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N-flips in 3-dimensional Small Covers

By

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Abstract

We study combinatorial construction of 3-dimensional small covers using the connected sum and the surgery in [11]. On the other hand, an operation called the N-flip in even triangulations on spheres has been studied in [8]. In this paper we study N-flips in 3-dimensional small covers and give other construction theorem for 3-dimensional small covers by the N-flip.

§ 1. Introduction

Small Covers were introduced by Davis and Januszkiewicz [1] as an n-dimensional closed manifold M^n with a locally standard $(\mathbb{Z}_2)^n$-action such that its orbit space is a simple convex polytope P (later we assume that polytopes are all simple and convex). Davis and Januszkiewicz showed that the (equivariant homeomorphic classes of) small covers over P are classified by $(\mathbb{Z}_2)^n$-colorings $\lambda : \mathcal{F}(P) \to (\mathbb{Z}_2)^n$ where $\mathcal{F}(P)$ is the set of facets of P (cf. [1, Proposition 1.8]). In other words a small cover is correspondent to a $(\mathbb{Z}_2)^n$-colored polytope (P, λ). Here we understand that two $(\mathbb{Z}_2)^n$-colorings on P are the same when one is correspondent to the other by a change of basis of $(\mathbb{Z}_2)^n$. In [3], [5], [7], [10] and [11] constructions of 3-dimensional small covers M^3 from basic small covers by some operations have been studied. At first Izmestiev [3] studied a class of 3-dimensional small covers M^3 which is called *linear models* which are correspondent to 3-colored polytopes. He proved that each linear model M^3 can be constructed from the 3-dimensional torus T^3 using the connected sum and the surgery (cf. [3, Theorem 3]). We generalized this result to general small covers M^3 which are correspondent to $(\mathbb{Z}_2)^3$-colored polytopes in [11, Theorems 1.4 and 1.6]. Partial results were shown in [5], [7] and [10].

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On the other hand in the view of the topological graph theory, some other operations called the N-flip and the P_2-flip on even triangulations of surfaces were studied in [4] and [8]. These operations both correspond to compositions of two surgeries on 3-dimensional linear models. In [8, Theorem 2], it was proved that two linear models over polytopes with same number of faces can be transformed into each other by these operations. In this paper we relate this result to our previous results, and we give other expressions of the construction theorem for 3-dimensional small covers by the N-flip.

§ 2. Basics of small covers

In this section we recall the definitions and basic facts of small covers (see [1] for detail). Let P be an n-dimensional simple convex polytope with the set of facets (i.e. codimension-one faces) $\mathcal{F} = \{F_1, \cdots, F_m\}$. A *small cover* M over P is an n-dimensional closed manifold with a locally standard $(\mathbb{Z}_2)^n$-action such that its orbit space is P. For a small cover M and a facet F of P, we set $\lambda(F)$ the generator of the isotropy subgroup at $x \in \pi^{-1}(\mathrm{int}F)$ where $\pi : M \to P$ is the orbit projection. Then a function $\lambda : \mathcal{F} \to (\mathbb{Z}_2)^n$ (which is defined up to change of basis of $(\mathbb{Z}_2)^n$) is called a *characteristic function* of M which satisfies the following condition.

($\star$) if $F_1 \cap \cdots \cap F_n \neq \emptyset$ then $\{\lambda(F_1), \cdots, \lambda(F_n)\}$ is linearly independent.

Therefore λ is a kind of face-coloring of P. Then we call a function satisfing ($\star$) a $(\mathbb{Z}_2)^n$*-coloring* of P. Here we say that two $(\mathbb{Z}_2)^n$-colored polytopes (P_i, λ_i) $(i = 1, 2)$ are *equivalent* when there exists a combinatorial equivalence of polytopes $\phi : P_1 \to P_2$ such that $\theta \circ \lambda_1 = \lambda_2 \circ \phi$ for some $\theta \in \mathrm{Aut}(\mathbb{Z}_2)^n$. Conversely given a simple convex polytope P and a $(\mathbb{Z}_2)^n$-coloring $\lambda : \mathcal{F} \to (\mathbb{Z}_2)^n$ satisfing ($\star$), we can construct a small cover M with a characteristic function λ as follows:

$$M(P, \lambda) := P \times (\mathbb{Z}_2)^n / \sim,$$

where $(x, t) \sim (y, s)$ is defined as $x = y \in P$ and $t^{-1}s$ is contained in the subgroup generated by $\lambda(F_1), \cdots, \lambda(F_k)$ such that $x \in \mathrm{int}(F_1 \cap \cdots \cap F_k)$. We say that two small covers M_i over P_i $(i = 1, 2)$ are $\mathrm{GL}(n, \mathbb{Z}_2)$-equivalent on a combinatorial equivalence of polytopes $\phi : P_1 \to P_2$ when there exists a θ-equivariant homeomorphism $f : M_1 \to M_2$ such that $\pi_2 \circ f = \phi \circ \pi_1$ and $f(g \cdot x) = \theta(g) \cdot f(x)$ $(g \in (\mathbb{Z}_2)^n, x \in M_1)$ for some $\theta \in \mathrm{Aut}(\mathbb{Z}_2)^n$. Moreover we say that two small covers are *equivalent* when they are $\mathrm{GL}(n, \mathbb{Z}_2)$-equivalent on some combinatorial equivalence of polytopes $\phi : P_1 \to P_2$. In [6] this equivalence and a $\mathrm{GL}(n, \mathbb{Z}_2)$-equivalence on the identity are called a *weakly equivariantly homeomorphism* and a *DJ-equivalence* respectively. Davis and Januszkiewicz proved that a small cover M over P with a characteristic function λ is DJ-equivalent to $M(P,\lambda)$ (cf. [1, Proposition 1.8]). Therefore we can identify an equivalence class of small cover $M(P,\lambda)$ with an equivalence class of $(\mathbb{Z}_2)^n$-colored polytope (P,λ).

Example 2.1. The real projective space $\mathbb{R}P^n$ and the n-dimensional torus T^n with the standard $(\mathbb{Z}_2)^n$-actions are examples of small covers over the n-simplex Δ^n and the n-cube I^n respectively. Figure 1 shows their characteristic functions on polytopes in the case of $n = 3$, where $\{\alpha,\beta,\gamma\}$ is a basis of $(\mathbb{Z}_2)^3$. We denote the associated $(\mathbb{Z}_2)^n$-colored simplex and cube by Δ^3 and (I^3,λ_0) respectively.

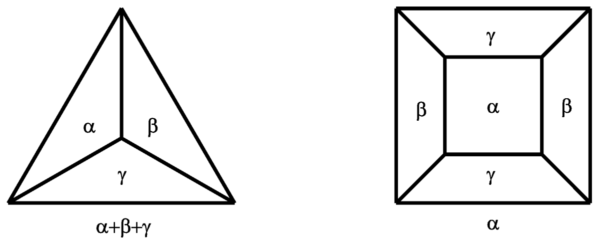

Figure 1. Characteristic functions of $\mathbb{R}P^3$ and T^3.

A small cover over P with an n-coloring (i.e. $\lambda(\mathcal{F})$ is a basis of $(\mathbb{Z}_2)^n$) is called a *linear model*. An example of a linear model is the torus T^n shown in Example 2.1. In this case the n-coloring of P (i.e. the linear model on P) is unique up to a change of colors (or basis of $(\mathbb{Z}_2)^n$). In case $n = 3$, it is well-known that a simple convex polytope is 3-colorable if and only if each facets contains an even number of edges. Such a polytope coincides the dual of an even triangulation of the sphere.

In [9, Theorem 1.7], we gave a criterion when a small cover is orientable. We recall its criterion in the special case $n = 3$.

Theorem 2.2. *A 3-dimensional small cover $M(P,\lambda)$ is orientable if and only if $\lambda(\mathcal{F})$ is contained in $\{\alpha,\beta,\gamma,\alpha+\beta+\gamma\}$ for a suitable basis $\{\alpha,\beta,\gamma\}$ of $(\mathbb{Z}_2)^3$.*

From the above theorem small covers $\mathbb{R}P^3$ and T^3 given in Figure 1 are both orientable. Since each triple of $\{\alpha,\beta,\gamma,\alpha+\beta+\gamma\}$ is linearly independent, a coloring which satisfies Theorem 2.2 is just a 4-coloring. We notice that the existence of an orientable small cover over every simple convex 3-polytope is guaranteed by the Four Color Theorem (cf. [9, Corollary 1.8]). Henceforth we assume that $n = 3$ and (P,λ) is a pair of a 3-dimensional simple convex polytope P with $(\mathbb{Z}_2)^3$-coloring λ, and $\{\alpha,\beta,\gamma\}$ is a basis of $(\mathbb{Z}_2)^3$. From the Steinitz's theorem (see [2] etc.) a 3-dimensional simple convex polytope is combinatorially equivalent to a 3-connected 3-regular planner graph which is the 1-skeleton of P.

