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Symplectic volumes of double weight varieties associated with $SU(3)$, I

By

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Abstract

We consider double weight varieties, that is, symplectic torus quotients for a direct product of two integral coadjoint orbits of $SU(3)$, and investigate their symplectic volumes. According to a fundamental theorem for symplectic quotients, it is equivalent to studying weight multiplicities in a tensor product of two irreducible representations of $SU(3)$, and their asymptotic behavior. We assume that both of two coadjoint orbits used to define the double weight variety are flag manifolds of $SU(3)$. As a main result, we obtain an explicit formula for the symplectic volumes of double weight varieties.

§ 1. Introduction

Let M be a symplectic manifold with a Hamiltonian action of a torus T and $\Phi : M \to \mathfrak{t}^*$ its moment map. For any regular value $\mu \in \mathfrak{t}^*$ of Φ, the symplectic quotient at μ is defined by

$$M//_{\mu}T := \Phi^{-1}(\mu)/T.$$

For example, let M be a coadjoint orbit $\mathcal{O}_\lambda$ of a compact semi-simple Lie group G through the point $\lambda \in \mathfrak{t}^*$, and consider the action of the maximal torus $T \subset G$ on $\mathcal{O}_\lambda$. Then $M//_{\mu}T = \mathcal{O}_\lambda//_{\mu}T$ is called a *weight variety*. Many results have been obtained about weight varieties. Knutson [15] studied the relation with weight spaces of an irreducible representation (thereby $\mathcal{O}_\lambda//_{\mu}T$ were named weight varieties), Guillemin-Lerman-Sternberg [6] gave some formulas for the volumes of weight varieties, and Goldin [4] gave an explicit expression for the cohomology rings of weight varieties for

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$G = SU(n)$. For compact semi-simple Lie groups except of type A, it is known that weight varieties are orbifolds in general (see [5]).

For $\lambda_1, \lambda_2 \in \mathfrak{t}^*$, let M be a direct product $\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2}$ of two coadjoint orbits of G, and consider the diagonal action of $T \subset G$ on $\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2}$. Then $M//_\mu T = (\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_\mu T$ is called a *double weight variety.*

In this paper, we consider the case where $G = SU(3)$. Except for the orbit consisting only of the origin, each coadjoint orbit of $SU(3)$ is diffeomorphic to either the flag manifold $SU(3)/T$ or the complex projective space $\mathbb{P}^2$. Our aim is to express the symplectic volume $\mathrm{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_\mu T)$ of the double weight variety in an explicit form.

First, we assume that $\lambda_1, \lambda_2 \in P_+$ and $\mu \in P$, where P denotes the weight lattice of G and P_+ denotes the set of dominant integral weights of G. As we will discuss in Section 2, under certain conditions on $\lambda_1, \lambda_2 \in P_+$ and $\mu \in P$, we can express the volume $\mathrm{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_\mu T)$ in terms of representations of T. Namely, $\mathrm{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_\mu T)$ is equal to

$$\lim_{k\to\infty} \frac{1}{k^d} \cdot [V_{k\lambda_1} \otimes V_{k\lambda_2} : W_{k\mu}],$$

where V_λ denotes the irreducible representation of G with highest weight $\lambda \in P_+$, and for a representation V of T, $[V : W_\mu]$ denotes the multiplicity of weight space W_μ ($\mu \in P$) in the weight decomposition of V. Moreover, k runs over positive integers and d is the complex dimension of $(\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_\mu T$. Here, the theorem of Guillemin-Sternberg (and its generalization) on the characteristic numbers of symplectic quotients (see, e.g., [8] and [17]) plays the key role, as well as the Borel-Weil theorem and the Hirzebruch-Riemann-Roch theorem. The argument above are essentially the same with those in [20] and [21].

