for BUGs

In this section, we will state and prove our new sufficient condition that a finite group becomes a BUG.

Let $\mu : \mathbb{N} \to \{-1, 0, 1\}$ be the Möbius function, that is, $\mu(1) = 1$, $\mu(n) = 0$ if n is divisible by the square of an integer > 1, $\mu(n) = (-1)^r$ if n is the product of r distinct prime numbers. Let $(P, \leqq)$ be a poset. As is well known, the Möbius function μ of P is defined inductively for $a, b \in P$ with $a \leqq b$ by

$$\mu(a, a) = 1 \text{ and}$$
$$\mu(a, b) = -\sum_{a \leqq z < b} \mu(a, z) = -\sum_{a < z \leqq b} \mu(z, b).$$

Let $\mathcal{S}(G)$ denote the set of all subgroups of G. It is made into a poset by defining $H \leqq K$ in $\mathcal{S}(G)$ if $H \subset K$. Let $\mathrm{Cycl}(G)$ be the full subposet of $\mathcal{S}(G)$ which contains all cyclic subgroups of G. It is well known that $\mu(H, K) = \mu(|K/H|)$ for $H, K \in \mathrm{Cycl}(G)$ with $H \leqq K$. For proving our main theorem, the following proposition plays an important role.

Proposition 3.1. *Let G be a finite group. If for any cyclic subgroup $D(\neq \{1\})$ of G*

$$\sum_{D \leqq C \in \mathrm{Cycl}(C_G(D))} \mu(D, C) \geqq 0$$

holds, then G is a BUG, where $C_G(D)$ is the centralizer of D in G.

Proof. Suppose that there is a G-isovariant map $V \to W$ between G-representations V and W. It is sufficient to prove

$$\sum_{D \leqq C \in \mathrm{Cycl}(G)} \mu(D, C) \geqq 0$$

since $\{C \in \mathrm{Cycl}(G) \mid D \leqq C \leqq \mathrm{Cycl}(C_G(D))\ \} = \{C \in \mathrm{Cycl}(G) \mid D \leqq C\ \}$. For $g \in G$, put

$$\alpha(g) = \chi_W(1) - \chi_W(g) - \chi_V(1) + \chi_V(g).$$

Since $\dim W = \dfrac{1}{|G|} \sum_{g \in G} \chi_W(1)$, we have

$$\dim W - \dim W^G - (\dim V - \dim V^G) = \frac{1}{|G|} \sum_{g \in G} \alpha(g).$$

For $H \in \mathcal{S}(G)$, put $h(H) = \sum_{g \in H} \alpha(g)$. For $C \in \mathrm{Cycl}(G)$, put $k(C) = \displaystyle\sum_{g \in gen(C)} \alpha(g)$, where $gen(C)$ is the set of all generators of C. Let D be a cyclic subgroup of G. If $D \neq \{1\}$, we see that

$$h(D) = \sum_{g \in D} \alpha(g) = \sum_{C \in \mathrm{Cycl}(D)} k(C).$$

If $D = \{1\}$, we have

$$h(D) = \alpha(1) = \chi_W(1) - \chi_W(1) - \chi_V(1) + \chi_V(1) = 0.$$

Then, for $C \in \mathrm{Cycl}(G)$ it follows from the Möbius inversion formula that

$$k(C) = \sum_{D \in \mathrm{Cycl}(C)} \mu(D, C) h(D).$$

Hence,

$$h(G) = \sum_{g \in G} \alpha(g) = \sum_{C \in \mathrm{Cycl}(G)} \sum_{g \in gen(C)} \alpha(g) = \sum_{C \in \mathrm{Cycl}(G)} k(C)$$

$$= \sum_{C \in \mathrm{Cycl}(G)} \sum_{D \in \mathrm{Cycl}(C)} \mu(D, C) h(D) = \sum_{D \in \mathrm{Cycl}(G)} \Big(\sum_{D \leqq C \in \mathrm{Cycl}(G)} \mu(D, C) \Big) h(D).$$

Since D is cyclic, D is a BUG, which means $h(D) \geqq 0$. Thus, by assumption $h(G) \geqq 0$. □

Definition 3.2. We say that a cyclic subgoup $D(\neq \{1\})$ of a finite group G satisfies *the Möbius condition* if D satisfies the inequality stated in Proposition 3.1. We say that a finite group G satisfies *the Möbius condition* if each cyclic subgroup $D(\neq \{1\})$ satisfies the Möbius condition.

