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# Seifert construction for nilpotent groups and Application to $S^1$-fibred nilBott Tower

By

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**Abstract**

We shall introduce a notion of $S^1$*-fibred nilBott tower*. It is an iterated $S^1$-bundle whose top space is called an $S^1$-fibred nilBott manifold. The nilBott tower is a generalization of *real Bott tower* from the viewpoint of fibration. We prove that any $S^1$-fibred nilBott manifold is *diffeomorphic* to an infranilmanifold. An $S^1$-fibred nilBott tower defines a sequence of group extensions. We study the group extension at each stage to apply Seifert rigidity for $S^1$-fibred nilBott manifolds.

## § 1. Introduction

Let $M$ be a closed aspherical manifold which is a top space of an iterated $S^1$-bundles over a point:

$$M = M_n \to M_{n-1} \to \ldots \to M_1 \to \{\mathrm{pt}\}. \tag{1.1}$$

Suppose $X$ is the universal covering of $M$ and each $X_i$ is the universal covering of $M_i$ and put $\pi_1(M_i) = \pi_i$ $(i = 1, \ldots, n-1)$ and $\pi_1(M) = \pi$.

**Definition 1.1.** An $S^1$*-fibred nilBott tower* is a sequence (1.1) which satisfies I, II and III below $(i = 1, \ldots, n-1)$. The top space $M$ is said to be an $S^1$*-fibred nilBott manifold (of depth $n$).*

I. $M_i$ is a fiber space over $M_{i-1}$ with fiber $S^1$.

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II. For the group extension

$$1 \to \mathbb{Z} \to \pi_i \longrightarrow \pi_{i-1} \to 1 \tag{1.2}$$

associated to the fiber space (I), there is an equivariant principal bundle:

$$\mathbb{R} \to X_i \xrightarrow{p_i} X_{i-1}. \tag{1.3}$$

III. Each $\pi_i$ normalizes $\mathbb{R}$.

The purpose of this paper is to announce the following result.

**Theorem 1.2.** *Suppose that $M$ is an $S^1$-fibred nilBott manifold.*

*(I) If every cocycle of $H^2_\phi(\pi_{i-1};\mathbb{Z})$ which represents a group extension* (1.2) *is* of finite order*, then $M$ is diffeomorphic to a Riemannian flat manifold.*

*(II) If there exists a cocycle of $H^2_\phi(\pi_{i-1};\mathbb{Z})$ which represents a group extension* (1.2) *is* of infinite order*, then $M$ is diffeomorphic to an infranilmanifold. In addition, $M$ cannot be diffeomorphic to any Riemannian flat manifold.*

## § 2. Preliminaries

### § 2.1. Infrahomogeneous space

Let $G$ be a (noncompact) simply connected Lie group, and $\mathrm{Aut}(G)$ denote the group of automorphisms of $G$ onto itself. Put $\mathrm{A}(G) = G \rtimes \mathrm{Aut}(G)$. $\mathrm{A}(G)$ becomes a group;

$$(g,\alpha)\cdot(h,\beta) = (g\cdot\alpha(h), \alpha\cdot\beta)$$

$(g, h \in G, \alpha, \beta \in \mathrm{Aut}(G))$. $A(G)$ is called the affine group of $G$. Here, letting $X = G$, an affine action $(\mathrm{A}(G), X)$ is obtained as follows:

$$((g,\alpha), x) = g\cdot\alpha(x).$$

Let $H \subset \mathrm{Aut}(G)$ be a compact subgroup (for example, maximal compact subgroup, finite groups). Form a subgroup $\mathrm{E}(G) = G \rtimes H \subset \mathrm{A}(G)$. Consider the action $(\mathrm{E}(G), X)$. We note that if $H$ is compact, then it is easy to check the following.

**Lemma 2.1** (Proper action). $(\mathrm{E}(G), X)$ *is a proper action.*

By Lemma 2.1, if $\pi \subset \mathrm{E}(G)$ is a discrete subgroup, we obtain a properly discontinuous action $(\pi, X)$.

**Definition 2.2.** The quotient space $X/\pi$ is said to be an infrahomogeneous orbifold. When $\pi$ has no elements of finite order, $\pi$ is said to be torsionfree, and $X/\pi$ is called an infrahomogeneous manifold.

**Example 2.3.**

(1) Taking the vector space $\mathbb{R}^n$ as $G$ it gives the usual affine group $\mathrm{A}(\mathbb{R}^n) = \mathbb{R}^n \rtimes \mathrm{GL}(n, \mathbb{R})$. If $H$ is a maximal compact subgroup $\mathrm{O}(n)$ of $\mathrm{GL}(n, \mathbb{R})$, we have the euclidean group $\mathrm{E}(\mathbb{R}^n) = \mathbb{R}^n \rtimes \mathrm{O}(n)$. A discrete uniform subgroup $\pi$ of $\mathrm{E}(\mathbb{R}^n)$ is called a crystallographic group. If $\pi \subset \mathrm{E}(\mathbb{R}^n)$ is a torsionfree crystallographic group, $\pi$ is called a Bieberbach group. Moreover, the infrahomogeneous space $\mathbb{R}^n/\pi$ is an Euclidean space form, i.e. a Riemannian flat manifold.

(2) When $G$ is a simply connected nilpotent Lie group $\mathcal{N}$, for any torsionfree discrete uniform subgroup $\pi \subset \mathrm{E}(\mathcal{N})$, $\mathcal{N}/\pi$ is called an infranilmanifold.

We have the fundamental classical result for crystallographic groups.

**Theorem 2.4** (Bieberbach first theorem). *Let $\pi \subset \mathrm{E}(\mathbb{R}^n)$ be a crystallographic group, then $\mathbb{R}^n \cap \pi \cong \mathbb{Z}^n$ and $\pi/\mathbb{R}^n \cap \pi$ is a finite group.*

The above theorem is extended to the almost crystallographic groups. See [4] for instance.