Example 2.3. We consider small covers on a 3-sided prism $P^3(3) = I \times \Delta^2$. There exists three types of $(\mathbb{Z}_2)^3$-coloring on $P^3(3)$ shown in Figure 2 up to equivalence. The first example $M(P^3(3), \lambda_1)$ is equivariantly homeomorphic to $S^1 \times \mathbb{R}P^2$. The second example $M(P^3(3), \lambda_2)$ is homeomorphic but not-equivariantly homeomorphic to $S^1 \times \mathbb{R}P^2$ (cf. [7, Lemmas 4.2 and 4.3]). We denote it by $S^1 \ltimes \mathbb{R}P^2$. The last example $M(P^3(3), \lambda_3)$ is orientable and homeomorphic to $\mathbb{R}P^3 \sharp \mathbb{R}P^3$.

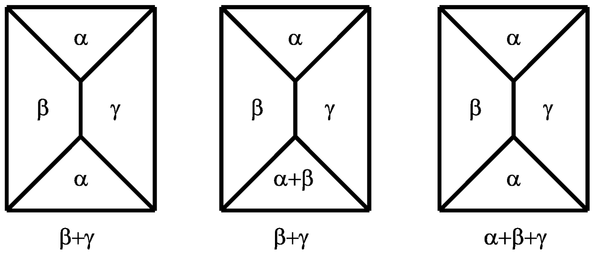

Figure 2. Three types of $(\mathbb{Z}_2)^3$-coloring on $P^3(3) = I \times \Delta^2$; λ_1, λ_2 and λ_3.

We recall some operations on $(\mathbb{Z}_2)^3$-colored polytopes (i.e. 3-dimensional small covers) which were introduced in [3], [5], [7] and [10].

Definition 2.4 (connected sum $\sharp$)**.** The operation $\sharp$ in Figure 3 is called the *connected sum (at vertices)*. We notice that the connected sum of colored polytopes $P_1 \sharp P_2$ is well-defined in every cases. Then the operation $\sharp$ corresponds with the equivariant connected sum $M(P_1, \lambda_1) \sharp M(P_2, \lambda_2)$ at fixed points (cf. [1, 1.11]).

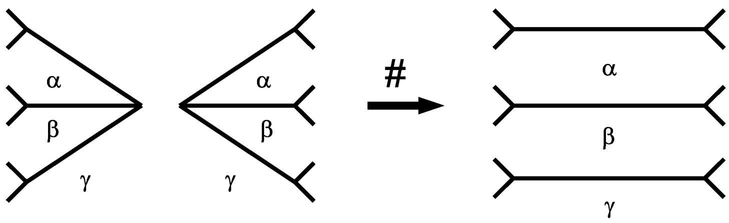

Figure 3. Connected sum $\sharp$.

Specifically the connected sum with Δ^3 and (I^3, λ_0) on polytopes, denoted by Bl_v and Bl_v^T, correspond to the operations called the *(T-)blow up* (at a fixed point) on small covers (Figures 4 and 5), respectively.

Definition 2.5 (surgery $\natural$)**.** The operation $\natural$ in Figure 6 (from left to right) is called the *surgery* along an edge e and its inverse $\natural^{-1}$ (from right to left) is called the

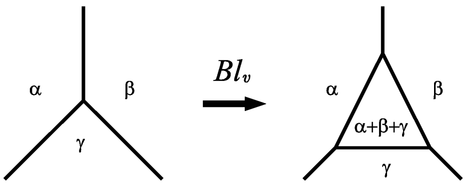

Figure 4. Blow up $Bl_v = \sharp\Delta^3$.

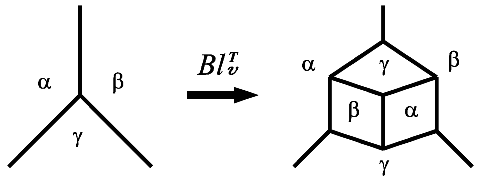

Figure 5. T-blow up $Bl_v^T = \sharp(I^3, \lambda_0)$.

inverse surgery along a pair of edges $\{e_1, e_2\}$. We denote the surgery along an edge e (resp. its inverse) as $\natural_e$ (resp. $\natural^{-1}_{\{e_1,e_2\}} = (\natural_e)^{-1}$), when an edge needs to be indicated. The operations $\natural$ and $\natural^{-1}$ both correspond to the equivariant surgeries on a small cover (cf. [3]). We do not allow the surgeries $\natural$ and $\natural^{-1}$ when the 3-connectedness of the 1-skeleton of P is destroyed after doing them.