There is an equivalent condition to the Möbius condition as follows.

Definition 3.3. Let G be a finite group. If for any element $g \in G$ of prime order, there exists a cyclic subgroup C of G such that C contains any $D \in \mathrm{Cycl}(G)$ with $g \in D$, we say that G satisfies *the maximality condition.*

Proposition 3.4. *Finite group G satisfies the Möbius condition if and only if G satisfies the maximality condition.*

Proof. Suppose G satisfies the maximality condition. Let $D \in \mathrm{Cycl}(G)$ with $D \neq \{1\}$. Applying the maximality condition for a generator g of D, we can find $\hat{D} \in \mathrm{Cycl}(G)$ such that $\hat{D}$ contains any $C \in \mathrm{Cycl}(G)$ with $D \leqq C$. If $D = \hat{D}$, then we have

$$\sum_{D \leqq C \in \mathrm{Cycl}(G)} \mu(D, C) = \mu(D, D) = 1$$

and otherwise

$$\sum_{D \leqq C \in \mathrm{Cycl}(G)} \mu(D, C) = \sum_{D \leqq C \in \mathrm{Cycl}(\hat{D})} \mu(D, C) = 0$$

because $\{C \in \mathrm{Cycl}(G) \mid D \leqq C\} = \{C \in \mathrm{Cycl}(G) \mid D \leqq C \leqq \hat{D}\}$ holds.

Conversely, suppose the maximality condition does not hold. There are a cyclic subgroup $D \neq \{1\}$ and maximal cyclic subgroups $\hat{D}_1$ and $\hat{D}_2$ of G such that $D < \hat{D}_1, D < \hat{D}_2$ and $\hat{D}_1 \neq \hat{D}_2$. Here a cyclic subgroup $\hat{D}$ is called maximal if there is no cyclic subgroup C of G with $\hat{D} < C$. By considering the maximality of such D, we may suppose that for any $D' \in \mathrm{Cycl}(G)$ with $D < D'$ there exists a unique maximal cyclic subgroup $\hat{D}'$ such that $D' \leqq \hat{D}'$. Let $\hat{D}_1, \ldots, \hat{D}_m$ be all distinct maximal cyclic sugroups of G containing D. Then, we have

$$\sum_{D \leqq C \in \mathrm{Cycl}(G)} \mu(D, C) = \sum_{1 \leqq j \leqq m} \sum_{D \leqq C \in \mathrm{Cycl}(\hat{D}_j)} \mu(D, C) - (m-1)\mu(D, D) = -(m-1) < 0.$$

□

§ 4. Proof of Main Thoerem

Let p be a prime number, and $q = p^r (r \in \mathbb{N})$. For simplicity, put $SL = SL(2, q)$, $PSL = PSL(2, q)$ and $GL = GL(2, q)$. For proving our main result, we review the conjugacy classes of SL. First, we prepare the following lemma for introducing an element B of GL which is necessary to describe the conjugacy classes of SL.

Lemma 4.1 ([3]). *For each $q = p^r (r \in \mathbb{N})$, GL has an element B of order $(q^2 - 1)$ which satisfies the following conditions:*

(1) $\langle B\rangle \supset Z(GL)=\left\{\begin{pmatrix} a & 0\\ 0 & a\end{pmatrix} \,\middle|\, a\in \mathbb{F}_q^*\right\}$, *the center of* GL.

(2) If $YXY^{-1}=X^r$ *for* $X\in\langle B\rangle - Z(GL)$ *and* $Y\in GL$, *then* $X^r=X$ *or* X^q.

(3) For any $X\in\langle B\rangle - Z(GL)$, $C_{GL}(X)=\langle B\rangle$.

(4) $|N_{GL}(\langle B\rangle)|=2(q^2-1)$.

The structure of the conjugacy classes of SL are different between the cases $p=2$ and odd p. For presenting its conjugacy classes, we introduce some elements of SL which are used in the both cases. Let B be the element in Lemma 4.1. Put,

$$I=\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix},\quad P=\begin{pmatrix} 1 & 1\\ 0 & 1\end{pmatrix},\quad D=\begin{pmatrix} \delta & 0\\ 0 & \delta^{-1}\end{pmatrix},\quad C=B^{q-1},$$

where δ is a generator of $\mathbb{F}_q^*$.