**Theorem 2.5** (Auslander-Bieberbach theorem). *Let $\pi$ be a torsionfree discrete uniform subgroup of $\mathrm{E}(\mathcal{N})$, then $\mathcal{N} \cap \pi$ is a maximal normal nilpotent subgroup of $\pi$ and $\pi/\mathcal{N} \cap \pi$ is a finite group.*

## § 3. Nil Geometry

Let

$$1 \to \Delta \to \pi \to F \to 1 \tag{3.1}$$

be a group extension where $\pi$ is a torsionfree group, $\Delta$ is a torsionfree finitely generated nilpotent group, and $F$ is a finite group. By Mal'cev's *existence* theorem, there is a (simply connected) nilpotent Lie group $\mathcal{N}$ containing $\Delta$ as a discrete uniform subgroup. The rest of this section is to review the following realization theorem obtained in [5].

**Theorem 3.1** (Realization). *There exists a discrete faithful representation $\rho : \pi \to \mathrm{E}(\mathcal{N})$ such that $\rho|\Delta = \mathrm{id}$. In particular, $\mathcal{N}/\rho(\pi)$ is an infranilmanifold.*

In order to prove this theorem, we need several facts. So we shall prepare them in turn.

## § 3.1. 2-cocycle

We recall the group cohomology. (Compare [10], [2] for example.)

Let $G, Q$ be groups and $\phi : Q \to \mathrm{Aut}(G)$ a function. Suppose there is a function $f : Q \times Q \to G$ which satisfies that

(i) $\phi(\alpha)(\phi(\beta)(n)) = f(\alpha, \beta)\phi(\alpha\beta)(n)f(\alpha, \beta)^{-1}$

(ii) $f(\alpha, 1) = f(1, \alpha) = 1$,

(iii) $\phi(\alpha)(f(\beta, \gamma))f(\alpha, \beta\gamma) = f(\alpha, \beta)f(\alpha\beta, \gamma)$,

where $n \in G$ and $\alpha, \beta, \gamma \in Q$. Then $f$ defines a group $E$ which is the product $G \times Q$ with the group law:

$$(n, \alpha)(m, \beta) = (n \cdot \phi(\alpha)(m) \cdot f(\alpha, \beta), \alpha\beta). \tag{3.2}$$

Then there is a $\phi$-group extension $1 \to G \to E \xrightarrow{\nu} Q \to 1$ where $\nu(n, \alpha) = \alpha$ and the group $E$ is denoted by $G \times_{(f,\phi)} Q$.

Conversely, given a group extension $1 \to G \to E \xrightarrow{\nu} Q \to 1$, we can associate $E$ with a $\phi$- group extension. Choose a section $q : Q \to E$ ($\nu \circ q = \mathrm{id}$), and $q(1) = 1$. A function $\phi : Q \to \mathrm{Aut}(G)$ is defined to be

$$\phi(\alpha)(n) = q(\alpha)nq(\alpha)^{-1} \quad (\forall\, \alpha \in Q, \forall\, n \in G).$$

Both $q(\alpha\beta)$, $q(\alpha)q(\beta)$ are mapped to $\alpha\beta \in Q$, so there is an element $f(\alpha, \beta) \in G$ such that $f(\alpha, \beta) \cdot q(\alpha\beta) = q(\alpha)q(\beta)$. Then it is easily checked that $f : Q \times Q \to G$ satisfies the above (i) (ii) (iii).

Let $\mathrm{Opext}(Q, G, \phi)$ be the set of all congruence classes of $\phi$- group extensions. Then an element $[f] \in \mathrm{Opext}(Q, G, \phi)$ is represented by an extension $1 \to G \to E \to Q \to 1$ with $E = G \times_{(f,\phi)} Q$. It is easy to check that $[f_1] = [f_2] \in \mathrm{Opext}(Q, A, \phi)$ if and only if there is a function $\lambda : Q \to \mathcal{C}(G)$ such that

$$f_1(\alpha, \beta) = \delta^1\lambda(\alpha, \beta) \cdot f_2(\alpha, \beta) \quad (\forall\ \alpha, \beta \in Q). \tag{3.3}$$

Here $\mathcal{C}(G)$ is the center of $G$ and $\delta^1$ is defined by $\delta^1\lambda(\alpha, \beta) = \phi(\alpha)(\lambda(\beta))\lambda(\alpha)\lambda(\alpha\beta)^{-1}$. For simplicity, we write it as $f_1 = \delta^1\lambda \cdot f_2$.

In particular, when $G$ is an abelian group $A$, $\phi : Q \to \mathrm{Aut}(A)$ is a homomorphism and hence $A$ is a $Q$-module. So there is the group cohomology $H^2_\phi(Q, A)$ and $f$ is a 2-cocycle by (iii), i.e. $[f] \in H^2_\phi(Q, A)$. Therefore any extension $1 \to A \to E \to Q \to 1$ corresponds to a cocycle $[f] \in H^2_\phi(Q, A)$. It is easy to check the following.

**Proposition 3.2.** *Suppose that $A$ is an abelian group. Then there is a one-to-one correspondence between $H^2_\phi(Q,A)$ and* $\mathrm{Opext}(Q,A,\phi)$.

*Remark.* Suppose $Q=F$ is a finite group and $f: F\times F\to\mathbb{R}^n$ is a 2-cocycle relative to $\phi: F\to\mathrm{Aut}(\mathbb{R}^n)$. Put $h: F\to\mathbb{R}^n$;

$$h(\alpha)=\sum_{\tau\in F} f(\alpha,\tau). \tag{3.4}$$

Then

$$\begin{aligned}
\delta^1 h(\alpha,\beta) &= \phi(\alpha)(h(\beta)) - h(\alpha\beta) + h(\alpha)\\
&= \sum_{\tau\in F}\phi(\alpha)(f(\beta,\tau)) - \sum_{\tau\in F} f(\alpha\beta,\tau) + \sum_{\tau\in F} f(\alpha,\tau)\\
&= \sum_{\tau\in F}(f(\alpha\beta,\tau) - f(\alpha,\beta\tau) + f(\alpha,\beta)) - \sum_{\tau\in F} f(\alpha\beta,\tau) + \sum_{\tau\in F} f(\alpha,\tau)\\
&= |F| f(\alpha,\beta)
\end{aligned}$$