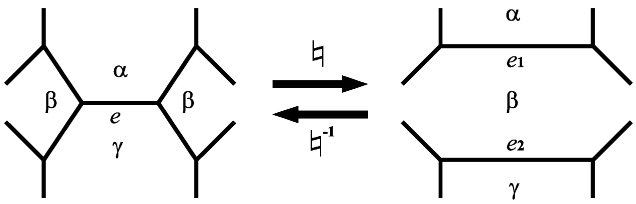

Figure 6. Surgery $\natural$ and its inverse $\natural^{-1}$.

Definition 2.6 (connected sum along edges $\sharp^e$). The operation $\sharp^e$ in Figure 7 is called the *connected sum along edges*. We notice that the operation $\sharp^e$ is obtained as a composition $\sharp^e = \natural \circ \sharp$ as shown in same figure (cf. [5, Theorem 4.1(2)]). The operation $\sharp^e$ corresponds to a connected sum around the circles $\pi^{-1}(e)$ of each small covers.

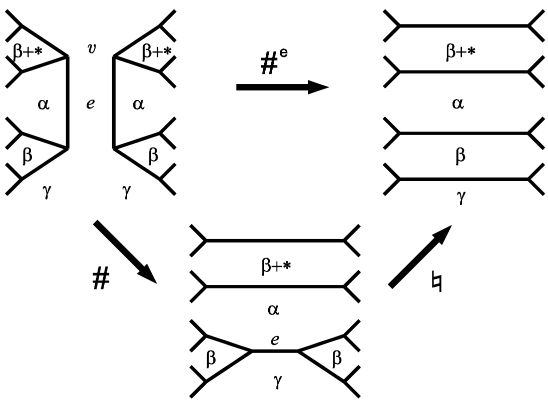

Figure 7. Connected sum along edges $\sharp^e = \natural \circ \sharp$.

Specifically the operations $\sharp^e P^3(3)$ and $\sharp^e \Delta^3$ are often called the *cutting edge* (Figure 8) and the *diagonal flip* (Figure 9), respectively. The former corresponds to a kind of blow up at the circle $\pi^{-1}(e)$ on small covers. This operation is denoted by Bl_e and is called the *blow up* along an edge. In Figure 8, as a color of center square we can choose not only $\beta+\gamma$ but also $\alpha+\beta+\gamma$ when $*=0$. The latter operation $\sharp^e\Delta^3 = \natural \circ (\sharp\Delta^3)$ corresponds to the Dehn surgery of type $\frac{2}{1}$ on small covers (cf. [10] or [5, 3.5]). This operation is denoted by $\natural^D$ and is called the *Dehn surgery.*

Moreover the operations $\sharp^e(I^3, \lambda_0)$ in Figure 10 is called the T-*blow up* along an edge and is denoted by Bl_e^T.

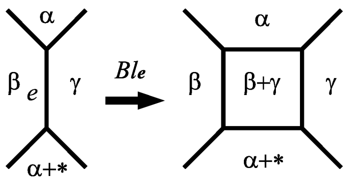

Figure 8. Cutting edge $Bl_e = \sharp^e P^3(3)$.

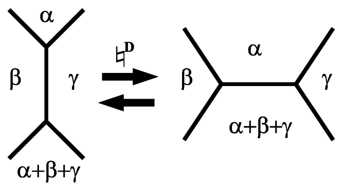

Figure 9. Dehn surgery $\natural^D = \sharp^e\Delta^3$.

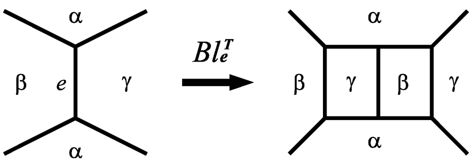

Figure 10. T-blow up along an edge $Bl_e^T = \sharp^e(I^3, \lambda_0)$.

Definition 2.7 (color change $\sharp_4^c$)**.** The operation $\sharp_4^c$ in Figure 11 is called the *color change* for a 2-independent quadrilateral. Here a face F is called 2-*independent* if the maximal number of linearly independent vectors of $\{\lambda(F_j) \mid F_j \text{ is adjacent to } F\}$ is two. (i.e. $\star = 0$ or γ in Figure 11). This operation is defined as the connected sum along a face with a $(\mathbb{Z}_2)^3$-colored cube (I^3, λ) (see [7]).

Combinatorial constructions of 3-dimensional small covers (i.e. $(\mathbb{Z}_2)^3$-colored polytopes) using the above operations have been studied in [3], [5], [7], [10] and [11]. We recall the conclusive theorem as follows (cf. [11, Theorems 1.4 and 1.6]).