Next, we assume that both of two coadjoint orbits $\mathcal{O}_{\lambda_1}$ and $\mathcal{O}_{\lambda_2}$ are flag manifolds of $SU(3)$. Namely, we assume $\lambda_1, \lambda_2 \in P_{++}$, where P_{++} denotes the set of dominant integral weights which belong to the interior of the Weyl chamber. The main result in this paper is an explicit formula for the volume $\mathrm{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_\mu T)$. The details of the notation will be given in Sections 2 and 3.

Theorem. (See Theorem 3.4 below.) *Let $\lambda_1, \lambda_2 \in P_{++}$ and $\mu \in P$ satisfy the assumptions* (A0), (A1) *and* (A2) *in Section 3.2. Then we have*

$$\mathrm{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_\mu T) = \sum_{w_1, w_2 \in W} \varepsilon(w_1)\varepsilon(w_2) F(\lambda_1, \lambda_2, \mu; w_1, w_2),$$

where W denotes the Weyl group of $SU(3)$ and $F(\lambda_1, \lambda_2, \mu; w_1, w_2)$ is given as follows.

(1) *If $w_1\lambda_1 + w_2\lambda_2 - \mu \in \gamma_1$,*

$$F(\lambda_1, \lambda_2, \mu; w_1, w_2) = \frac{1}{12} \langle w_1\lambda_1 + w_2\lambda_2 - \mu, \Lambda_2\rangle^3 \langle w_1\lambda_1 + w_2\lambda_2 - \mu, 2\Lambda_1 - \Lambda_2\rangle.$$

(2) *If* $w_1\lambda_1 + w_2\lambda_2 - \mu \in \gamma_2$,

$$F(\lambda_1, \lambda_2, \mu; w_1, w_2) = \frac{1}{12}\langle w_1\lambda_1 + w_2\lambda_2 - \mu, \Lambda_1\rangle^3 \langle w_1\lambda_1 + w_2\lambda_2 - \mu, -\Lambda_1 + 2\Lambda_2\rangle.$$

(3) *Otherwise,*

$$F(\lambda_1, \lambda_2, \mu; w_1, w_2) = 0.$$

§ 2. Preliminaries

§ 2.1. The representation theory of $SU(3)$ and notation

We review some standard facts about the representation theory of $SU(3)$ in order to fix our notation. We refer to [2] for the generalities on compact Lie groups and their representations.

Let $G = SU(3)$, $\mathfrak{g} = \mathfrak{su}(3)$, T the standard maximal torus of G consisting of diagonal matrices in G, and $\mathfrak{t}$ its Lie algebra. Let $\mathfrak{g}^*$ and $\mathfrak{t}^*$ be the duals of $\mathfrak{g}$ and $\mathfrak{t}$, respectively. We denote by $\langle\ ,\ \rangle$ the pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$, or $\mathfrak{t}^*$ and $\mathfrak{t}$. Let $W \cong \mathfrak{S}_3$ be the Weyl group of $G = SU(3)$ with respect to T. We define an AdG-invariant positive definite inner product $(\ ,\ )$ on $\mathfrak{g}$ by

$$(X, Y) := -\frac{1}{4\pi^2}\mathrm{Tr}(XY) \qquad (X, Y \in \mathfrak{g}).$$

We identify $\mathfrak{g}^*$ with $\mathfrak{g}$ by the inner product $(\ ,\ )$. We regard $\mathfrak{t}^*$ as a subspace of $\mathfrak{g}^*$ by the identification

$$\mathfrak{t}^* = \{f \in \mathfrak{g}^* | t \cdot f = f\ (\forall t \in T)\}.$$

