RIMS Kôkyûroku Bessatsu
B39 (2013), 149–165

Smith sets of non-solvable groups whose nilquotients are cyclic groups of order 1, 2, or 3

By

Toshio SUMI*

Abstract

Let G be a finite group. Two real G-modules U and V are called Smith equivalent if there exists a smooth action of G on a sphere with exactly two fixed points at which tangential representations are isomorphic to U and V respectively. The Smith set of G is the subset of the real representation ring of G consisting differences of Smith equivalent G-modules. We discuss the question when the Smith set of an Oliver group becomes a group and give several examples of classes of non-solvable groups of which the Smith sets are groups.

§ 1. Introduction

Let G be a finite group. Two real G-modules U and V are called *Smith equivalent* if there exists a smooth action of G on a sphere Σ such that $\Sigma^G = \{x, y\}$, and $T_x(\Sigma) \cong U$ and $T_y(\Sigma) \cong V$ as a real G-module. Let $Sm(G)$, called the Smith set of G, be the subset of the real representation ring $RO(G)$ of G consisting of all differences $[U] - [V]$ for real G-modules U and V which are Smith equivalent. If Σ^P is connected for any subgroup P of G of prime power order, then we call that U and V are *c-primary Smith equivalent*. Let $PSm^c(G)$ be the subset of $Sm(G)$ consisting of all differences for c-primary Smith equivalent real G-modules. The set $Sm(G)$ always contains the zero but the set $PSm^c(G)$ contains the zero if and only if G is not of prime power order.

Atiyah and Bott [2] showed that the Smith set of a cyclic group of prime order is zero, namely, Smith equivalent real modules are isomorphic. Cappell and Shaneson [4] showed that the Smith set of any cyclic group C_{4n} of order $4n \geq 8$ is not zero, namely, there is a pair of non-isomorphic Smith equivalent real modules. Dovermann and Petrie

Received September 20, 2011. Revised April 5, 2012.
2000 Mathematics Subject Classification(s): 57S17, 57S25, 20C15
Key Words: tangential representation, Smith equivalent, Oliver groups
*Faculty of Arts and Science, Kyushu University, 744, Motooka, Nishi-ku, Fukuoka 819-0395, Japan.
e-mail: sumi@artsci.kyushu-u.ac.jp

[12] gave infinitely many cyclic groups of odd order of which the Smith sets are not zero. For a perfect group G, Laitinen and Pawałowski [19] showed that $PSm^c(G)$ is not zero if and only if the number, denoted by r_G, of real conjugacy classes of elements of G not of prime power is greater than or equal to 2. Pawałowski and Solomon [26] showed that for a gap Oliver group G with $r_G \geq 2$, if $G \not\cong \mathrm{Aut}(A_6), \mathrm{P\Sigma L}(2, 27)$, then $PSm^c(G) \neq 0$. Note that $r_G = 2$ for $G = \mathrm{Aut}(A_6), \mathrm{P\Sigma L}(2, 27)$. Morimoto determined $PSm^c(\mathrm{Aut}(A_6)) = Sm(\mathrm{Aut}(A_6)) = 0$ [21] and $PSm^c(\mathrm{P\Sigma L}(2, 27)) \neq 0$ [22]. Pawałowski and Sumi [28] showed that $PSm^c(G) \neq 0$ for a non-solvable group G with $r_G \geq 2$ which which is not isomorphic neither to $\mathrm{Aut}(A_6)$ nor to $\mathrm{P\Sigma L}(2, 27)$. Therefore, for a non-solvable group G, $PSm^c(G) \neq 0$ if and only if $r_G \geq 2$ and $G \not\cong \mathrm{Aut}(A_6)$. Many researchers have studied the problem [32] whether the Smith set is zero or not [5, 11, 6, 7, 9, 12, 20, 34, 8, 14, 13].

In general, the Smith set is not a group. For example, Morimoto pointed out that the Smith set of a cyclic 2-group is a finite set by Bredon's inspection [3] and therefore, it is not a group (see [24, Theorem 1] for details). In this paper, we try to find classes of finite non-solvable groups G such that $Sm(G)$ is a group.

Main Theorem. *The Smith set $Sm(G)$ is a group for groups G such as*

- *the alternating groups A_n,*
- *the symmetric groups S_n,*
- *the projective special linear groups $\mathrm{PSL}(2, q)$ and $\mathrm{PSL}(3, q)$,*
- *the projective general linear groups $\mathrm{PGL}(2, q)$ and $\mathrm{PGL}(3, q)$,*
- *the projective special unitary groups $\mathrm{PSU}(3, q^2)$, and*
- *the sporadic groups and their automorphism groups.*

This paper is organized as follows. In Section 2, we prepare notations and results as tools to determine the Smith sets for appropriate groups. In Section 3, we completely determine the Smith sets of the alternating groups and the symmetric groups. In Section 4, we consider certain classes of perfect groups and determine the Smith sets of $\mathrm{PSL}(2, q)$, $\mathrm{PSL}(3, q)$ and $\mathrm{PSU}(3, q^2)$. In Section 5, we treat some classes of non-perfect non-solvable groups and determine the Smith sets of $\mathrm{PGL}(2, q)$ and $\mathrm{PGL}(3, q)$. In Section 6, we discuss the sporadic groups and their extensions, and compute the related Smith sets.

§ 2. Preliminaries and basic results

In this paper, every group G that we consider is finite. Also, we assume that 1 is a power of a prime. First we introduce notations. Let $RO(G)$ be the real representation ring of G, $\mathcal{P}(G)$ the set of all subgroups of G of prime power order, and put

$$\mathcal{P}_e(G) = \{P \in \mathcal{P}(G) \mid |P| = 2^a,\ a \geq 3\} \ \text{ and } \ \mathcal{P}_o(G) = \mathcal{P}(G) \smallsetminus \mathcal{P}_e(G).$$

Also we put

$$\mathrm{Elm}_e(G) = \{g \in G \mid \langle g \rangle \in \mathcal{P}_e(G)\} \ \text{ and } \ \mathrm{Elm}_o(G) = \{g \in G \mid \langle g \rangle \in \mathcal{P}_o(G)\}.$$

Let $O^p(G)$ be the smallest normal subgroup of G of order a power of p for a prime p and $\mathcal{L}(G)$ the set of all subgroups L of G such that $L \geq O^p(G)$ for some prime p. Let G^{nil} be the smallest normal subgroup of G such that the quotient G/G^{nil} is nilpotent. Then

$$G^{\mathrm{nil}} = \bigcap_p O^p(G).$$

For subsets $\mathcal{F}_1$ and $\mathcal{F}_2$ of subgroups of G and a subset $\mathcal{A}$ of $RO(G)$, we put

$$\mathcal{A}_{\mathcal{F}_1} = \bigcap_{P \in \mathcal{F}_1} \ker(\mathrm{Res}^G_P \colon RO(G) \to RO(P)) \cap \mathcal{A},$$
$$\mathcal{A}^{\mathcal{F}_2} = \bigcap_{L \in \mathcal{F}_2} \ker(\mathrm{Fix}^L \colon RO(G) \to RO(N_G(L)/L)) \cap \mathcal{A}$$

and $\mathcal{A}^{\mathcal{F}_2}_{\mathcal{F}_1} = \mathcal{A}_{\mathcal{F}_1} \cap \mathcal{A}^{\mathcal{F}_2}$. By Smith theory, it holds that $PSm^c(G) \subset RO(G)_{\mathcal{P}(G)}$.