i.e. $\delta^1\frac{1}{|F|}h=f$. It implies that

$$H^2_\phi(F;\mathbb{R}^n)=0. \tag{3.5}$$

## § 3.2. Pushout

Let $\pi$, $\Delta$ and $\mathcal{N}$ be as before and $1\to\Delta\to\pi\to Q\to 1$ a group extension which is represented by $[f]\in\mathrm{Opext}(Q,\Delta,\phi)$. Given a function $\phi: Q\to\mathrm{Aut}(\Delta)$, Mal'cev's unique extension theorem implies that each automorphism $\phi(\alpha):\Delta\to\Delta$ extends uniquely to an automorphism $\bar\phi(\alpha):\mathcal{N}\to\mathcal{N}$. In particular, this gives a correspondence $\bar\phi: Q\to\mathrm{Aut}(\mathcal{N})$. Note that it is not necessarily a homomorphism. In general it satisfies

$$\bar\phi(\alpha)(\bar\phi(\beta)(x)) = f(\alpha,\beta)\bar\phi(\alpha\beta)(x)f(\alpha,\beta)^{-1}\ (x\in\mathcal{N}). \tag{3.6}$$

Then the "pushout" $\pi\mathcal{N}=\{(x,\alpha)\mid x\in\mathcal{N},\alpha\in Q\}$ can be constructed. Its group law is defined by $(x,\alpha)\cdot(y,\beta)=(x\bar\phi(\alpha)(y)f(\alpha,\beta),\alpha\beta)$;

$$\begin{tikzcd}
1 \arrow[r] & \mathcal{N} \arrow[r] & \pi\mathcal{N} \arrow[r] & Q \arrow[r] & 1\\
1 \arrow[r] & \Delta \arrow[r] \arrow[u] & \pi \arrow[r] \arrow[u] & Q \arrow[r] \arrow[u, equal] & 1.
\end{tikzcd} \tag{3.7}$$

This group (extension) $\pi\mathcal{N}$ is also represented by $[f]\in\mathrm{Opext}(Q,\mathcal{N},\bar\phi)$.

## § 3.3. Existence of the Seifert construction

Let $W$ be a contractible smooth manifold. Suppose that a group $Q$ acts properly discontinuously on $W$ such that the quotient space $W/Q$ is compact. Given a group extension:

$$1 \longrightarrow \Delta \longrightarrow \pi \xrightarrow{\nu} Q \longrightarrow 1, \tag{3.8}$$

we shall show that there is an action of $\pi$ on $\mathcal{N} \times W$ which is compatible with the left translations of $\mathcal{N}$. Let $\mathrm{Diff}(\mathcal{N} \times W)$ be the group of all diffeomorphisms of $\mathcal{N} \times W$ onto itself. $\mathcal{N}$ is a subgroup of $\mathrm{Diff}(\mathcal{N} \times W)$ via an embedding: $l(n)(m, \alpha) = (nm, \alpha)$.

We denote $\mathrm{Diff}^{\mathrm{F}}(\mathcal{N} \times W)$ the normalizer of $l(\mathcal{N})$ in $\mathrm{Diff}(\mathcal{N} \times W)$. Let $\mathrm{Map}(W, \mathcal{N})$ be the set of smooth maps from $W$ into $\mathcal{N}$. Then $\mathrm{Diff}^{\mathrm{F}}(\mathcal{N} \times W)$ coincides with the group $\mathrm{Map}(W, \mathcal{N}) \rtimes (\mathrm{Aut}(\mathcal{N}) \times \mathrm{Diff}(W))$ with the group law:

$$(\lambda_1, g_1, h_1)(\lambda, g, h) = ((g_1 \circ \lambda \circ h_1^{-1}) \cdot \lambda_1, g_1 g, h_1 h)$$

and

$$(\lambda, g, h)(x, w) = (g(x) \cdot \lambda(hw), hw)$$

for $(x, w) \in \mathcal{N} \times W$, defines an action on $\mathcal{N} \times W$. See [5].

We call the set $(\Delta, \pi, Q, W)$ a smooth data for the group extension (3.8). The following theorem is obtained in [5].

**Theorem 3.3.** *For any smooth data $(\Delta, \pi, Q, W)$, there exists a continuous homomorphism $\Psi : \pi \to \mathrm{Diff}^{\mathrm{F}}(\mathcal{N} \times W)$ such that $\Psi|_{\Delta} = l$.*

$\Psi$ is called the Seifert construction of the smooth data $(\Delta, \pi, Q, W)$. We shall review the proof of [5].

*Proof.* Using the pushout (3.6) in § 3.2, if we show that there exists a continuous homomorphism $\bar{\Psi} : \pi\mathcal{N} \to \mathrm{Diff}^{\mathrm{F}}(\mathcal{N} \times W)$ such that $\bar{\Psi}|_{\mathcal{N}} = l$, then a Seifert construction $\Psi : \pi \to \mathrm{Diff}^{\mathrm{F}}(\mathcal{N} \times W)$ is obtained as a restriction. Suppose there exists a $\bar{\Psi}$. For $(n, \alpha) \in \pi\mathcal{N}$, if we put $\bar{\Psi}(1, \alpha) = (\lambda, g, h) \in \mathrm{Map}(W, \mathcal{N}) \rtimes (\mathrm{Aut}(\mathcal{N}) \times \mathrm{Diff}(W))$, then $\bar{\Psi}(n, \alpha) = \ell(n)\bar{\Psi}(1, \alpha) = (n \cdot \lambda, g, h)$. Then it is easy to check that

$$\bar{\Psi}(n, \alpha) = (n \cdot \lambda(\alpha), \mu(n) \circ \bar{\phi}(\alpha), \alpha)$$

where $\lambda : Q \to \mathrm{Map}(W, \mathcal{N})$ satisfies

$$f(\alpha, \beta) = (\bar{\phi}(\alpha) \circ \lambda(\beta) \circ \alpha^{-1}) \cdot \lambda(\alpha) \cdot \lambda(\alpha\beta)^{-1} \quad (\alpha, \beta \in Q), \tag{3.9}$$

where $f$ be a function representing the group extension (3.8). Therefore to guarantee the existence of such $\bar{\Psi}$, we have only to find a map $\lambda$ satisfying the condition (3.9).