When $p=2$, the conjugacy classes of SL are as follows:

Proposition 4.2 ([3])**.** *Let* $q=2^r (r\in\mathbb{N})$. *Then,* $SL(2,q)$ *has the conjugacy classes whose representatives are*

$$I,\quad P,\quad D^i\ (1\leqq i\leqq \frac{q-2}{2}),\quad C^j\ (1\leqq j\leqq \frac{q}{2}).$$

When p is an odd prime, the conjugacy classes of SL are as follows:

Proposition 4.3 ([3])**.** *Let* p *be an odd prime, and* $q=p^r (r\in\mathbb{N})$. *Pick a non-square element* $\nu\in\mathbb{F}_q^*\setminus(\mathbb{F}_q^*)^2$, *and fix it. Put*

$$Z_0=\begin{pmatrix} -1 & 0\\ 0 & -1\end{pmatrix},\quad P'=\begin{pmatrix} 1 & \nu\\ 0 & 1\end{pmatrix}.$$

Then, $SL(2,q)$ *has the conjugacy classes whose representatives are*

$$I,\quad Z_0,\quad P,\quad PZ_0,\quad P',\quad P'Z_0,\quad D^i\ (1\leqq i\leqq \frac{q-3}{2}),\quad C^j\ (1\leqq j\leqq \frac{q-1}{2}).$$

We will prove our main result by checking that the cyclic subgroups generated by each element presented in Propositions 4.2 and 4.3 satisfies the Möbius condition. At first, we show that $SL:=SL(2,2^r)$ is a BUG.

(i) The case of $\langle P\rangle$ By considering the elements of SL which is commutative to $\begin{pmatrix} 1 & 1\\ 0 & 1\end{pmatrix}$, we see that

$$C_{SL}(\langle P\rangle)=\left\{\begin{pmatrix} 1 & b\\ 0 & 1\end{pmatrix} \,\middle|\, b\in\mathbb{F}_q\right\}\cong (C_2)^r,$$

where C_2 is a cyclic group of order 2. Since $\langle P\rangle \cong C_2$, the only cyclic subgroup between $\langle P\rangle$ and $C_{SL}(\langle P\rangle)$ is $\langle P\rangle$. Thus, $\langle P\rangle$ satisfies the Möbius condition.

(ii) The case of $\langle D\rangle$ First, we note that such a cyclic subgroup D exists when $r \geqq 2$ and in this case $|\langle D\rangle| = q-1$. Let $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element $C_{SL}(\langle D\rangle)$. By $DX = XD$, we have $c\delta = c\delta^{-1}$ and $b\delta = b\delta^{-1}$. Since δ is a generator of $\mathbb{F}_q^* \cong C_{q-1}$, $b = c = 0$ holds. Hence,

$$C_{SL}(\langle D\rangle) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \,\middle|\, a \in \mathbb{F}_q^* \right\} = \langle D\rangle.$$

Thus, $\langle D\rangle$ satisfies the Möbius condition.

(iii) The case of $\langle D^i\rangle$ As similar as the case of $\langle D\rangle$, $C_{SL}(\langle D^i\rangle) = \langle D\rangle$. If $\langle D^i\rangle = \langle D\rangle$, then

$$\sum_{\langle D^i\rangle \leqq H \in \mathrm{Cycl}(\langle D\rangle)} \mu(\langle D^i\rangle, H) = \mu(\langle D\rangle, \langle D\rangle) = 1.$$

If $\langle D^i\rangle < \langle D\rangle$,the definition of the Möbius function

$$\mu(\langle D^i\rangle, \langle D\rangle) = -\sum_{\langle D^i\rangle \leqq H:\text{cyclic} < \langle D\rangle} \mu(\langle D^i\rangle, H)$$

yields

$$\sum_{\langle D^i\rangle \leqq H \in \mathrm{Cycl}(\langle D\rangle)} \mu(\langle D^i\rangle, H) = 0.$$

Thus, $\langle D^i\rangle$ satisfies the Möbius condition.