Theorem 2.1 ([29]). *It holds that $Sm(G) \subset RO(G)_{\mathcal{P}_o(G)}$.*

Note that

$$RO(G)_{\mathcal{P}_o(G)} = RO(G)_{\{\langle g \rangle \mid g \in \mathrm{Elm}_o(G)\}}.$$

In particular if G has no element of order 8 then $Sm(G)_{\mathcal{P}(G)} = Sm(G)$. For example, the Ree group ${}^2G_2(3^{2n+1})$ [37] and the Suzuki group ${}^2B_2(2^{2n+1})$ [36] are simple groups which have no element of order 8.

Put

$$\mathcal{N}_2(G) := \{H \leq G \mid [G : H] \leq 2\}$$

and

$$N_2(G) := \bigcap_{H \in \mathcal{N}_2(G)} H.$$

The group $N_2(G)$ is the smallest normal subgroup H so that G/H is an elementary abelian 2-group.

Theorem 2.2 ([21]).

$$Sm(G) \subset RO(G)^{N_2(G)}.$$

Theorem 2.3 ([27, Theorem 2.5]). *If $g\langle g^2\rangle \subset \langle g\rangle$ for $g \in \mathrm{Elm}_e(G)$, then it holds that $Sm(G)_{\mathcal{P}(G)} = Sm(G)$.*

Let $\mathrm{Irr}(G)$ be the set of representatives of isomorphism classes of the irreducible real G-modules and put

$$i_{\mathbb{R}}(g,G) = \left|\{[V] \in \mathrm{Irr}(G) \mid \dim V^g = 0 = \dim V^{N_2(G)}\}\right|$$

for $g \in G$ and

$$i_{\mathbb{R}}(G) = \max\left(\{i_{\mathbb{R}}(g,G) \mid g \in \mathrm{Elm}_e(G)\} \cup \{0\}\right).$$

Proposition 2.4. *For a group G and its quotient group K, it holds that $i_{\mathbb{R}}(K) \leq i_{\mathbb{R}}(G)$.*

Proof. Let $f\colon G \to K$ be a canonical epimorphism. Note that $f^*\colon RO(K)^{N_2(K)} \to RO(G)^{N_2(G)}$ is injective since $f(N_2(G)) = N_2(K)$, and $\dim(f^*V)^x = \dim V^{f(x)}$ for $x \in G$ and a real K-module V. Let a be an element of $\mathrm{Elm}_e(K)$ such that $i_{\mathbb{R}}(K) = i_{\mathbb{R}}(a,K)$ and let b be an element of $f^{-1}(a) \cap \mathrm{Elm}_e(G)$. If a real K-module V satisfies that $V^{N_2(K)} = 0 = \dim V^a$ then $(f^*V)^{N_2(K)} = 0 = \dim(f^*V)^b$. For an irreducible real K-module V, f^*V is an irreducible real G-module. Therefore $i_{\mathbb{R}}(K) \leq i_{\mathbb{R}}(G)$. □

We can slightly extend Theorem 2.5 [27].

Proposition 2.5. *If $i_{\mathbb{R}}(G) \leq 1$ then $Sm(G)_{\mathcal{P}(G)} = Sm(G)$.*

Proof. Let $[U] - [V] \in Sm(G)$. By Theorem 2.2, we may assume that $U^{N_2(G)} = 0 = V^{N_2(G)}$. In this situation, according to the proof of Theorem 2.5 [27], if $i_{\mathbb{R}}(G) = 0$ then $\dim U^g > 0$ for $g \in \mathrm{Elm}_e(G)$ and if $i_{\mathbb{R}}(G) = 1$ then $\chi_U(g) = \chi_V(g)$ for $g \in \mathrm{Elm}_e(G)$. In the both cases $[U] - [V] \in RO(G)_{\mathcal{P}(G)}$. Therefore $Sm(G)_{\mathcal{P}(G)} = Sm(G)$ holds. □

We obtain the characters of irreducible real G-modules from the characters of irreducible complex G-modules. The indicator ι_V of an irreducible complex G-module V is given by

$$\frac{1}{|G|}\sum_{g\in G}\chi_V(g^2).$$

It takes a value 0, 1 or -1. The value is 1 if and only if the character χ_V is a character of an irreducible real G-module. If the value is -1 (resp. 0) then $2\chi_V$ (resp. $\chi_V + \overline{\chi}_V$) is a character of an irreducible real G-module. We frequently use the formula

$$\dim_{\mathbb{C}} V^g = \frac{1}{|g|}\sum_{c\in\langle g\rangle}\chi_V(c)$$

for $g \in \mathrm{Elm}_e(G)$. In particular, if $\dim_{\mathbb{C}} V^g > 0$ then we do not have to take care about the indicator of V.

Lemma 2.6. *If there is not an irreducible complex G-module V such that* $\dim V^g$ *is zero for any* $g \in \mathrm{Elm}_e(G)$*, then* $i_{\mathbb{R}}(G) = 0$.

Let $\mathrm{Irr}_{\mathbb{C}}(G)$ be the set of representatives of isomorphism classes of the irreducible complex G-modules. Let G be a central extension of K by H, that is, $H \le Z(G)$ and $1 \to H \to G \xrightarrow{\pi} K \to 1$ is an exact sequence. Fix an injective map $f\colon K \to G$ such that $\pi \circ f = id$ and if elements x and y of K are conjugate then $f(x)$ and $f(y)$ are conjugate in G. The map f is not necessary a homomorphism. An element of G is expressed as $hf(k)$ for $h \in H$ and $k \in K$. Let V be an irreducible complex G-module. The character value $\chi_V(hf(k))$ of V at $hf(k)$ can be described as $\chi_\xi(h)\chi_\eta(k)$ for some $[\xi] \in \mathrm{Irr}_{\mathbb{C}}(H)$ and some $[\eta] \in \mathrm{Irr}_{\mathbb{C}}(K)$. In such a case, we denote V by $V(\xi, \eta)$. This notation depends on a choice of f. It holds that

$$\mathrm{Irr}_{\mathbb{C}}(G) = \{[V(\xi, \eta)] \mid [\xi] \in \mathrm{Irr}_{\mathbb{C}}(H), [\eta] \in \mathrm{Irr}_{\mathbb{C}}(K)\}.$$

We canonically extend it to a bilinear map $V(\cdot, \cdot)\colon R(H) \times R(K) \to R(G)$, where $R(G)$ is the complex representation ring of G. Similarly as for $i_{\mathbb{R}}(G)$, we put

$$i_{\mathbb{C}}(g, G) = \left|\{[V] \in \mathrm{Irr}_{\mathbb{C}}(G) \mid \dim V^g = 0 = \dim V^{N_2(G)}\}\right|$$

for $g \in G$ and

$$i_{\mathbb{C}}(G) = \max\left(\{i_{\mathbb{C}}(g, G) \mid g \in \mathrm{Elm}_e(G)\} \cup \{0\}\right).$$

Lemma 2.7. $i_{\mathbb{C}}(G) = 0$ *if and only if* $i_{\mathbb{R}}(G) = 0$.