Remark that if $\mathcal{N}$ is a vector space $V$ then $\mathrm{Map}(W, V)$ is a topological group with $Q$-action by

$$\alpha \cdot \lambda(w) = \bar{\phi}(\alpha)(\lambda(\alpha^{-1}w)). \tag{3.10}$$

So we have a group cohomology $H^2_{\bar{\phi}}(Q, \mathrm{Map}(W, V))$. First note that $H^2_{\bar{\phi}}(Q, \mathrm{Map}(W, V)) = 0$ for any vector space $V$. This vanishing is obtained by using Shapiro's lemma. (See [3], page 251, Lemma 8.4.)

By induction, we suppose that the statement is true for any nilpotent Lie group whose dimension is less than $\dim \mathcal{N}$. Let $\mathcal{C}$ be the center of $\mathcal{N}$ and put $\mathcal{N}_1 = \mathcal{N}/\mathcal{C}$, $\pi\mathcal{N}_1 = \pi\mathcal{N}/\mathcal{C}$. Consider the group extension

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{N} & \longrightarrow & \pi\mathcal{N} & \xrightarrow{\nu} & Q & \longrightarrow & 1 \\ & & \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle p} & & \big\| & & \\ 1 & \longrightarrow & \mathcal{N}_1 & \longrightarrow & \pi\mathcal{N}_1 & \xrightarrow{\nu_1} & Q & \longrightarrow & 1, \end{array} \tag{3.11}$$

with a section $q_1 = p \circ q$ of $\nu_1$ where $q$ is a section to $\nu$. The section $q_1$ determines $f_1 : Q \times Q \to \mathcal{N}_1$ and $\bar{\phi}_1 : Q \to \mathrm{Aut}(\mathcal{N}_1)$ as in §3.1. We suppose by induction on the dimension of $\mathcal{N}$ that there exists $\lambda_1 : Q \to \mathrm{Map}(W, \mathcal{N}_1)$ such that

$$f_1(\alpha, \beta) = (\bar{\phi}_1(\alpha) \circ \lambda_1(\beta) \circ \alpha^{-1}) \cdot \lambda_1(\alpha) \cdot \lambda_1(\alpha\beta)^{-1}$$

Choose any lift $\lambda' : Q \to \mathrm{Map}(W, \mathcal{N})$ of $\lambda_1$ so that $\lambda_1 = p \circ \lambda'$. Put

$$g(\alpha, \beta) = (\bar{\phi}(\alpha) \circ \lambda'(\beta) \circ \alpha^{-1}) \cdot \lambda'(\alpha) \cdot \lambda'(\alpha\beta)^{-1},$$

then there exists an element $c(\alpha, \beta) \in \mathrm{Map}(W, \mathcal{C})$ such that

$$f(\alpha, \beta) = c(\alpha, \beta) \cdot g(\alpha, \beta).$$

Since both $f$ and $g$ satisfy (iii) in §3.1, $c$ is also a 2-cocycle i.e. $[c] \in H^2_{\bar{\phi}}(Q, \mathrm{Map}(W, \mathcal{C}))$ which vanishes because $\mathcal{C}$ is a vector space. So there is a function $\eta : Q \to \mathrm{Map}(W, \mathcal{C})$ such that

$$c(\alpha, \beta) = (\bar{\phi}_1(\alpha) \circ \eta(\beta) \circ \alpha^{-1}) \cdot \eta(\alpha) \cdot \eta(\alpha\beta)^{-1}.$$

Put $\lambda = \eta \cdot \lambda' : Q \to \mathrm{Map}(W, \mathcal{N})$, then $\lambda$ satisfies (3.9). □

*Remark.* Let $1 \to \mathbb{Z} \to \pi_i \longrightarrow \pi_{i-1} \to 1$ be a group extension as in (1.2). Then $\pi_i$ acts on the universal cover $X_i$ of $M_i$ as freely. Assume that $\Psi_i : \pi_i \to \mathrm{Diff}(X_i)$ is the representation homomorphism for this action $(\pi_i, X_i)$, then $\Psi_i : \pi_i \to \Psi_i(\pi_i)$ is the Seifert construction of the smooth data $(\mathbb{Z}, \pi_i, \pi_{i-1}, X_{i-1})$.

## § 3.4. Infranilmanifold

Let $(\Delta, \pi, F, \{pt\})$ be a smooth data with finite group $F$ and $f$ a function representing the given group extension $1 \to \Delta \to \pi \to F \to 1$. In the same way as the proof of Theorem 3.3, we can obtain a 1-chain $\chi : F \to \mathcal{N}$ such that $f = \delta^1\chi$;

$$f(\alpha, \beta) = \bar{\phi}(\alpha)(\chi(\beta))\chi(\alpha)\chi(\alpha\beta)^{-1} \ (\alpha, \beta \in F). \tag{3.12}$$

We shall repeat the construction of $\chi$ for our use. Let $\bar{f} : F \times F \to \mathcal{N}/\mathcal{C}$ be a function which represents $1 \to \mathcal{N}_1 \to \pi\mathcal{N}_1 \to F \to 1$, then we suppose $\bar{f} = \delta^1\bar{\lambda}$ for some function $\bar{\lambda} : F \to \mathcal{N}/\mathcal{C}$ by induction. Choose a lift $\lambda : F \to \mathcal{N}$ of $\bar{\lambda}$. It is easy to see the function $g = f \cdot (\delta^1\lambda)^{-1}$ is a cocycle lying in $\mathcal{C}$, that is $[g] \in H^2_{\bar{\phi}}(F, \mathcal{C})$. As $H^2_{\bar{\phi}}(F, \mathcal{C}) = 0$ from (3.5), there is a map $\mu : F \to \mathcal{C}$ such that $\delta^1\mu = g$. Then $f = \delta^1(\mu \cdot \lambda)$ and the 1-chain $\chi$ denoted by $\mu \cdot \lambda$.