Theorem 2.8. *(1) Each small cover M^3 can be constructed from $\mathbb{R}P^3$, T^3, $S^1 \times \mathbb{R}P^2$ and $S^1 \ltimes \mathbb{R}P^2$ by using operations $\sharp$ and $\natural$.*
(2) Each small cover M^3 can be constructed from $\mathbb{R}P^3$, $S^1 \times \mathbb{R}P^2$ and $S^1 \ltimes \mathbb{R}P^2$ by using operations $\sharp$, $\sharp^e$, $\natural^{-1}$ and $\sharp_4^c$.

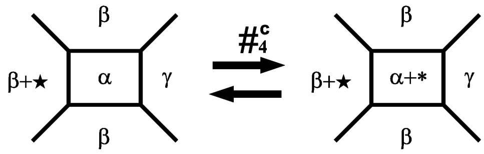

Figure 11. Color change $\natural_4^c$ for a 2-independent quadrilateral.

Remark. In the above theorem the later is a non-decreasing construction, i.e. all operations do not decrease the number of faces. We can restrict the above theorem to 3- (resp. 4-)colored polytopes, and obtain the construction theorem for linear models [11, Corollary 4.10] (resp. orientable small covers [11, Propositions 5.2 and 5.3]).

§ 3. N-flips in 3-dimensional small covers

In the view of topological graph theory, Nakamoto, Sakuma and Suzuki [8] gave a characterization of even triangulations on sphere (i.e. linear models) using the operations called the *N-flip* and the *P_2-flip*. In this section we discuss N-flips in $(\mathbb{Z}_2)^3$-colored polytopes.

Definition 3.1 (N-flip and P_2-flip)**.** The operations $\natural^N$ in Figure 12 and $\natural^P$ in Figure 13 are called the *N-flip* and the *P_2-flip* respectively. These operation act on a (locally) even triangulation of sphere. We consider their duals and generalize to $(\mathbb{Z}_2)^3$-colored polytopes as Figures 14 and 15. We notice that the N-flip $\natural^N$ and the P_2-flip $\natural^P$ can be also obtained as compositions of surgeries $\natural_e \circ \natural^{-1}_{\{e'_1, e'_2\}}$ as shown in same figures. Moreover $\natural^P$ can be also obtained as a composition $Bl_e^T \circ (Bl_{e'}^T)^{-1}$ as shown in Figure 15 when colors around v are linearly independent.

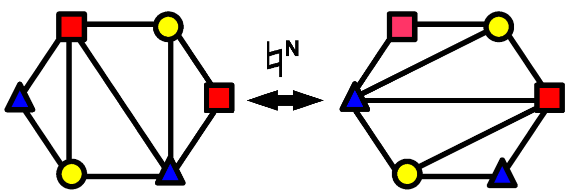

Figure 12. N-flip $\natural^N$ (dual).

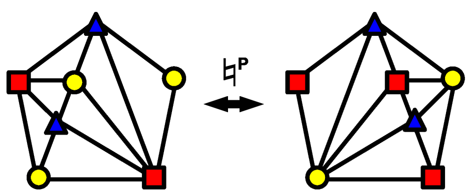

Figure 13. P_2-flip $\natural^P$ (dual).

We notice that the N-flip preserves the partition of faces by colors. Although the P_2-flip also preserves the total number of faces, it changes the partition of faces by colors. In [8] a characterization of 3-colored polytopes under these operations was given as follows.

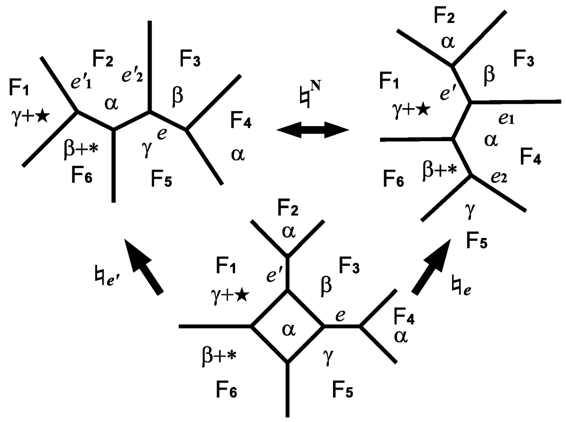

Figure 14. N-flip $\natural^N$.

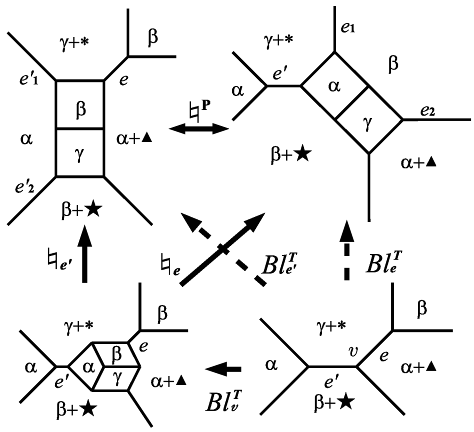

Figure 15. P_2-flip $\natural^P$.