(iv) The case of $\langle C^j\rangle$ Assume that $C^j = B^{j(q-1)}$ belongs to $Z(GL)$ for $1 \leqq j \leqq \frac{q}{2}$. Then, since C is an element of $SL(2,q)$, C must be equal to I, which contradicts to the order of B. Thus, $C^j \in \langle B\rangle - Z(GL)$. Hence, it follows from Lemma 4.1 that $C_{GL}(\langle C^j\rangle) = \langle B\rangle$ for $1 \leqq j \leqq \frac{q}{2}$. Therefore,

$$C_{SL}(\langle C^j\rangle) = \langle B\rangle \cap SL = \langle C\rangle \quad (1 \leqq j \leqq \frac{q}{2}).$$

By a similar discussion as the previous case, we obtain

$$\sum_{\langle C^j\rangle \leqq H \in \mathrm{Cycl}(\langle C\rangle)} \mu(\langle C^j\rangle, H) = \begin{cases} 0 & \text{if } \langle C^j\rangle < \langle C\rangle \\ 1 & \text{if } \langle C^j\rangle = \langle C\rangle. \end{cases}$$

Thus, $\langle C^j\rangle$ satisfies the Möbius condition.

Next, we present outline of the proof that $PSL := SL(2,p^r)/\{\pm I\}$ is a BUG for any odd prime p and $r \in \mathbb{N}$. Let $\overline{X}$ denote the element of PSL represented by $X \in SL$.

(i) The case of $\langle \overline{P} \rangle$

By a similar discussion as $SL(2,2^r)$ case, we see that

$$C_{PSL}(\langle \overline{P} \rangle) = \left\{ \overline{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}} \;\middle|\; b \in \mathbb{F}_q \right\} \cong (C_p)^r.$$

Since $\langle \overline{P} \rangle \cong C_p$, the only cyclic subgroup between $\langle \overline{P} \rangle$ and $C_{PSL}(\langle \overline{P} \rangle)$ is $\langle \overline{P} \rangle$. Thus, $\langle \overline{P} \rangle$ satisfies the Möbius condition.

(ii) The case of $\langle \overline{P'} \rangle$

We can check that $\langle \overline{P'} \rangle$ satisfies the Möbius condition as similar as case (i).

(iii) The case of $\langle \overline{D^i} \rangle$

Suppose that $q \equiv 1 \mod 4$. If $i \neq \dfrac{q-1}{4}$, then we have

$$C_{PSL}(\langle \overline{D^i} \rangle) = \left\{ \overline{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}} \;\middle|\; a \in \mathbb{F}_q^* \right\} = \langle \overline{D} \rangle \cong C_{\frac{q-1}{2}},$$

hence

$$\sum_{\langle \overline{D^i} \rangle \leqq H \in \mathrm{Cycl}(\langle \overline{D} \rangle)} \mu(\langle \overline{D^i} \rangle, H) = 0$$

holds. On the other hand, if $i = \dfrac{q-1}{4}$, then we have

$$C_{PSL}(\langle \overline{D^i} \rangle) = \left\{ \overline{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}}, \overline{\begin{pmatrix} 0 & b \\ -b^{-1} & 0 \end{pmatrix}} \;\middle|\; a, b \in \mathbb{F}_q^* \right\} \cong D_{q-1},$$

hence

$$\sum_{\langle \overline{D^i} \rangle \leqq H \in \mathrm{Cycl}(C_{PSL}(\langle \overline{D^i} \rangle))} \mu(\langle \overline{D^i} \rangle, H) = 0.$$

holds, where D_{q-1} is a dihedral group of order $q-1$. Suppose that $q \equiv 3 \mod 4$. Then, we have

$$C_{PSL}(\langle \overline{D^i} \rangle) = \left\{ \overline{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}} \;\middle|\; a \in \mathbb{F}_q^* \right\} = \langle \overline{D} \rangle \cong C_{\frac{q-1}{2}},$$

hence

$$\sum_{\langle \overline{D^i} \rangle \leqq H \in \mathrm{Cycl}(\langle \overline{D} \rangle)} \mu(\langle \overline{D^i} \rangle, H) = 0$$

holds. Thus, $\langle \overline{D^i} \rangle$ satisfies the Möbius condition.