Now define an automorphism of $\mathcal{N}$ $h(\alpha) : \mathcal{N} \to \mathcal{N}$ for each $\alpha \in F$ to be

$$h(\alpha)(x) = \chi(\alpha)^{-1} \cdot \bar{\phi}(\alpha)(x) \cdot \chi(\alpha) \ (x \in \mathcal{N}).$$

Using (3.6), we can prove that $h(\alpha\beta) = h(\alpha)h(\beta)$ for $\alpha, \beta \in F$. Therefore $h : F \to \mathrm{Aut}(\mathcal{N})$ is a homomorphism. Since $\mathrm{Aut}(\mathcal{N})$ is a noncompact Lie group, it has a maximal compact group $\mathcal{K}$. Then the finite subgroup $h(F)$ is conjugate to a subgroup of $\mathcal{K}$. We can assume that $h(F) \subset \mathcal{K}$.

Define $\rho : \pi \to E(\mathcal{N})$ to be

$$\rho((n, \alpha)) = (n\chi(\alpha), h(\alpha)) \ (n \in \Delta, \alpha \in F). \tag{3.13}$$

It is easy to check that $\rho$ is a homomorphism. We define an action of $\pi$ on $\mathcal{N}$ to be

$$((n, \alpha), x) = \rho(n, \alpha)(x) = n\bar{\phi}(\alpha)(x)\chi(\alpha) \quad ((n, \alpha) \in \pi). \tag{3.14}$$

Theorem 3.1 is obtained by the following proposition.

**Proposition 3.4.** *The action $(\pi, \mathcal{N})$ is a properly discontinuous free action. In particular, $\rho$ is a faithful representation.*

*Proof.* First note that $\rho|_\Delta = id$, so $\Delta$ is contained in $\rho(\pi)$. Since $\Delta$ acts as left translations of $\mathcal{N}$ from (3.13), it acts properly discontinuously and freely. Moreover since $\Delta$ is a finite index subgroup of $\rho(\pi)$ from (3.1), $\rho(\pi)$ acts properly discontinuously on $\mathcal{N}$.

Let $(n, \alpha) \in \mathrm{Ker}\,\rho$ be an element of $\pi$. Then $((n, \alpha), x) = x$ $(\forall x \in \mathcal{N})$ by (3.14). As $\pi$ acts properly discontinuously, $(n, \alpha)$ is of finite order. On the other hand, $\pi$ is torsionfree, we obtain $(n, \alpha) = 1$ and so $\rho$ is faithful. □

The following remark shows that $\rho$ is a Seifert construction (cf. Theorem 3.3).

*Remark.* Let $\mathrm{A}(\mathcal{N})^*$ be a group which is the product $\mathcal{N}\times\mathrm{Aut}(\mathcal{N})$ with the group law:

$$(n,\alpha)\cdot(m,\beta)=(\alpha(m)\cdot n,\alpha\cdot\beta)$$

for $n,m\in\mathcal{N}$, and $\alpha,\beta\in\mathrm{Aut}(\mathcal{N})$. The action $(\mathrm{A}(\mathcal{N})^*,\mathcal{N})$ is obtained as follows:

$$((n,\alpha),x)=\alpha(x)\cdot n$$

for $x\in\mathcal{N}$. Then there is an isomorphism $\delta:\mathrm{A}(\mathcal{N})^*\to\mathrm{A}(\mathcal{N})$ defined by $\delta(n,\alpha)=(n,\mu(n^{-1})(\alpha))$. Here $\mu:\mathcal{N}\to\mathrm{Aut}(\mathcal{N})$ is the conjugation homomorphism: $\mu(n)(x)=nxn^{-1}$. It is easily checked that

$$((n,\alpha),x)=(\delta(n,\alpha),x)$$

This shows that the affine action $(\mathrm{A}(\mathcal{N}),\mathcal{N})$ coincides with the above action $(\mathrm{A}(\mathcal{N})^*,\mathcal{N})$.

*Remark.* There is a commutative diagram.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{N} & \longrightarrow & \mathrm{E}(\mathcal{N}) & \longrightarrow & \mathcal{K} & \longrightarrow & 1\\
 & & \uparrow & & \uparrow & & \cup & & \\
1 & \longrightarrow & \mathcal{N}\cap\rho(\pi) & \longrightarrow & \rho(\pi) & \longrightarrow & H & \longrightarrow & 1.
\end{array}\tag{3.15}$$

By the theorem of Auslander-Bieberbach, $\mathcal{N}\cap\rho(\pi)$ is a maximal normal nilpotent subgroup of $\rho(\pi)$. Note that $\Delta\subset\mathcal{N}\cap\rho(\pi)$, so if $\Delta$ is maximal, then $\Delta=\mathcal{N}\cap\rho(\pi)$.

## § 3.5. Seifert rigidity

Let $\Delta_i$ be a discrete uniform subgroup of a simply connected nilpotent Lie group $\mathcal{N}_i$ $(i=1,2)$ respectively. Let $\Psi_1$, $\Psi_2$ be Seifert constructions for smooth data $(\Delta_1,\pi_1,Q_1,W_1)$, $(\Delta_2,\pi_2,Q_2,W_2)$ respectively. Suppose there exists an isomorphism $\theta:\pi_1\to\pi_2$ inducing isomorphisms $\bar\theta:\Delta_1\to\Delta_2$, $\hat\theta:Q_1\to Q_2$. Furthermore $(Q_1,W_1)$ is equivariantly diffeomorphic to $(Q_2,W_2)$. with respect to $\hat\theta$. Then *Seifert rigidity* shows that $(\Psi_2(\pi_1),\mathcal{N}_1\times W_1)$ is equivariantly diffeomorphic to $(\Psi_1(\pi_2),\mathcal{N}_2\times W_2)$. See [5], page 441.