Proof. Let W be an irreducible real G-module which comes from an complex G-module V with $\dim V^g = 0 = \dim V^{N_2(G)}$. Since χ_W is equal to χ_V, $2\chi_V$ or $\chi_V + \overline{\chi_V}$, $\dim W^g$ is equal to $\dim_{\mathbb{C}} V^g$ or $2\dim_{\mathbb{C}} V^g$. Therefore the statements $i_{\mathbb{C}}(G) = 0$ and $i_{\mathbb{R}}(G) = 0$ are equivalent. □

Lemma 2.8. *Let G be a central extension of a group K by a group H of odd order and $\pi\colon G \to K$ a canonical epimorphism. For $x \in \mathrm{Elm}_e(G)$, it holds that $i_{\mathbb{C}}(x, G) \ge |\mathrm{Irr}(H)| i_{\mathbb{C}}(\pi(x), K)$ and $i_{\mathbb{C}}(G) \ge |\mathrm{Irr}(H)| i_{\mathbb{C}}(K)$. Furthermore, if $O^2(K) = K$ then $i_{\mathbb{C}}(G) = i_{\mathbb{C}}(x, G) = |\mathrm{Irr}_{\mathbb{C}}(H)| i_{\mathbb{C}}(\pi(x), K)$ and $i_{\mathbb{C}}(K) = i_{\mathbb{C}}(\pi(x), K)$ for some $x \in \mathrm{Elm}_e(G)$.*

Proof. Let $f\colon K \to G$ be an injective map such that $\pi \circ f = id$, if elements x and y of K are conjugate then $f(x)$ and $f(y)$ are conjugate in G, and $f(b)$ is an element of order a power of 2 for any element b of order a power of 2. Let $x \in \mathrm{Elm}_e(G)$. Since $|H|$ is odd, $\pi(x)$ is an element of $\mathrm{Elm}_e(K)$. Since $f(\pi(x)) \in \pi^{-1}(\pi(x))$ and x have order a power of 2, it holds that $f(\pi(x)) = x$. Let $[\xi] \in \mathrm{Irr}_{\mathbb{C}}(H)$ and $[\eta] \in \mathrm{Irr}_{\mathbb{C}}(K)$

with $\eta^{N_2(K)} = 0$. Since $\chi_{V(\xi,\eta)}(x) = (\dim_{\mathbb{C}} \xi)\chi_\eta(\pi(x))$, $\dim_{\mathbb{C}} \eta^{\pi(x)} = 0$ if and only if $\dim_{\mathbb{C}} V(\xi,\eta)^x = 0$, and then $i_{\mathbb{C}}(x,G) \geq |\operatorname{Irr}_{\mathbb{C}}(H)| i_{\mathbb{C}}(\pi(x),K)$.

Now suppose that $K = O^2(K)$. Then $G = O^2(G) = N_2(G)$. Since $\dim_{\mathbb{C}} V(\xi,\eta)^G = (\dim_{\mathbb{C}} \xi^H)(\dim_{\mathbb{C}} \eta^K)$, if $\dim_{\mathbb{C}} \eta^{\pi(x)} = 0$ then $\dim_{\mathbb{C}} \eta^K = 0 = \dim_{\mathbb{C}} V(\xi,\eta)^G$. Therefore $i_{\mathbb{C}}(x,G) = |\operatorname{Irr}_{\mathbb{C}}(H)| i_{\mathbb{C}}(\pi(x),K)$. □

Proposition 2.9. *Let G be a central extension of a group K with $O^2(K) = K$ by a group H of odd order. Then $i_{\mathbb{R}}(G) = i_{\mathbb{R}}(K) + (|\operatorname{Irr}(H)| - 1) i_{\mathbb{C}}(K)/2$. In particular, $i_{\mathbb{R}}(G) = 0$ if and only if $i_{\mathbb{R}}(K) = 0$.*

Proof. Let g be an element of G such that $i_{\mathbb{R}}(g,G) = i_{\mathbb{R}}(G)$. Note that $\iota_{V(\xi,\eta)} = \iota_\xi \iota_\eta$. There are $i_{\mathbb{R}}(K)$ elements $[V]$ of $\operatorname{Irr}(G)$ which come from $V(\mathbb{R},\cdot)$ such that $\dim V^g = 0 = \dim V^{N_2(G)}$. Let $[\xi]$ be an element of $\operatorname{Irr}(H)$ with $\xi \neq \mathbb{R}$. Then $\iota_\xi = 0$ and for $[\eta] \in \operatorname{Irr}_{\mathbb{C}}(K)$, there is one element of $\operatorname{Irr}(G)$ whose character is $\chi_{V(\xi,\eta)} + \chi_{\overline{V(\xi,\eta)}}$. Therefore $i_{\mathbb{R}}(G) = i_{\mathbb{R}}(K) + (|\operatorname{Irr}(H)| - 1) i_{\mathbb{C}}(K)/2$. By Lemma 2.7, $i_{\mathbb{R}}(G) = 0$ and $i_{\mathbb{R}}(K) = 0$ are equivalent. □

For an element $h \in G$, let

$$FO(G,h) = \langle [V] \mid V \in \operatorname{Irr}(G),\ \dim V^h = 0 \rangle \subset RO(G).$$

The subset $FO(G,h)$ is a subgroup of $RO(G)$.

Theorem 2.10. *If $FO(G,h)_{\mathcal{P}_o} = 0$ for all $h \in \operatorname{Elm}_e(G)$, then $Sm(G)_{\mathcal{P}(G)} = Sm(G)$.*

Proof. Let $[U] - [V] \in Sm(G)$. Suppose that $\chi_U(h) \neq \chi_V(h)$ for some $h \in \operatorname{Elm}_e(G)$. It holds that $U^h = 0 = V^h$ by Smith theory. Then $[U] - [V] \in FO(G,h)$. Further, it follows from Theorem 2.1 that $0 \neq [U] - [V] \in FO(G,h)_{\mathcal{P}_o(G)}$. □

Let $\mathcal{D}(G)$ be the set of pairs (P,H) of subgroups P and H of G such that $P \in \mathcal{P}(G)$ and $P < H$. A finite group G is called a *gap group* [25] if there is a real G-module V such that $\dim V^L = 0$ for $L \in \mathcal{L}(G)$ and $\dim V^P > 2 \dim V^H$ for $(P,H) \in \mathcal{D}(G)$. For example, a non-trivial perfect group is a gap group. A finite group G is called an *Oliver group* [18] if there does not exist a pair (P,H) of normal subgroups P in H and H in G such that $P \in \mathcal{P}(G)$, H/P is a cyclic group and G/H is a group of prime power order. In particular, a non-solvable group is an Oliver group.