Theorem 3.2 (Nakamoto, Sakuma and Suzuki). *(1) Let (P_i, λ_i) $(i = 1, 2)$ be 3-colored polytopes such that $|\lambda_1^{-1}(x)| = |\lambda_2^{-1}(x)|$ for $x = \alpha, \beta, \gamma$. Then P_1 and P_2 can be transformed into each other by a sequence of N-flips $\natural^N$.*
(2) Any two 3-colored polytopes with the same number of faces can be transformed into each other by a sequence of N-flips $\natural^N$ and P_2-flips $\natural^P$.

Remark. The above theorem is a 3-colored analogue of the famous theorem "two (non-colored) simple convex polytopes with the same number of faces can be transformed into each other by a sequence of diagonal flips $\natural^D$".

Izmestiev's theorem [3, Theorem 3] "Each linear model M^3 can be constructed from T^3 by using $\sharp$, $\natural$ and $\natural^{-1}$" can be led from Theorem 3.2 immediately because $\natural^N$ and $\natural^P$ are both compositions of $\natural$ and $\natural^{-1}$. Moreover we may obtain another expression of the construction theorem for linear models by using the N-flip as follows.

Theorem 3.3. *Each linear model M^3 can be constructed from T^3 by using three operations Bl_v^T, Bl_e^T and $\natural^N$.*

Proof. Notice that the number of faces of a 3-colored polytope is six (when $P = I^3$) or more than seven. We shall prove by the induction on the number of faces. When the number of faces is six, theorem is trivial because (P, λ) is just the 3-colored cube (I^3, λ_0). We assume that the number of faces is more than seven. Since the operations Bl_v^T or Bl_e^T increase the number of faces by three or two respectively, it is easy to construct a 3-colored polytope P' with same number of faces as P. From the Theorem 3.2 (2), there exists a sequence $P' = P_0 \to P_1 \to P_2 \to \cdots \to P_n = P$ such that $P_i = \natural^N P_{i-1}$ or $P_i = \natural^P P_{i-1}$ $(i = 1, 2, \cdots, n)$. If there is no P_2-flip in this sequence, then the proof finishes. Let $P_{m-1} \to P_m$ be the last P_2-flip in this sequence. Here we notice that $P_m = Bl_e^T \circ (Bl_{e'}^T)^{-1}P_{m-1}$ and $(Bl_{e'}^T)^{-1}P_{m-1}$ has less faces than P. Therefore $(Bl_{e'}^T)^{-1}P_{m-1}$ can be constructed from (I^3, λ_0) by using Bl_v^T, Bl_e^T and $\natural^N$ because of the induction hypothesis. Then the proof is complemented. $\square$

Corollary 3.4. *For each 3-colored polytope (P, λ), the following inequality holds:*

$$|\lambda^{-1}(\alpha)| \leq |\lambda^{-1}(\beta)| + |\lambda^{-1}(\gamma)| - 2.$$

Proof. The above inequality holds for the 3-colored cube (I^3, λ_0). For a 3-colored polytople (P, λ), we denote 3-colorings of $Bl_v^T P$ and $Bl_e^T P$ by λ_v and λ_e respectively. Then it is easy to see that $|\lambda_v^{-1}(x)| = |\lambda^{-1}(x)| + 1$ for any $x = \alpha, \beta, \gamma$ and $|\lambda_e^{-1}(x)| = |\lambda^{-1}(x)|$, $|\lambda_e^{-1}(y)| = |\lambda^{-1}(y)| + 1$ for some $x \in \{\alpha, \beta, \gamma\}$ and any $y \neq x$. Therefore if the above inequality holds for (P, λ) then it also holds for $(Bl_v^T P, \lambda_v)$ and $(Bl_e^T P, \lambda_e)$. $\square$

Next we consider the construction of orientable small covers by the N-flip. We recall that each 4-colored polytope (P^3, λ) can be constructed from 3-colored polytopes and Δ^3 by using $\sharp$ and $\natural^D$ (cf. [11, Proposition 4.2] or [10]). We combine this fact with Theorem 3.3 and prove immediately that each orientable small cover M^3 can be constructed from T^3 and $\mathbb{R}P^3$ by using operations $\sharp$, Bl_e^T, $\natural^D$ and $\natural^N$. Moreover we can restrict $\sharp$ to Bl_v and Bl_v^T and show the following theorem.