(iv) The case of $\langle \overline{C^j} \rangle$

Suppose that $q \equiv 3 \mod 4$. If $j \neq \dfrac{q+1}{4}$, then we have

$$C_{PSL}(\langle \overline{C^j} \rangle) = \langle \overline{C} \rangle \cong C_{\frac{q+1}{2}},$$

hence

$$\sum_{\langle \overline{C^j} \rangle \leqq H \in \mathrm{Cycl}(\langle \overline{C} \rangle)} \mu(\langle \overline{C^j} \rangle, H) = 0$$

holds. On the other hand, if $j = \dfrac{q+1}{4}$, then we have

$$C_{PSL}(\langle \overline{C^j} \rangle) = \left\{ \overline{\begin{pmatrix} a & -b \\ b & a \end{pmatrix}}, \overline{\begin{pmatrix} c & d \\ d & -c \end{pmatrix}} \;\middle|\; a^2 + b^2 = 1, c^2 + d^2 = -1 \right\} \cong D_{q+1},$$

hence

$$\sum_{\langle \overline{C^j} \rangle \leqq H \in \mathrm{Cycl}(C_{PSL}(\langle \overline{C^j} \rangle))} \mu(\langle \overline{C^j} \rangle, H) = 0$$

holds, where D_{q+1} is a dihedral group of order $q+1$. Suppose that $q \equiv 1 \mod 4$. Then, we obtain

$$C_{PSL}(\langle \overline{C^j} \rangle) = \langle \overline{C} \rangle \cong C_{\frac{q+1}{2}},$$

hence

$$\sum_{\langle \overline{C^j} \rangle \leqq H \in \mathrm{Cycl}(\langle \overline{C} \rangle)} \mu(\langle \overline{C^j} \rangle, H) = 0$$

holds. Now, we have checked that $SL(2, 2^r)$ and $PSL(2, p^r)$ (p:odd prime) satisfy the Möbius condition, thereby they are BUGs. By applying Lemma 2.3 to the following exact sequences

$$1 \longrightarrow \{\pm I\} \xrightarrow{incl} SL(2, p^r) \xrightarrow{proj} PSL(2, p^r) \longrightarrow 1 \quad \text{(exact)}$$

$$1 \longrightarrow SL(2, q) \xrightarrow{incl} GL(2, q) \xrightarrow{det} \mathbb{F}_q^* \longrightarrow 1 \quad \text{(exact)},$$

we complete the proof.

§ 5. Our results and the prime condition

Let p be a prime number. For $q = p^r, (r > 1)$, $PSL(2, q)$ is simple. In this section, by constructing examples which do not satisfy the prime condition, we show that our results truly give new examples of BUGs.

Example 5.1. For $r \geqq 2$, put $q = 2^r$. We consider the element $D \in SL(2,q) = PSL(2,q)$ defined in Proposition 4.2. Its order is $q-1$. If $r = 180m$ $(m \in \mathbb{N})$, $q-1$ does not satisfy the prime condition, thereby, such a $SL(2,q)$ is a new example of a BUG. In fact, the set of prime factors of $2^{180}-1$ is $\{3, 5, 7, 11, 13, 19, 31, 37, 41, 61, 73, 109, 151, 181, 331, 631, 1321, 23311, 54001, 18837001, 29247661\}$ and

$$\sum_{\substack{p \mid 2^{180}-1 \\ p:\text{prime}}} \frac{1}{p} = \frac{3532122154914024131931168790209409121445196817323781 3}{340554564636864190746299339366735374137497582707467 15} > 1.$$

Example 5.2. Let p be an odd prime number. For $r \geqq 2$, put $q = p^r$. We take the element $D \in SL(2,q)$ defined in Proposition 4.3. Then the order of $\overline{D} \in PSL(2,q)$ is $\dfrac{q-1}{2}$. As a similar discussion as Example 5.1, we obtain :

(1) If $p = 3$ and $r = 60m$ $(m \in \mathbb{N})$, $\dfrac{q-1}{2}$ does not satisfy the prime condition because it is divisible by $10010 = 2 \cdot 5 \cdot 7 \cdot 11 \cdot 13$.

(2) If $p = 5$ and $r = 30m$ $(m \in \mathbb{N})$, $\dfrac{q-1}{2}$ does not satisfy the prime condition because it is divisible by $462 = 2 \cdot 3 \cdot 7 \cdot 11$.

(3) If $p \geqq 7$ and $r = 4m$ $(m \in \mathbb{N})$, $\dfrac{q-1}{2}$ does not satisfy the prime condition because it is divisible by 30.

Thus, for these p and r, $PSL(2,p^r)$ is a new example of a BUG.

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