Lemma 2.11. *Let G be a non-trivial perfect group. Then G is a gap Oliver group with $PSm^c(G) = RO(G)^{\{G\}}_{\mathcal{P}(G)}$. Furthermore, if $i_{\mathbb{R}}(G) \leq 1$ then*

$$PSm^c(G) = RO(G)^{\{G\}}_{\mathcal{P}(G)} = Sm(G).$$

Lemma 2.12. *Let G be a gap Oliver group. If G/G^{nil} is an elementary abelian 2-group, then $PSm^c(G) = RO(G)^{\{G^{\mathrm{nil}}\}}_{\mathcal{P}(G)}$. Furthermore, if $i_{\mathbb{R}}(G) \le 1$ then*

$$PSm^c(G) = RO(G)^{\{G^{\mathrm{nil}}\}}_{\mathcal{P}(G)} = Sm(G).$$

Proof of Lemmas 2.11 and 2.12. If G is perfect then $N_2(G) = G = G^{\mathrm{nil}}$. If G/G^{nil} is a 2-group then $O^2(G) = G^{\mathrm{nil}}$. By the Realization Theorem [26, p.850], we have

$$PSm^c(G) = RO(G)^{\{G^{\mathrm{nil}}\}}_{\mathcal{P}(G)}$$

and by Proposition 2.5,

$$Sm(G) \subset RO(G)^{\{G^{\mathrm{nil}}\}}_{\mathcal{P}(G)}.$$

Therefore the equality holds. □

Let r_G be the number of real conjugacy classes represented by elements not of prime power order. Then $RO(G)^{\{G\}}_{\mathcal{P}(G)}$ is a free abelian group of rank $\max(r_G - 1, 0)$. In particular, $PSm^c(G) = 0$ if $r_G \le 1$.

In the case where G is not a gap group, we check the weak gap condition. We say that a G-module V satisfies the *weak gap condition* if the following conditions all hold (cf. [23]).

- $\dim V^P \ge 2 \dim V^H$ for $(P, H) \in \mathcal{D}(G)$.
- If $\dim V^P = 2 \dim V^H$ for some $(P, H) \in \mathcal{D}(G)$, then $[H : P] = 2$ and $\dim V^H > \dim V^K + 1$ for every $H < K \le G$.
- If $\dim V^P = 2 \dim V^H$ for some $(P, H) \in \mathcal{D}(G)$ with $[H : P] = 2$, then the map $g_* \colon V^H \to V^H$ is orientation preserving for any $g \in N_G(H)$.
- If $\dim V^P = 2 \dim V^H$ and $\dim V^P = 2 \dim V^{H'}$ for some $(P, H), (P, H') \in \mathcal{D}(G)$, then the smallest subgroup of G containing H and H' does not lie in $\mathcal{L}(G)$.

Lemma 2.13. *Let H be a subgroup of G, V a G-module and $gH \in N_G(H)/H$. If $(xH)_* \colon V^H \to V^H$ preserves the orientation, where xH is a generator of a Sylow 2-subgroup of $\langle gH \rangle$, then $(gH)_* \colon V^H \to V^H$ preserves the orientation.*

Proof. It is clear that $(yH)_* \colon V^H \to V^H$ preserves the orientation for any $yH \in N_G(H)/H$ of odd order. Therefore the assertion follows. □

Let $\mathcal{D}^{(2)}(G)$ be the subset of $\mathcal{D}(G)$ consisting of (P, H) such that the following properties hold.

(1) $[H:P]=[O^2(G)H:O^2(G)P]=2$ and $O^p(G)P=G$ for every odd prime p,

(2) $O^2(C_G(h))$ is a subgroup of P for any element $h \in H \smallsetminus P$ of order 2, and

(3) $C_G(h)$ is a 2-group for any 2-element $h \in H \smallsetminus P$ of order > 2.

We consider the following partial condition (PWGC) of the weak gap condition:

- $\dim V^P \geq 2 \dim V^H$ for $(P,H) \in \mathcal{D}^{(2)}(G)$.
- If $\dim V^P = 2\dim V^H$ for some $(P,H) \in \mathcal{D}^{(2)}(G)$, then the map $(gH)_* \colon V^H \to V^H$ is orientation preserving for any $gH \in N_G(H)/H$ of order a power of 2.

Let $\mathbb{R}[G]_{\mathcal{L}(G)}$ be a real G-module defined as

$$(\mathbb{R}[G]-\mathbb{R}[G]^G) - \bigoplus_p (\mathbb{R}[G]-\mathbb{R}[G]^G)^{O^p(G)}.$$

A real G-module V is called $\mathcal{L}(G)$-free if $[V] \in RO(G)^{\mathcal{L}(G)}$ and called nonnegative if $\dim V^P \geq 2\dim V^H$ for $(P,H) \in \mathcal{D}(G)$. Suppose that $\mathcal{P}(G) \cap \mathcal{L}(G) = \varnothing$. In their paper [18], Laitinen and Morimoto essentially used the $\mathcal{L}(G)$-free and nonnegative real G-module $\mathbb{R}[G]_{\mathcal{L}(G)}$, denoted by $V(G)$. It also holds that $\dim \mathbb{R}[G]^P_{\mathcal{L}(G)} > 2\dim \mathbb{R}[G]^H_{\mathcal{L}(G)}$ if (P,H) does not satisfy (1). By [35, Theorem C and Lemma 4.3], there exists an $\mathcal{L}(G)$-free nonnegative real G-module V fulfilling that $\dim V^P > 2\dim V^H$ if (P,H) does not satisfy (2) or (3). Therefore, there exists an $\mathcal{L}(G)$-free nonnegative real G-module W such that W contains $\mathbb{R}[G]_{\mathcal{L}(G)}$ as a submodule and $\dim W^P > 2\dim W^H$ if (P,H) does not satisfy (1), (2) or (3). By [18, Theorem 3.2] with Lemma 2.13, for an Oliver group G, if a real G-module V satisfies (PWGC) then $V \oplus W^{\oplus n}$ satisfies the weak gap condition for some even integer $n>0$. In particular, if G is a gap group then taking W as a gap real G-module, for any G-module V, $V \oplus W^{\oplus n}$ satisfies the weak gap condition for sufficient large integer $n>0$.