## § 4. $S^1$-fibred nilBott tower

This section is to give an idea of proof of Theorem 1.2. The details will appear in [13]. (See also [6].) Let $M_i$ be an $S^1$-*fibred nilBott manifold* $(i=1,\dots,n)$. Let

$$1\to\mathbb{Z}\to\pi_i\longrightarrow\pi_{i-1}\to 1,\tag{4.1}$$

be a group extension associated with a fiber space:

$$S^1 \to M_i \to M_{i-1}. \tag{4.2}$$

The conjugate by each element of $\pi_i$ defines a homomorphism $\phi : \pi_{i-1} \to \mathrm{Aut}(\mathbb{Z}) = \{\pm 1\}$, so that the above group extension represents a 2-cocycle in $H^2_\phi(\pi_{i-1}; \mathbb{Z})$. Then we can find the commutative diagram of central extensions for each $i$:

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d] & \tilde{\Delta}_i \arrow[r] \arrow[d] & \Delta_{i-1} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathsf{R} \arrow[r] & \tilde{\mathcal{N}_i} \arrow[r] & \mathcal{N}_{i-1} \arrow[r] & 1,
\end{tikzcd} \tag{4.3}$$

where $\tilde{\Delta}_i$ and $\Delta_{i-1}$ are torsionfree finitely generated normal nilpotent subgroups of finite index in $\pi_i$ and $\pi_{i-1}$ respectively. And $\tilde{\mathcal{N}_i}$, $\mathcal{N}_{i-1}$ are simply connected nilpotent Lie groups containing $\tilde{\Delta}_i$ and $\Delta_{i-1}$ as a discrete cocompact subgroup, respectively:

$$\begin{tikzcd}
 & & 1 & 1 & \\
 & & F \arrow[u] & F \arrow[u] & \\
1 \arrow[r] & \mathbb{Z} \arrow[r] & \pi_i \arrow[r] \arrow[u] & \pi_{i-1} \arrow[r] \arrow[u] & 1 \\
1 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[u, equal] & \tilde{\Delta}_i \arrow[r] \arrow[u] & \Delta_{i-1} \arrow[r] \arrow[u] & 1 \\
 & & 1 \arrow[u] & 1 \arrow[u] &
\end{tikzcd} \tag{4.4}$$

From Theorem 3.1 (see [5]) there exists a faithful representation

$$\rho_i : \pi_i \longrightarrow \mathrm{E}(\tilde{\mathcal{N}_i}) \tag{4.5}$$

for which $\rho_i|_{\tilde{\Delta}_i} = \mathrm{id}$ and the quotient $\tilde{\mathcal{N}_i}/\rho_i(\pi_i)$ is an infranilmanifold. On the other hand, (4.5) induces the following group extension:

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, equal] & \pi_i \arrow[r] \arrow[d, "\rho_i"'] & \pi_{i-1} \arrow[r] \arrow[d, "\hat{\rho}_i"'] & 1 \\
1 \arrow[r] & \mathbb{Z} \arrow[r] & \rho_i(\pi_i) \arrow[r] & \hat{\rho}_i(\pi_{i-1}) \arrow[r] & 1.
\end{tikzcd} \tag{4.6}$$

Since $\tilde{\Delta}_i$ centralizes $\mathbb{Z}$, $\tilde{\mathcal{N}_i}$ centralizes $\mathsf{R}$ in (4.3). And $\hat{\rho}_i$ is a monomorphsim from $\pi_{i-1}$ into $\mathrm{E}(\mathcal{N}_{i-1})$. Thus we have two Seifert fibrations

$$(\mathbb{Z}, \mathsf{R}) \to (\rho_i(\pi_i), \tilde{\mathcal{N}_i}) \xrightarrow{\nu_i} (\hat{\rho}_i(\pi_{i-1}), \mathcal{N}_{i-1})$$

and

$$(\mathbb{Z}, \mathsf{R}) \to (\pi_i, X_i) \xrightarrow{p_i} (\pi_{i-1}, X_{i-1})$$

(cf. (1.3)).

By induction, assume that the isomorphism $\hat{\rho}$ induces an equivariant diffeomorphism of $(\pi_{i-1}, X_{i-1})$ onto $(\hat{\rho}_i(\pi_{i-1}), \mathcal{N}_{i-1})$. Then Seifert rigidity implies that $(\pi_i, X_i)$ is equivariantly diffeomorphic to $(\rho_i(\pi_i), \tilde{\mathcal{N}}_i)$. Let $M = X_n/\pi_n$. As a consequence, $M$ is diffeomorphic to an infranilmanifold $\tilde{\mathcal{N}}_n/\rho_n(\pi_n)$.

We conclude that any $S^1$-fibred nilBott manifold $M$ is diffeomorphici to an infranilmanifold. According to Cases I, II stated in Theorem 1.2, we prove that $\tilde{\mathcal{N}}_n$ is isomorphic to a vector space or $\tilde{\mathcal{N}}_n$ is a nilpotent Lie group but not a vector space respectively (cf. [13]).

In order to study $S^1$-fibred nilBott manifolds further, we introduce the following definition:

**Definition 4.1.** If an $S^1$-fibred nilBott manifold $M$ satisfies Case I (respectively Case II) of Theorem 1.2, then $M$ is said to be an $S^1$-fibred nilBott manifold of finite type (respectively of infinite type). Apparently there is no intersection between finite type and infinite type. And $S^1$-fibred nilBott manifolds are of finite type until dimension 2.

*Remark.* Let $M$ be an $S^1$-fibred nilBott manifold of finite type, then $\rho(\pi)$ is a Bieberbach group (cf. Theorem 1.2). By the Bieberbach Theorem, $\rho(\pi)$ satisfies a group extension

$$1 \to \mathbb{Z}^n \to \rho(\pi) \longrightarrow H \to 1 \tag{4.7}$$

where $\mathbb{Z}^n = \rho(\pi) \cap \mathbb{R}^n$, and $H$ is the holonomy group of $\rho(\pi)$. By Proposition 3.4, we may identify $\rho(\pi)$ with $\pi$ whenever $\pi$ is torsionfree.

The following Proposition 4.2 and Corollary 4.3 have been proved. See [13] for details.