Theorem 3.5. *Each orientable small cover M^3 can be constructed from T^3 and $\mathbb{R}P^3$ by using operations Bl_v, Bl_v^T, Bl_e^T, $\natural^D$ and $\natural^N$.*

Proof. By induction on the number of faces of P, it is sufficient to prove the following assertion:
$(*)$ There exists a sequence $P_0 \to P_1 \to \cdots \to P_n = P$ such that P_0 is a 3-colored polytope and P_i is one of $\natural^D P_{i-1}$, $\natural^N P_{i-1}$ and $Bl_v P_{i-1}$.

We have already known that $(*)$ holds for each 4-colored polytope P whose faces of P are less than seven (cf. [7] or [11]). Then we assume that a 4-colored polytope P has at least seven faces. Moreover we may assume that P has no triangular face because if P has a triangular face then we obtain $P = Bl_v P'$ for some P' immediately. Let F be a 3-independent face which has the least edges. Here a face F is 3-*independent* means that the maximal number of linearly independent vectors of $\{\lambda(F_j) \mid F_j \text{ is adjacent to } F\}$ is three. From the proof of [11, Proposition 4.2], there exists an edge e of F such that the 3-independence of F is preserved under the Dehn surgery $\natural^D$ along e. If the Dehn surgery $\natural^D$ along this edge e does not destroy the 3-connectedness of the 1-skeleton of P then we can decrease the number of edges of F and prove $(*)$ by induction on the number of edges of F. We consider the case that the 3-connectedness of the 1-skeleton of P is destroyed after doing $\natural^D$ along the edge e. In this case the situation around

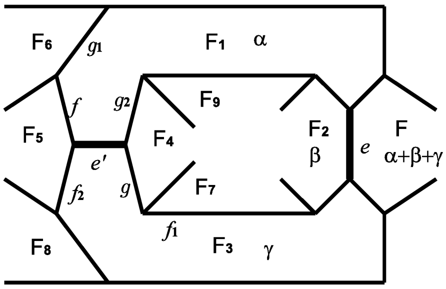

Figure 16. Obstacle for the Dehn surgery $\natural^D$ along the edge e.

F is shown as Figure 16 ($F_1 \cap F_3 = e' \neq \emptyset$). In Figure 16, $\lambda(F_4)$ and $\lambda(F_5)$ are β or $\alpha+\beta+\gamma$. Since there is no triangular face of P, the three faces F_2, F_4, F_7 in Figure 16 are different to each other (F_9 may coincide with F_2).

(i) When $\lambda(F_4) \neq \lambda(F_5)$, we can do the Dehn surgery $\natural^D$ along the edge e' and dissolve the obstacle for $\natural^D$ along the edge e.

(ii) When $\lambda(F_4) = \lambda(F_5)$ (we may assume its value to be β), we notice that $\lambda(F_7)$ and $\lambda(F_8)$ are α or $\alpha+\beta+\gamma$, $\lambda(F_6)$ and $\lambda(F_9)$ are γ or $\alpha+\beta+\gamma$.

(a) In case $\lambda(F_6) = \gamma$ or $\lambda(F_7) = \alpha$, we can do $\natural^N = \natural_f \circ \natural^{-1}_{\{f_1,f_2\}}$ or $\natural^N = \natural_g \circ \natural^{-1}_{\{g_1,g_2\}}$ and dissolve the obstacle for $\natural^D$ along the edge e respectively (see Figure 17). The case when $\lambda(F_8) = \alpha$ or $\lambda(F_9) = \gamma$ is similar.

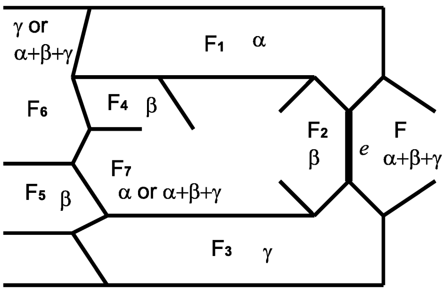

Figure 17. After doing $\natural^N$ in Figure 16.

(b) In case $\lambda(F_6) = \lambda(F_7) = \lambda(F_8) = \lambda(F_9) = \alpha+\beta+\gamma$ (in this case F_5 has at least five edges), we can do $\natural^D$ along one of edges f or f_2 because either $F_3 \cap F_6$ or $F_1 \cap F_8$ must be empty. Next we can do $\natural^D$ along the edge e' and dissolve the obstacle for $\natural^D$ along the edge e.