Proposition 2.14. *If U and V are $\mathcal{L}(G)$-free real G-modules satisfying* (PWGC) *then there are $\mathcal{L}(G)$-free real G-modules U' and V' satisfying the weak gap condition such that $[U']-[V']=[U]-[V]$.*

Proof. For each $X=U,V$, let n_X be an even positive integer such that $X \oplus W^{\oplus n}$ satisfies the weak gap condition. Put $m=\max(n_U,n_V)$ and $X'=X\oplus W^{\oplus m}$ for $X=U$ or V. Since W is nonnegative, the assertion follows. □

Therefore, by Proposition 2.14 and [28, Theorem 3.9], the following theorem holds.

Theorem 2.15. *Let G be an Oliver group. If U and V are G-modules such that $[U]-[V] \in RO(G)_{\mathcal{P}(G)}$, $U^L = 0 = V^L$ for $L \in \mathcal{L}(G)$, and both U and V satisfy* (PWGC)*, then $[U]-[V] \in PSm^c(G)$.*

From the next section, we use these results to determine $Sm(G)$.

§ 3. Alternating groups and symmetric groups

In $G = A_n$ or S_n, it holds that $g\langle g^2\rangle \subset (g)^{\pm}$ for any $g \in \mathrm{Elm}_e(G)$. This allows to determine $Sm(G)$ for $G = A_n$ or S_n.

Proposition 3.1. *For the alternating group A_n,*

$$PSm^c(A_n) = Sm(A_n) = RO(A_n)^{\{A_n\}}_{\mathcal{P}(A_n)}$$

which is a free abelian group of rank $\max(r_{A_n} - 1, 0)$. *In particular,* $Sm(A_n) = 0$ *if and only if* $n \le 7$.

Proof. By Theorem 2.3, $Sm(A_n)$ is a subset of $RO(A_n)^{\{A_n\}}_{\mathcal{P}(A_n)}$. For $n \le 7$, A_n has no element of order 8 and then $PSm^c(A_n) = Sm(A_n)$ [19]. Furthermore, for $n \le 7$, $r_{A_n} \le 1$ and then $PSm^c(A_n) = 0$ by [26, Theorem B3]. Let $n \ge 8$. Since A_n is a gap Oliver group, Theorem 2.11 implies $PSm^c(A_n) = Sm(A_n) = RO(A_n)^{\{A_n\}}_{\mathcal{P}(A_n)}$. □

Proposition 3.2. *For the symmetric group S_n,*

$$PSm^c(S_n) = Sm(S_n) = RO(S_n)^{\{A_n\}}_{\mathcal{P}(S_n)}$$

which is a free abelian group of rank

$$\begin{cases} 0, & n = 2, 3, 4, 5, \\ 1, & n = 6, \\ r_{S_n} - 2\ (\ge 3), & n \ge 7. \end{cases}$$

Proof. By Theorem 2.3, $Sm(S_n)$ is a subset of $RO(S_n)_{\mathcal{P}(S_n)}$ and by Theorem 2.2, $Sm(S_n)$ is a subset of $RO(S_n)^{\{A_n\}}$. Therefore $Sm(S_n)$ is a subset of $RO(S_n)^{\{A_n\}}_{\mathcal{P}(S_n)}$.

For $n \le 7$, S_n has no element of order 8 and then $PSm^c(S_n) = Sm(S_n)$ [19]. Furthermore, for $n \le 5$, $r_{S_n} \le 1$ and then $PSm^c(S_n) = 0$ by [26, Theorem B3]. For $n \ge 6$, S_n is a gap Oliver group and then $PSm^c(G) = Sm(G) = RO(S_n)^{\{A_n\}}_{\mathcal{P}(S_n)}$ by Theorem 2.12. □

§ 4. Projective special linear groups

The next targets are the projective special linear groups $\mathrm{PSL}(2, q)$ which are simple groups for $q \ge 4$.

Since $\mathrm{PSL}(2, 2) \cong D_6 \cong S_3$ and $\mathrm{PSL}(2, 3) \cong A_4$, it holds that $Sm(\mathrm{PSL}(2, q)) = 0$ for $q = 2, 3$.

Theorem 4.1. *For $G = \mathrm{PSL}(2,q)$, $i_{\mathbb{R}}(G) = 0$ and $PSm^c(G) = Sm(G) = RO(G)^{\{G\}}_{\mathcal{P}(G)}$ which is a free abelian group of rank* $\max(r_{C_{q-1}} + r_{C_{q+1}} - 1, 0)$.

Proof. If q is a power of 2 then G has no element of order 4 and then $i_{\mathbb{R}}(G) = 0$. Let q be a power of an odd prime. It is easy to see that $r_G = r_{C_{q-1}} + r_{C_{q+1}}$. By Lemmas 2.7 and 2.11, it suffices to show that $i_{\mathbb{C}}(G) = 0$. The character table for G is known (cf. [1]) and it is induced from the character table for $\mathrm{SL}(2,q)$ [17, 30]. First suppose that q is congruent to 1 modulo 4. Put $q = 2^{s+1}t + 1$, where t is odd. Let x be an element of G of order $(q-1)/2$ and $y = x^t$. An element of order a power of 2 greater than or equal to 4 is conjugate to an element of the cyclic group $\langle y \rangle$. The following table is a part of the character table for G and the dimensions of the $\langle y \rangle$-fixed point sets of the irreducible complex G-modules:

V	χ_1	χ_q	$\chi^{(i)}_{q+1}$	$\chi^{(j)}_{q-1}$	$\chi_{(q+1)/2}$	$\chi'_{(q+1)/2}$
(e)	1	q	$q-1$	$q+1$	$(q+1)/2$	$(q+1)/2$
(x^a)	1	1	0	$\eta^{ja} + \eta^{-ja}$	$(-1)^a$	$(-1)^a$
$\dim V^y$	1	$2t+1$	$2t$	$2t \quad (2^s \nmid j)$ $2t+2 \quad (2^s \mid j)$	t	t

where $\eta = \exp(\pi\sqrt{-1}/(q-1))$, $1 \le i \le (q-1)/4$ and $1 \le j \le (q-5)/4$.

Therefore $i_{\mathbb{C}}(G) = 0$.

Next suppose that q is congruent to 3 modulo 4. Then $q - 1$ is not divisible by 4. Put $q = 2^{s+1}t - 1$, where $s \ge 1$ and t is odd. Let x be an element of G of order $q + 1$ and $y = x^t$. An element of order a power of 2 greater than or equal to 4 is conjugate to an element of the cyclic group $\langle y \rangle$.

V	χ_1	χ_q	$\chi^{(i)}_{q+1}$	$\chi^{(j)}_{q-1}$	$\chi_{(q+1)/2}$	$\chi'_{(q+1)/2}$
(e)	1	q	$q-1$	$q+1$	$(q+1)/2$	$(q+1)/2$
(x^b)	1	-1	$-\sigma^{ib} - \sigma^{-ib}$	0	1	1
$\dim V^y$	1	$2t-1$	$2t \quad (2^s \nmid i)$ $2t-2 \quad (2^s \mid i)$	$2t$	t	t

where $\sigma = \exp(\pi\sqrt{-1}/(q+1))$, $1 \le i \le (q-3)/4$ and $1 \le j \le (q-3)/4$.