**Proposition 4.2.** *Suppose $M$ is an $S^1$-fibred nilBott manifold of finite type. Then the holonomy group of $\pi$ is isomorphic to the power of cyclic group of order two $(\mathbb{Z}_2)^s$ in $(0 \le s \le n)$.*

**Corollary 4.3.** *Each $S^1$-fibred nilBott manifold of finite type $M$ admits a homologically injective $T^k$-action where $k = \operatorname{Rank} H_1(M)$. Moreover, the action is maximal, i.e. $k = \operatorname{Rank} C(\pi)$.*

### § 4.1. $S^1$-fibred nilBott manifolds of depth 3 (Case I).

By the definition of $S^1$-fibred nilBott manifold $M_n$ of depth $n$, $M_2$ is either a torus or a Klein bottle. In particular, $M_2$ is a Riemannian flat manifold. A 3-dimensional $S^1$-fibred nilBott manifold $M_3$ is either a Riemannian flat manifold or an infranil-Heisenberg manifold in accordance with the cases I (finite type) or II (infinite type) of Theorem 1.2.

On the other hand, there are 10-isomorphism classes $\mathcal{G}_1, \dots, \mathcal{G}_6$, $\mathcal{B}_1, \dots, \mathcal{B}_4$ of 3-dimensional Riemannian flat manifolds. (Refer to Wolf [18] for the classification of 3-dimensional Riemannian flat manifolds.) Among these, real Bott manifolds consist of 4; $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{B}_1$, $\mathcal{B}_3$. (See [15].) We shall show that $\mathcal{B}_2$, $\mathcal{B}_4$ are $S^1$-fibred nilBott manifolds.

$\mathcal{B}_2$: $T^3/\mathbb{Z}_2$ whose holonomy group $\mathbb{Z}_2 = \langle\alpha\rangle$ acts on $T^3$;

$$\alpha(z_1, z_2, z_3) = (-z_1 z_3, z_2 z_3, \bar{z}_3).$$

Define an $S^1$-action on $T^3$ by

$$t(z_1, z_2, z_3) = (tz_1, tz_2, z_3).$$

Then it is easy to see that the $S^1$-action induces an $S^1$-action on $T^3/\mathbb{Z}_2$ naturally. This gives a principal bundle

$$S^1 \to T^3/\mathbb{Z}_2 \longrightarrow K$$

where $K$ is a Klein bottle. So the tower of $S^1$-fiber bundles

$$T^3/\mathbb{Z}_2 \to K \to S^1 \to \{\mathrm{pt}\}$$

is an $S^1$-fibred nilBott tower.

$\mathcal{B}_4$: $T^3/(\mathbb{Z}_2)^2$ whose holonomy group $(\mathbb{Z}_2)^2 = \langle\alpha, \beta\rangle$ acts on $T^3$;

$$\alpha(z_1, z_2, z_3) = (-z_1, \bar{z}_2, \bar{z}_3),$$
$$\beta(z_1, z_2, z_3) = (z_1, -z_2, -\bar{z}_3).$$

Denote an action of $(\mathbb{Z}_2)^2$ on $T^2$ by

$$\hat{\alpha}(z_1, z_2) = (-z_1, \bar{z}_2),$$
$$\hat{\beta}(z_1, z_2) = (z_1, -z_2).$$

The quotient manifold is the Klein bottle $T^2/(\mathbb{Z}_2)^2 = (S^1 \times \mathbb{RP}^1)/\mathbb{Z}_2 = K$. The projection $P(z_1, z_2, z_3) = (z_1, z_2)$ is equivariant with respect to the $(\mathbb{Z}_2)^2$-action on $T^2$. So the tower

$$T^3/(\mathbb{Z}_2)^2 \to K \to S^1 \to \{\mathrm{pt}\}$$

is an $S^1$-fibred nilBott tower.

**Proposition 4.4.** *The 3-dimensional $S^1$-fibred nilBott manifold of finite type are those of $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$, $\mathcal{B}_4$.*

*Proof.* As any real Bott manifold is an $S^1$-fibred nilBott manifold of finite type (cf. [7]), consider Reimannian flat manifolds $\mathcal{G}_3$, $\mathcal{G}_4$, $\mathcal{G}_5$, $\mathcal{G}_6$, $\mathcal{B}_2$, $\mathcal{B}_4$ which are not real Bott manifolds. Since holonomy groups are the product of $\mathbb{Z}_2$ by Proposition 4.2, the remaining cases are either $\mathcal{G}_6$, $\mathcal{B}_2$, or $\mathcal{B}_4$ from the list [18]. Moreover, by Corollary 4.3, an $S^1$-fibred nilBott manifold $M$ of finite type admits a homologically injective $T^k$-action for $k = \mathrm{Rank}\, H_1(M)$ $(k \geq 1)$. In particular, $\mathbb{Z}^k$ is a direct summand of $H_1(M)$. By the classification of the first homology (cf. [18]), $H_1(M;\mathbb{Z}) = \mathbb{Z}_4 + \mathbb{Z}_4$ for $\mathcal{G}_6$. So it cannot admit a structure of $S^1$-fibred nilBott manifold. For Riemannian flat 3-manifolds corresponding to $\mathcal{B}_2$ and $\mathcal{B}_4$, we have shown that they admit $S^1$-fibred nilBott tower. □

## § 4.2. $S^1$-fibred nilBott manifolds of depth 3 (CaseII).

The 3-dimensional simply connected nilpotent Lie group $\mathcal{N}$ is isomorphic to the Heisenberg Lie group $N_3$ which is the product $R \times \mathbb{C}$ with group law:

$$(x, z) \cdot (y, w) = (x + y - \mathrm{Im}\bar{z}w, z + w).$$

Then the maximal compact Lie subgroup of $\mathrm{Aut}(N_3)$ is $\mathrm{U}(1) \rtimes \langle\tau\rangle$ which acts on $N_3$

$$\begin{aligned} e^{\mathbf{i}\theta}(x, z) &= (x, e^{\mathbf{i}\theta} z)\ (e^{\mathbf{i}\theta} \in \mathrm{U}(1)), \\ \tau(x, z) &= (-x, \bar{z}). \end{aligned} \tag{4.8}$$

A 3-dimensional compact infranilmanifold is obtained as a quotient $N_3/\Gamma$ where $\Gamma$ is a torsionfree discrete uniform subgroup of $\mathrm{E}(N_3) = N_3 \rtimes (\mathrm{U}(1) \rtimes \langle\tau\rangle)$. (See [4].)