In all cases we can decrease the number of edges of F by operations $\natural^D$ and if necessary $\natural^N$ until F is transformed into a triangle. Then we can decrease the number of faces of P by the blow down Bl_v^{-1} until P becomes a 3-colored polytope. Namely the proof of $(*)$ can be done by the double induction on the numbers of edges of F and faces of P. $\square$

Next we consider the constructions of general $(\mathbb{Z}_2)^3$-colored polytopes by the N-flip. We say that (P,λ) is *quasi-decomposable* when there exist two $(\mathbb{Z}_2)^3$-colored polytopes (P_i,λ_i) $(i=1,2)$ such that $(P,\lambda)=(P_1,\lambda_1)\sharp(P_2,\lambda_2)$ or $(P,\lambda)=(P_1,\lambda_1)\sharp^e(P_2,\lambda_2)$ except $P=P_1\sharp^e\Delta^3(=\natural^D P_1)$.

Theorem 3.6. *Each small cover M^3 can be constructed from $\mathbb{R}P^3$, $S^1\times\mathbb{R}P^2$ and $S^1\ltimes\mathbb{R}P^2$ by using operations $\sharp$, $\sharp^e$, $\natural^N$ and $\sharp_4^c$.*

Proof. We shall prove by induction on the number of faces of P. We have already known the theorem is stated for a polytope which has at most six faces. Then we assume that the faces of P are more than six. We notice that P has a small face F (which has at most five edges). From [11, Proposition 4.5] if P has a 3-independent small face then either P or $\natural^D P$ is quasi-decomposable. Moreover when P has a 2-independent triangular face, P is also quasi-decomposable (cf. [11, Proposition 3.5]). In these cases we can reduce P to polytopes which decreased in the number of faces by using the inverses of operations $\sharp$ and $\sharp^e$. It is sufficient to consider the case when P has a 2-independent quadrilateral or pentagon.

(i) When F is a 2-independent pentagon, the situation around F is shown as the first diagram in Figure 18. We can assume that P has no triangle. We do the N-flip $\natural^N(=\natural_e\circ\natural^{-1}_{\{e_1,e_2\}})$ and transform F into a triangle (the second diagram). Then $\natural^N P$ has a 2-independent triangle and therefore it is quasi-decomposable.

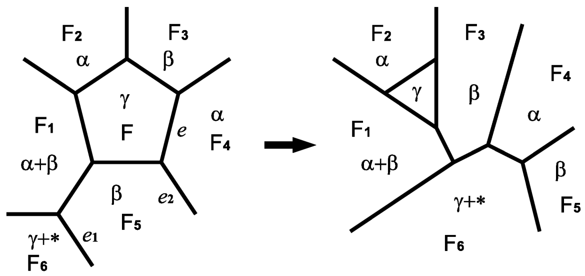

Figure 18. 2-independent pentagon.

(ii) When F is a 2-independent quadrilateral, the situation around F is shown as the first diagram in Figure 19. In this diagram we can assume that P has no triangle and no pentagon. First we do the color change $\sharp_4^c$ if necessary and change the color of F to $\alpha+\beta+\gamma$. Next we do the Dehn surgery $\natural^D$ along an edge e and transform F into a pentagon (second diagram). Then $\natural^D\circ\sharp_4^c P$ has a 3-independent pentagon. Therefore we can reduce this case to the previous case.

$\square$

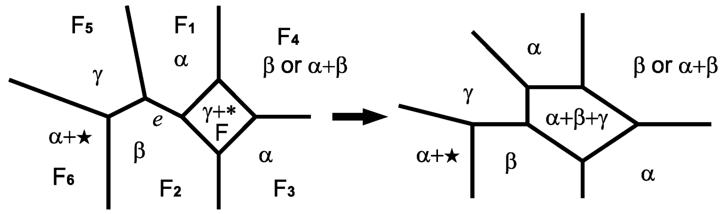

Figure 19. 2-independent quadrilateral.

Remark. We expect that the above theorem is improvable like Theorems 3.3 and 3.5 as follows: Each small cover M^3 can be constructed from $\mathbb{R}P^3$, $S^1 \times \mathbb{R}P^2$ and $S^1 \ltimes \mathbb{R}P^2$ by using operations Bl_v, Bl_e, Bl_v^T, Bl_e^T, $\natural^D$, $\natural^N$ and $\sharp_4^c$ (i.e. $\sharp$ and $\sharp^e$ are restricted to $\{Bl_v,\ Bl_v^T\}$ and $\{\natural^D,\ Bl_e,\ Bl_e^T\}$ respectively). However the proof seems rather delicate.

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