Note that if $t = 1$ then $(q-3)/4 < 2^s$. Thus if i is divisible by 2^s then $t > 1$ and $2t - 2 > 0$. Therefore $i_{\mathbb{C}}(G) = 0$. ☐

Theorem 4.2. *Let $G = \mathrm{PSL}(3,q)$ or $\mathrm{PSU}(3,q^2)$. Then $i_{\mathbb{R}}(G) = 0$ and $PSm^c(G) = Sm(G) = RO(G)^{\{G\}}_{\mathcal{P}(G)}$ which is a free abelian group.*

Proof. The group G is a simple group. By using the character table [31], we straightforwardly see that $i_{\mathbb{C}}(G) = 0$. By Lemmas 2.7 and 2.11 we get the assertion. $\square$

We have the same result for certain perfect groups.

Theorem 4.3. *Let $G = \mathrm{SL}(3, q)$ or $\mathrm{SU}(3, q^2)$. Then $i_{\mathbb{R}}(G) = 0$ and $PSm^c(G) = Sm(G) = RO(G)^{\{G\}}_{\mathcal{P}(G)}$.*

Proof. $\mathrm{SL}(3, q)$ is a central extension of the simple group $\mathrm{PSL}(3, q)$ by a cyclic group of order $\gcd(3, q - 1)$. Also, $\mathrm{SU}(3, q^2)$ is a central extension of the simple group $\mathrm{PSU}(3, q^2)$ by a cyclic group of order $\gcd(3, q+1)$. Therefore the assertion follows from Theorem 4.2 and Lemma 2.8. $\square$

§ 5. Projective general linear groups

We consider the projective general linear group $\mathrm{PGL}(2, q)$. For even q, it is isomorphic to $\mathrm{PSL}(2, q)$ which is a perfect group for $q \geq 4$. For odd $q \geq 5$, it has a subgroup $\mathrm{PSL}(2, q)$ with index 2.

Theorem 5.1. *Let q be a power of an odd prime and $G = \mathrm{PGL}(2, q)$. It holds that $i_{\mathbb{R}}(G) \leq 1$. If G is a gap group, then $PSm^c(G) = Sm(G) = RO(G)^{\{\mathrm{PSL}(2,q)\}}_{\mathcal{P}(G)}$ which is a free abelian group of rank $r_{C_{q-1}} + r_{C_{q+1}} - 2$.*

Proof. Let x_{q-1}, x_q and x_{q+1} be elements of G of order $q - 1$, q and $q + 1$, respectively. The character table of $\mathrm{PGL}(2, q)$ is known (cf. [33]):

	χ_1	χ_1'	χ_q	χ_q'	$\chi_{q+1}^{(i)}$	$\chi_{q-1}^{(j)}$
(e)	1	1	q	q	$q+1$	$q-1$
(x_q)	1	1	0	0	1	-1
(x_{q-1}^a)	1	$(-1)^a$	1	$(-1)^a$	$\epsilon^{ia(q+1)} + \epsilon^{-ia(q+1)}$	0
(x_{q+1}^b)	1	$(-1)^b$	-1	$(-1)^{b+1}$	0	$-\epsilon^{jb(q-1)} - \epsilon^{-jb(q-1)}$

Here $1 \leq a \leq (q-1)/2$, $1 \leq b \leq (q+1)/2$, $1 \leq i < (q-1)/2$, $1 \leq j < (q+1)/2$, and $\epsilon = \exp(2\pi\sqrt{-1}/(q^2 - 1))$.

Since the indicator of each irreducible complex G-module is one, the table is also a character table of the irreducible real G-modules. Note that $\dim(\chi_1)^{N_2(G)} = 1 = \dim(\chi_1')^{N_2(G)}$. Now we show that $i_{\mathbb{R}}(G) \leq 1$. If G has no element of order 8 then we have nothing to do. The other cases are the case where $q-1$ is divisible by 8 and the case where $q+1$ is divisible by 8.

Suppose that $q-1$ is divisible by 8. Let $q-1=2^s t$, where t is odd. An element of order 8 is conjugate to an element of the subgroup $\langle x_{q-1}\rangle$. Let y be an element of the group $\langle x_{q-1}\rangle$ of order 2^s. Then we have

V	χ_q	χ'_q	$\chi^{(i)}_{q+1}$	$\chi^{(j)}_{q-1}$
$\dim V^y$	$t+1$	t	$t \quad (2^s \nmid i)$ $t+2 \quad (2^s \mid i)$	t

and then $i_{\mathbb{R}}(G))=0$.

Next suppose that $q+1$ is divisible by 8. Similarly as above, letting $q+1=2^s t$, for an element y of the group $\langle x_{q+1}\rangle$ of order 2^s, we have the data of dimensions of the fixed point sets:

V	χ_q	χ'_q	$\chi^{(i)}_{q+1}$	$\chi^{(j)}_{q-1}$
$\dim V^y$	$t-1$	t	t	$t \quad (2^s \nmid j)$ $t-2 \quad (2^s \mid j)$

Note that if $2^s \mid j$ occurs then $t \geq 3$, since $j \leq 2^{s-1}t$. Therefore $i_{\mathbb{R}}(G) \leq 1$. We have done to show that $i_{\mathbb{R}}(G) \leq 1$ for any q.

Thus, by Lemma 2.12, $PSm^c(G)=Sm(G)=RO(G)^{\{\mathrm{PSL}(2,q)\}}_{\mathcal{P}(G)}$ and since x_{q-1} and x_{q+1} are elements not of prime power order, $\operatorname{rank} RO(G)^{\{\mathrm{PSL}(2,q)\}}_{\mathcal{P}(G)} = r_{C_{q-1}} + r_{C_{q+1}} - 2$. □

For odd q, if $\mathrm{PGL}(2,q)$ is a gap group then $q \neq 3,5,7,9,17$ [35]. If $q=3,5,7$ then $PSm^c(G)=Sm(G)=0$ since $r_G \leq 1$.