The elements

$$H_1 = 2\pi\sqrt{-1}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ H_2 = 2\pi\sqrt{-1}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{2.1}$$

in $\mathfrak{t}$ are generators of the integral lattice $\mathrm{Ker}(\exp : \mathfrak{t} \to T)$ and form a basis of $\mathfrak{t}$. Under the identification by the inner product, we define the simple roots α_1, α_2 in $\mathfrak{t}^*$ by the elements which correspond to $H_1, H_2 \in \mathfrak{t}$, respectively. Let $\triangle_+ := \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$ be the positive root system. Let us set

$$Q := \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2,$$

$$\gamma_1 := \mathbb{R}_{>0}\alpha_1 + \mathbb{R}_{>0}(\alpha_1 + \alpha_2),\ \gamma_2 := \mathbb{R}_{>0}(\alpha_1 + \alpha_2) + \mathbb{R}_{>0}\alpha_2,$$

$$\bar{\gamma_1} := \mathbb{R}_{\geq 0}\alpha_1 + \mathbb{R}_{\geq 0}(\alpha_1 + \alpha_2),\ \bar{\gamma_2} := \mathbb{R}_{\geq 0}(\alpha_1 + \alpha_2) + \mathbb{R}_{\geq 0}\alpha_2.$$

We define the fundamental weights $\Lambda_1, \Lambda_2 \in \mathfrak{t}^*$ by

$$\Lambda_1 := \frac{2\alpha_1 + \alpha_2}{3}, \Lambda_2 := \frac{\alpha_1 + 2\alpha_2}{3}.$$

Under the identification by the inner product, $\Lambda_1, \Lambda_2 \in \mathfrak{t}^*$ correspond to the elements

$$I_1 = \frac{2\pi\sqrt{-1}}{3} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \; I_2 = \frac{2\pi\sqrt{-1}}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

in $\mathfrak{t}$, respectively. Define

$$\mathfrak{t}^*_+ := \mathbb{R}_{\geq 0}\Lambda_1 + \mathbb{R}_{\geq 0}\Lambda_2, \; \mathfrak{t}^*_{++} := \mathbb{R}_{>0}\Lambda_1 + \mathbb{R}_{>0}\Lambda_2,$$
$$P := \mathbb{Z}\Lambda_1 + \mathbb{Z}\Lambda_2, \; P_+ := \mathbb{Z}_{\geq 0}\Lambda_1 + \mathbb{Z}_{\geq 0}\Lambda_2, \; P_{++} := \mathbb{Z}_{>0}\Lambda_1 + \mathbb{Z}_{>0}\Lambda_2.$$

The set $\mathfrak{t}^*_+$ is a fundamental domain of the action of the Weyl group W on $\mathfrak{t}^*$. Let us set

$$\rho := \frac{1}{2} \sum_{\alpha \in \triangle_+} \alpha = \Lambda_1 + \Lambda_2.$$

According to the representation theory of compact Lie groups, irreducible representations of G are, by assigning their highest weights, in one-to-one correspondence with elements in P_+. For $\lambda \in P_+$, we denote by V_λ the irreducible representation of G with the highest weight $\lambda \in P_+$, and by $\chi_\lambda : G \to \mathbb{C}$ the character of V_λ. For $\mu \in P$, we define $e^\mu : T \to \mathbb{C}$ by $e^\mu(t) := e^{2\pi\sqrt{-1}\langle \mu, X \rangle}$ for $t = \exp X \in T$ $(X \in \mathfrak{t})$. By the Weyl character formula, χ_λ is given as a function on T by

$$\chi_\lambda(t) = \frac{\sum_{w \in W} \varepsilon(w) e^{w(\lambda+\rho)}(t)}{e^\rho(t) \prod_{\alpha \in \triangle_+} (1 - e^{-\alpha}(t))},$$

where $\varepsilon(w) = \pm 1$ is the signature of $w \in W$.

§ 2.2. Coadjoint orbits

Although we mainly consider the case $G = SU(3)$, most of the following still holds when G is a general compact Lie group. For further details on coadjoint orbits, see, e.g., [13] and [16]. We also refer to [12] for the Borel-Weil theorem.