Let

$$S^1 \to M_3 \to M_2$$

be an $S^1$-fibred nilBott manifold of infinite type which has a group extension $1 \to \mathbb{Z} \to \pi_3 \to \pi_2 \to 1$. Since $\mathsf{R} \subset N_3$ is the center of $N_3$, there is a commutative diagram of central extensions:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \tilde{\Delta}_3 & \longrightarrow & \Delta_2 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathsf{R} & \longrightarrow & N_3 & \longrightarrow & \mathbb{C} & \longrightarrow & 1 \end{array} \tag{4.9}$$

(cf. (4.3)). Using this, we obtain an embedding:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_3 & \longrightarrow & \pi_2 & \longrightarrow & 1 \\ & & \iota\big\downarrow & & \rho\big\downarrow & & \hat{\rho}\big\downarrow & & \\ 1 & \longrightarrow & \mathsf{R} & \longrightarrow & \mathrm{E}(N_3) & \longrightarrow & \mathbb{C} \rtimes (\mathrm{U}(1) \rtimes \langle\tau\rangle) & \longrightarrow & 1. \end{array} \tag{4.10}$$

Note that $\mathbb{C} \rtimes (\mathrm{U}(1) \rtimes \langle\tau\rangle) = \mathbb{R}^2 \rtimes \mathrm{O}(2) = \mathrm{E}(2)$. Since $\mathsf{R} \cap \pi_3 = \mathbb{Z}$ from (4.10), $\hat{\rho}(\pi_2)$ is a Bieberbach group in $\mathrm{E}(2)$ so that $\mathbb{R}^2/\hat{\rho}(\pi_2)$ is either $T^2$ or $K$.

We shall consider the following two cases.
**Case (i):** The holonomy group of $\pi_3$ is trivial.
Let $k \in \mathbb{Z}$ and define $\Delta(k)$ to be a subgroup of $N_3$ generated by

$$c = (2k, 0), a = (0, k), b = (0, k\mathbf{i}).$$

Put $\mathsf{Z} = \langle c\rangle$ which is a central subgroup of $\Delta(k)$. It is easy to see that

$$[a, b] = c^{-k}. \tag{4.11}$$

Since $\mathsf{R}$ is the center of $N_3$, we have a principal bundle

$$S^1 = \mathsf{R}/\mathsf{Z} \to N_3/\Delta(k) \longrightarrow \mathbb{C}/\mathbb{Z}^2.$$

Then the euler number of the fibration is $\pm k$. (See [12] for example.)

**Case (ii):** The holonomy group is nontrivial.
Let $\Gamma(k)$ be a subgroup of $\mathrm{E}(N_3)$ generated by

$$n = ((k, 0), I)\,,\ \alpha = \left((0, \frac{k}{2}), \tau\right),\ \beta = ((0, k\mathbf{i}), I)\,.$$

Note that $\alpha^2 = ((0, k), I)$. Then it is easy to check that

$$\alpha n \alpha^{-1} = n^{-1},\ \alpha\beta\alpha^{-1} = n^k\beta^{-1},\ \beta n \beta^{-1} = n. \tag{4.12}$$

Then $M_3 = N_3/\Gamma(k)$ is an $S^1$-fibred nilBott manifold:

$$S^1 \to N_3/\Gamma(k) \to K$$

where $S^1 = \mathsf{R}/\langle n\rangle$ is the fiber (but not an action).

**Proposition 4.5.** *A 3-dimensional $S^1$-fibred nilBott manifold $M_3$ of infinite type is either a Heisenberg nilmanifold $N_3/\Delta(k)$ or an infranilmanifold $N_3/\Gamma(k)$.*

The details of Proposition 4.5 will appear in [13]. Propositions 4.4 and 4.5 are also obtained independently by Lee and Masuda (cf. [11]).

*Remark.* Originally, the $3\times 3$-unipotent upper traingular matrices $N$ is called the Heisenberg nilpotent Lie group. Of course $N_3$ is isomorphic to $N$. We use $N_3$ because it is easy to see the automorphism group $\mathrm{E}(N_3)$.

## § 5. Further remarks

Let $Q = \pi_1(K)$ be the fundamental group of the Klein bottle $K$. $Q$ has a presentation:

$$\{g, h \mid ghg^{-1} = h^{-1}\}. \tag{5.1}$$

A group extension $1 \to \mathbb{Z} \to \pi \to G \to 1$ for any 3-dimensional $S^1$-fibred nilBott manifold over $K$ represents a 2-cocycle in $H^2_\phi(Q, \mathbb{Z})$ for some representation $\phi$. Conversely, given a representation $\phi$, we prove in [13] that any elements of $H^2_\phi(Q, \mathbb{Z})$ can be realized as an $S^1$-fibred nilBott manifold, we have obtained the following table.

| | | Case 1 | Case2 | Case3 | Case4 |
|---|---|---|---|---|---|
| | $H^2_\phi(Q,\mathbb{Z})$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ |
| $\pi_1(M)$ | $[f]=0$ | $\pi_1(\mathcal{B}_1)$ | $\pi_1(\mathcal{B}_3)$ | $\pi_1(\mathcal{G}_2)$ | $\pi_1(\mathcal{B}_3)$ |
| | $[f]\neq 0:$ torsion | $\pi_1(\mathcal{B}_2)$ | $\pi_1(\mathcal{B}_4)$ | - | $\pi_1(\mathcal{B}_4)$ |
| | $[f]\neq 0:$ torsionfree | - | - | $\Gamma(k)$ | - |

Here

**Case1.** : $\phi(g) = 1,\ \phi(h) = 1,$

**Case2.** : $\phi(g) = 1,\ \phi(h) = -1,$

**Case3.** : $\phi(g) = -1,\ \phi(h) = 1,$

**Case4.** : $\phi(g) = -1,\ \phi(h) = -1.$

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