Now we discuss the Smith sets of $\mathrm{PGL}(2,9)$ and $\mathrm{PGL}(2,17)$. Since their groups are not gap groups, we confirm the assumption of Theorem 2.15. Let $q=9,17$ and let x_{q+1} and x_{q-1} be an element of $\mathrm{PGL}(2,q)$ of order $q+1$ and $q-1$ respectively. An element of $\mathrm{PGL}(2,q) \smallsetminus \mathrm{PSL}(2,q)$ of order 2 is conjugate to an element of the cyclic group $\langle x_{q+1}\rangle$. Then for $(P,H) \in \mathcal{D}^{(2)}(G)$, if $H \smallsetminus P$ has the element $x_{q+1}^{(q+1)/2}$ of order 2 then P contains $\langle x_{q+1}^2\rangle$. Thus, any element of $\mathcal{D}^{(2)}(G)$ is conjugate to $(\langle x_{q-1}^2\rangle, \langle x_{q-1}\rangle)$, $(\langle x_{q+1}^2\rangle, \langle x_{q+1}\rangle)$, or $(\langle x_{q+1}^2\rangle, D_{q+1})$, where D_{q+1} is a dihedral group generated by x_{q+1}^2 and y which is conjugate to $x_{q+1}^{(q+1)/2}$. For the groups H in the above pairs of subgroups, $N_G(H)/H$ is a cyclic group C_2 of order 2 generated by an element gH such that g is conjugate in G to $x_{q+1}^{(q+1)/2}$, $x_{q-1}^{(q-1)/2}$, and $x_{q-1}^{(q-1)/2}$, respectively.

Proposition 5.2. *It holds that*

$$PSm^c(\mathrm{PGL}(2,9)) = Sm(\mathrm{PGL}(2,9)) = RO(\mathrm{PGL}(2,9))^{\{\mathrm{PSL}(2,9)\}}_{\mathcal{P}(\mathrm{PGL}(2,9))},$$

which is a free abelian group of rank 1.

Proof. We use the notation in the proof of Theorem 5.1. Let $G = \mathrm{PGL}(2,9)$, $U = \chi_8^{(1)} \oplus \chi_8^{(2)}$ and $V = \chi_8^{(3)} \oplus \chi_8^{(4)}$. By the character table for G, we have

$$RO(G)^{\{\mathrm{PSL}(2,9)\}}_{\mathcal{P}(G)} = \langle [U] - [V] \rangle.$$

We show that both $U \oplus V$ and $2V$ satisfy the weak gap condition. Recall a non-solvable group is an Oliver group. Then G is an Oliver group. For each $(P,H) \in \mathcal{D}^{(2)}(G)$, we have U^H and V^H are both isomorphic to $\mathbb{R}[N_G(H)/H]$ as $N_G(H)/H$-modules and $\dim U^P = \dim V^P = 4$. Therefore $U \oplus V$ and $2V$ satisfy (PWGC) and then $RO(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} \subset PSm^c(G)$ by Theorem 2.15. Since $\mathcal{L}(G) = \{\mathrm{PSL}(2,9), G\}$, we have $RO(G)^{\{\mathrm{PSL}(2,9),G\}}_{\mathcal{P}(G)} = RO(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)}$. By Theorem 2.2 with $i_{\mathbb{R}} = 0$, the equalities $RO(G)^{\{\mathrm{PSL}(2,9),G\}}_{\mathcal{P}(G)} = PSm^c(G) = Sm(G)$ hold. □

Proposition 5.3. *It holds that*

$$PSm^c(\mathrm{PGL}(2,17)) = Sm(\mathrm{PGL}(2,17)) = RO(\mathrm{PGL}(2,17))^{\{\mathrm{PSL}(2,17)\}}_{\mathcal{P}(\mathrm{PGL}(2,17))},$$

which is a free abelian group of rank 3.

Proof. By the notation in the proof of Theorem 5.1, let $G = \mathrm{PGL}(2,17)$, $U_1 = \chi_{16}^{(3)} + \chi_{16}^{(4)}$, $V_1 = \chi_{16}^{(5)} + \chi_{16}^{(6)}$, $U_2 = \chi_{16}^{(1)} + \chi_{16}^{(4)}$, $V_2 = \chi_{16}^{(5)} + \chi_{16}^{(8)}$, $U_3 = \chi_{16}^{(2)} + \chi_{16}^{(5)}$, $V_3 = \chi_{16}^{(4)} + \chi_{16}^{(7)}$, and $W_j = U_j \oplus V_j$ and $W_j' = 2V_j$, for $j = 1,2,3$. Then the three elements $[W_j] - [W_j']$ span $RO(G)^{\{\mathrm{PGL}(2,17),G\}}_{\mathcal{P}(G)}$. For each $(P,H) \in \mathcal{D}^{(2)}(G)$ we have $\dim U^P = 4 = \dim V^P$, and U^H and V^H are isomorphic to $\mathbb{R}[C_2]$ as $N_G(H)/H$-modules. Therefore W_j and W_j' satisfy (PWGC). By Theorems 2.2 and 2.15 with $i_{\mathbb{R}} = 0$, the equalities $RO(G)^{\{\mathrm{PGL}(2,9),G\}}_{\mathcal{P}(G)} = PSm^c(G) = Sm(G)$ hold. □

Therefore the Smith set of $\mathrm{PGL}(2,q)$ is a free abelian group for any q, a power of a prime.

Now we discuss the Smith sets of Oliver groups G with $[G : O^3(G)] = 3$. The following theorem is a key result for $\mathrm{PGL}(3,q)$, obtained essentially by Theorem 1.3 [22].

Theorem 5.4. *If G is an Oliver group such that $G/G^{\mathrm{nil}} \cong C_3$ and G^{nil} has a subquotient group isomorphic to D_{2st} for distinct primes s and t, then*

$$PSm^c(G) = RO(G)^{\{G\}}_{\mathcal{P}(G)}.$$

Furthermore, if $i_{\mathbb{R}}(G) \leq 1$ then $PSm^c(G) = Sm(G) = RO(G)^{\{G\}}_{\mathcal{P}(G)}$.

Proof. The Realization Theorem [26, p.850] implies that $RO(G)^{\{G^{\mathrm{nil}}\}}_{\mathcal{P}(G)} \subset PSm^c(G)$, since G is a gap group. If $G \setminus G^{\mathrm{nil}}$ has no element not of prime power order, then $RO(G)^{\{G\}}_{\mathcal{P}(G)} = RO(G)^{\{G^{\mathrm{nil}}\}}_{\mathcal{P}(G)}$ and $PSm^c(G) = RO(G)^{\{G\}}_{\mathcal{P}(G)}$ holds. Otherwise, by Theorem 1.3 [22], $PSm^c(G) = RO(G)^{\{G\}}_{\mathcal{P}(G)}$ holds. Therefore, if $i_{\mathbb{R}}(G) \leq 1$, then Proposition 2.5 implies that $PSm^c(G) = RO(G)^{\{G\}}_{\mathcal{P}(G)} = Sm(G)$. □

Note that $\mathrm{PGL}(3,2) = \mathrm{PSL}(3,2) \cong \mathrm{PSL}(2,7)$ and $r_{\mathrm{PSL}(2,7)} = 0$, which implies $Sm(\mathrm{PGL}(3,2)) = 0$. Also note that $Sm(\mathrm{PGL}(3,3)) = 0$ since $\mathrm{PGL}(3,3) = \mathrm{PSL}(3,3)$ and $r_{\mathrm{PSL}(3,3)} = 1$. If $\mathrm{PGL}(3,q) \neq \mathrm{PSL}(3,q)$ then $q-1$ is divisible by 3. Hence, if $q-1$ is not divisible by 3 then the Smith set of $\mathrm{PGL}(3,q)$ has been already obtained in Theorem 4.2.