The left coadjoint action of G on $\mathfrak{g}^*$ is defined by $g \cdot f := \mathrm{Ad}^*(g)f$ for $g \in G$ and $f \in \mathfrak{g}^*$, where

$$\langle \mathrm{Ad}^*(g)f, X \rangle = \langle f, \mathrm{Ad}(g^{-1})X \rangle \qquad (X \in \mathfrak{g}).$$

We denote by $\mathcal{O}_\lambda = G \cdot \lambda$ the coadjoint orbit through $\lambda \in \mathfrak{t}^* \subset \mathfrak{g}^*$. Then the intersection $\mathcal{O}_\lambda \cap \mathfrak{t}^*$ is the W-orbit through λ, and $\mathcal{O}_\lambda \cap \mathfrak{t}^*_+$ consists of a single point. In other words, coadjoint orbits are parametrized by elements in $\mathfrak{t}^*_+$.

In particular, for $G = SU(3)$, coadjoint orbits $\mathcal{O}_\lambda$ are classified as follows, where G_λ denotes the isotropy subgroup at $\lambda \in \mathfrak{t}_+^*$ for the coadjoint action of G on $\mathfrak{g}^*$.

(1) If $\lambda \in \mathfrak{t}_{++}^*$, then $G_\lambda \cong T$ and $\mathcal{O}_\lambda \cong SU(3)/T$.

(2) If $\lambda \in \mathfrak{t}_+^* - \mathfrak{t}_{++}^* - \{0\}$, then $G_\lambda \cong S(U(1) \times U(2))$ and $\mathcal{O}_\lambda \cong \mathbb{P}^2$.

(3) If $\lambda = 0$, then $G_\lambda = G$ and $\mathcal{O}_\lambda = \{0\}$.

On each coadjoint orbit $\mathcal{O}_\lambda$, there exists a natural G-invariant symplectic structure ω_λ, called the Kirillov-Kostant-Souriau symplectic form, defined by

$$(\omega_\lambda)_x(\tilde{X}_x, \tilde{Y}_x) = \langle x, [X, Y] \rangle \quad (x \in \mathcal{O}_\lambda, X, Y \in \mathfrak{g}),$$

where $\tilde{X}$ is the vector field on $\mathcal{O}_\lambda$ given by

$$\tilde{X}_x := \left. \frac{d}{dt} \right|_{t=0} (\exp\, tX) \cdot x.$$

The action of G on $\mathcal{O}_\lambda$ is Hamiltonian and the associated moment map is given by the inclusion $\iota_\lambda : \mathcal{O}_\lambda \hookrightarrow \mathfrak{g}^*$, that is, we have $d\langle \iota_\lambda, X \rangle(\cdot) = \omega_\lambda(\tilde{X}, \cdot)$.

In addition, there exists a G-invariant complex structure J_λ on $\mathcal{O}_\lambda$, which is compatible with the symplectic structure ω_λ, that is, $\omega_\lambda(\cdot, J_\lambda \cdot)$ becomes a Riemannian metric on $\mathcal{O}_\lambda$, and makes $\mathcal{O}_\lambda$ into a Kähler manifold.

Moreover, in the case $\lambda \in P_+$, there exists a G-equivariant holomorphic line bundle L_λ over $\mathcal{O}_\lambda$ such that $c_1(L_\lambda) = [\omega_\lambda]$. The Borel-Weil theorem shows that

$$H^0(\mathcal{O}_\lambda, \mathcal{O}(L_\lambda)) \cong V_\lambda \ , \ H^k(\mathcal{O}_\lambda, \mathcal{O}(L_\lambda)) = 0 \ \ (k > 0)$$

as representations of G, where $\mathcal{O}(L_\lambda)$ denotes the sheaf of germs of holomorphic sections of L_λ.

Remark 1. (1) For $k \in \mathbb{R}_{>0}$ and $\lambda \in \mathfrak{t}_+^*$, $\mathcal{O}_{k\lambda} = \mathcal{O}_\lambda$ as complex manifolds. If we compare the symplectic forms and the moment maps under this identification, we have $\omega_{k\lambda} = k\omega_\lambda$ and $\iota_{k\lambda} = k\iota_\lambda$. In the case $k \in \mathbb{Z}_{>0}$ and $\lambda \in P_+$, we have $L_{k\lambda} \cong L_\lambda^{\otimes k}$.

(2) For $\lambda \in P_+$, the action on $\mathcal{O}_\lambda$ of the center $Z(G) \cong \mathbb{Z}/3\mathbb{Z}$ of $G = SU(3)$ is trivial, while that on L_λ is not trivial in general. However, by the construction of L_λ (see, e.g., [12]), if we suppose $\lambda \in P_+ \cap Q$, this action becomes trivial, too.

§ 2.3. Double weight varieties

For general properties of symplectic and Kähler quotients, see, e.g., [10], [14] and [18]. The following still holds for a general compact Lie group G.

Let $\lambda_1, \lambda_2 \in \mathfrak{t}_+^*$. The diagonal action of the maximal torus T of G on the direct product $\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2}$ is also Hamiltonian and the moment map $\Phi : \mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2} \to \mathfrak{t}^*$ is given by the composition of the map

$$\begin{aligned} \mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2} \to & \quad \mathfrak{g}^* \\ (x_1, x_2) \quad & \to x_1 + x_2 \end{aligned}$$

and the projection $\mathfrak{g}^* \to \mathfrak{t}^*$. For $\mu \in \mathfrak{t}^*$, we define the symplectic (or Kähler) quotient at μ by

$$(\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_\mu T := \Phi^{-1}(\mu)/T.$$

Here we assume that for $\mu \in \mathfrak{t}^*$,

(a0) $\Phi^{-1}(\mu) \neq \emptyset$,

(a1) μ is a regular value of the moment map Φ, and $(\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_\mu T$ is a smooth manifold.

Then there exist a natural symplectic structure $\omega = \omega(\lambda_1, \lambda_2, \mu)$ and a compatible complex structure J on $(\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_\mu T$, induced from those on $\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2}$, which make $(\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_\mu T$ a Kähler manifold. The complex dimension d of $(\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_\mu T$ is

$$d = \dim_{\mathbb{R}} G - \frac{1}{2}(\dim_{\mathbb{R}} G_{\lambda_1} + \dim_{\mathbb{R}} G_{\lambda_2}) - \dim_{\mathbb{R}} T.$$

We call $(\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_\mu T$ a *double weight variety.*

Now, suppose $\lambda_1, \lambda_2 \in P_+$. Let L_{λ_i} be the T-equivariant holomorphic line bundle over $\mathcal{O}_{\lambda_i}$ as in Section 2.2, and let us set

$$\begin{aligned} \mathcal{L} = (L_{\lambda_1} \boxtimes L_{\lambda_2})//_\mu T &:= ((\mathrm{pr}_1^* L_{\lambda_1} \otimes \mathrm{pr}_2^* L_{\lambda_2})|_{\Phi^{-1}(\mu)})/T, \\ \mathcal{L}_i = L_{\lambda_i}//_\mu T &:= (\mathrm{pr}_i^* L_{\lambda_i}|_{\Phi^{-1}(\mu)})/T \quad (i = 1, 2), \end{aligned}$$

where

$$\mathrm{pr}_i : \mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2} \to \mathcal{O}_{\lambda_i} \quad (i = 1, 2)$$

is the i-th projection.

By the assumption (a1), the isotropy subgroup at each point in $\Phi^{-1}(\mu)$ is a finite group. Since its action on $\mathrm{pr}_i^* L_{\lambda_i}$ $(i = 1, 2)$ and $L_{\lambda_1} \boxtimes L_{\lambda_2} = \mathrm{pr}_1^* L_{\lambda_1} \otimes \mathrm{pr}_2^* L_{\lambda_2}$ may not be trivial, $\mathcal{L}$ and $\mathcal{L}_i$ $(i = 1, 2)$ are, in general, orbifold holomorphic line bundles over $(\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_\mu T$. We assume that

(a2) $\mathcal{L}_i$ is a genuine holomorphic line bundle over $(\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_\mu T$ for $i = 1, 2$.