Theorem 5.5. *Let $q \geq 4$ be a power of a prime such that $q-1$ is divisible by 3 and let $G = \mathrm{PGL}(3,q)$. It holds that $i_{\mathbb{R}}(G) = 0$. Furthermore, $PSm^c(G) = Sm(G) = RO(G)^{\{G\}}_{\mathcal{P}(G)}$ which is a free abelian group of rank $r_G - 1$.*

Proof. The character table for G is well-known [33]. The equality $i_{\mathbb{C}}(G) = 0$ follows from it. Then, by Lemma 2.7, $i_{\mathbb{R}}(G) = 0$ holds.

We show that $PSm^c(G) = Sm(G) = RO(G)^{\{G\}}_{\mathcal{P}(G)}$. First suppose that $q = 4$. We have $RO(G)^{\{\mathrm{PSL}(3,4)\}}_{\mathcal{P}(G)} \subset PSm^c(G)$ by the Realization Theorem [26, p.850]. Since $r_{\mathrm{PSL}(3,4)} = 0$ and $[\mathrm{PGL}(3,4) : \mathrm{PSL}(3,4)] = 3$, or by the character table of G, we see that the equalities $RO(G)^{\{\mathrm{PSL}(3,4)\}}_{\mathcal{P}(G)} = RO(G)^{\{G\}}_{\mathcal{P}(G)}$ and then $PSm^c(G) = RO(G)^{\{G\}}_{\mathcal{P}(G)}$ hold. Suppose that $q > 4$. It suffices to see that the equation $PSm^c(G) = RO(G)^{\{G\}}_{\mathcal{P}(G)}$ holds. Since G is a non-solvable group, it is an Oliver group with $[G : \mathrm{PSL}(3,q)] = 3$. The group $\mathrm{PSL}(3,q)$ has a subgroup isomorphic to $\mathrm{PGL}(2,q)$ which has a subgroup isomorphic to $D_{2(q-1)}$ of order $2(q-1)$. If q is odd, then $q-1$ is divisible by 6 and otherwise $q-1$ is not a power of 3 by Lemma 3.3 [35]. Therefore, by Theorem 5.4, $PSm^c(G) = RO(G)^{\{G\}}_{\mathcal{P}(G)}$ holds. □

Proposition 5.6. *For $G = \mathrm{GL}(2,2^a)$ or $\mathrm{GL}(3,2^a)$, $i_{\mathbb{R}}(G) = 0$.*

Proof. $\mathrm{GL}(n,2^a)$ is a central extension of $\mathrm{PGL}(n,2^a)$ by a cyclic group of order $2^a - 1$, $\mathrm{PGL}(2,2^a) = \mathrm{PSL}(2,2^a)$ and $[\mathrm{PGL}(3,2^a) : \mathrm{PSL}(3,2^a)] = \gcd(3, 2^a - 1)$. Therefore, Theorems 4.1 and 5.5 and Proposition 2.9, $i_{\mathbb{R}}(G) = 0$ for $G = \mathrm{GL}(2,2^a)$ or $\mathrm{GL}(3,2^a)$. □

Theorem 5.7. *Let $a \geq 4$ be an even integer. Put $n = 2,3$ and $q = 2^a$. For a subgroup G of $\mathrm{GL}(n,q)$ with $[G : \mathrm{SL}(n,q)] = 3$, it holds that $PSm^c(G) = Sm(G) = RO(G)^{\{G\}}_{\mathcal{P}(G)}$.*

Proof. By Proposition 2.9, we have $i_{\mathbb{R}}(G) = 0$. $\mathrm{PSL}(2,q) \cong \mathrm{PGL}(2,q)$ has a subgroup isomorphic to a dihedral subgroup of $D_{2(q-1)}$. Since $q-1$ is an odd integer divisible by 3, Lemma 3.3 (1) [35] implies that $q-1$ is not a power of 3. Therefore $G^{\mathrm{nil}} = \mathrm{PSL}(n,q)$ has a dihedral subgroup of order $2st$ where s and t are distinct odd primes. By Theorem 5.4 we have $PSm^c(G) = Sm(G) = RO(G)^{\{G\}}_{\mathcal{P}(G)}$. □

§ 6. Sporadic groups and their automorphism groups

There are 26 simple groups which are not classical groups, called sporadic groups. Atlas of finite groups [10] has an information of prime power maps. The prime power map for a prime p sends a conjugacy class (x) to (x^p). Therefore the numbers $i_{\mathbb{R}}(G)$ and $i_{\mathbb{C}}(G)$ for a group G described in [10] can be computed.

Theorem 6.1. *For a sporadic group G, $i_{\mathbb{R}}(G) = 0$ and*

$$PSm^c(G) = Sm(G) = RO(G)^{\{G\}}_{\mathcal{P}(G)}.$$

If G is a sporadic group, then $[\mathrm{Aut}(G) : G] \leq 2$, and $\mathrm{Aut}(G) \not\cong G$ implies that G is isomorphic to

$$M_{12},\ M_{22},\ J_2,\ J_3,\ Suz,\ HS,\ M^cL,\ He,\ Fi_{22},\ Fi'_{24},\ HN, \text{ or } O'N$$

(cf. [10]). The automorphism groups of these groups are all gap groups [35, Corollary 3.6].

Theorem 6.2. *For a sporadic group G, $i_{\mathbb{R}}(\mathrm{Aut}(G)) = 0$ and*

$$PSm^c(\mathrm{Aut}(G)) = Sm(\mathrm{Aut}(G)) = RO(\mathrm{Aut}(G))^{\{G\}}_{\mathcal{P}(\mathrm{Aut}(G))}.$$

We denote by $K.n$ the extension of C_n by K and by $m.K.n$ the extension of $K.n$ by C_m where C_k is a cyclic group of order k. All groups $L = m.K.n$ listed below are gap groups, since K and $K.2 \cong \mathrm{Aut}(K)$ are gap groups, and the number $i_{\mathbb{R}}(L) = 0$, which is also computable by using the software GAP [15]. Therefore we have the following theorem.

Theorem 6.3. *$Sm(G)_{\mathcal{P}(G)} = Sm(G)$ with $L \geq G \geq L^{\mathrm{nil}}$ for the following groups L:*

$3.McL.2$, $3.Suz.2$, $3.J_3.2$, $3.Fi'_{24}.2$, $3.O'N.2$, $6.Fi_{22}.2$, $6.M_{22}.2$,
$2.M_{12}$, $2.HS$, $2.Ru$

We need more inspection to determine whether $Sm(G)$ is zero for an Oliver group G such that G/G^{nil} is not of prime power order.

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