Then we have $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2$ and $c_1(\mathcal{L}) = c_1(\mathcal{L}_1) + c_1(\mathcal{L}_2) = [\omega(\lambda_1, \lambda_2, \mu)]$.

In general, let (M, ω) be a compact Kähler manifold and suppose that there exists a holomorphic line bundle L over M such that $c_1(L) = [\omega]$. In this case, we define

$$\chi(M, L) := \sum_i (-1)^i H^i(M, \mathcal{O}(L))$$

as a virtual vector space. The dimension $\dim_{\mathbb{C}} \chi(M, L)$ is called the Riemann-Roch number of (M, L). We define the symplectic volume $\mathrm{vol}(M)$ of (M, ω) by

$$\mathrm{vol}(M) := \int_M \frac{\omega^d}{d!},$$

where d is the complex dimension of M.

Lemma 2.1. *Suppose that (M, ω) and L satisfy the assumptions above. Then we have*

$$\mathrm{vol}(M) = \lim_{k \to \infty} \frac{1}{k^d} \cdot \dim_{\mathbb{C}} \chi(M, L^{\otimes k}).$$

Proof. By the Hirzebruch-Riemann-Roch theorem, we have

$$\dim_{\mathbb{C}} \chi(M, L) = \int_M \mathrm{Ch}(L)\mathrm{Td}(M) = \int_M e^{c_1(L)}\mathrm{Td}(M),$$

where $\mathrm{Ch}(L)$ is the Chern character of L and $\mathrm{Td}(M)$ is the Todd class of M. Hence we have

$$\lim_{k \to \infty} \frac{1}{k^d} \cdot \dim_{\mathbb{C}} \chi(M, L^{\otimes k}) = \lim_{k \to \infty} \int_M \frac{e^{kc_1(L)}}{k^d}\mathrm{Td}(M) = \int_M \frac{c_1(L)^d}{d!} = \mathrm{vol}(M).$$

□

In addition, we assume that a torus T acts holomorphically on M, and this action lifts to $L \to M$. Then the action of T on M is Hamiltonian, and the moment map $\Phi : M \to \mathfrak{t}^*$ is determined. For suitably chosen $\mu \in P$, we obtain the Kähler manifold $M/\!/_\mu T = \Phi^{-1}(\mu)/T$, the line bundle $L/\!/_\mu T = (L|_{\Phi^{-1}(\mu)})/T$ over $M/\!/_\mu T$ and $\chi(M/\!/_\mu T, L/\!/_\mu T)$ (see, e.g., [8] and [17]).

Remark 2. (1) It follows from the multiplicative property of χ (see the appendix in [11]) that $\chi(\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2}, L_{\lambda_1} \boxtimes L_{\lambda_2}) = \chi(\mathcal{O}_{\lambda_1}, L_{\lambda_1}) \otimes \chi(\mathcal{O}_{\lambda_2}, L_{\lambda_2})$.

(2) As in Remark 1 (1), $(\mathcal{O}_{k\lambda_1} \times \mathcal{O}_{k\lambda_2})/\!/_{k\mu} T = (\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})/\!/_\mu T$ as complex manifolds, and $\omega(k\lambda_1, k\lambda_2, k\mu) = k\omega(\lambda_1, \lambda_2, \mu)$ for $k \in \mathbb{R}_{>0}$, $\lambda_1, \lambda_2 \in \mathfrak{t}^*_+$ and $\mu \in \mathfrak{t}^*$. In particular, it follows that $\mathrm{vol}((\mathcal{O}_{k\lambda_1} \times \mathcal{O}_{k\lambda_2})/\!/_{k\mu} T) = k^d \cdot \mathrm{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})/\!/_\mu T)$. In the case $k \in \mathbb{Z}_{>0}$, $\lambda_1, \lambda_2 \in P_+$ and $\mu \in P$, we have $(L_{k\lambda_1} \boxtimes L_{k\lambda_2})/\!/_{k\mu} T = ((L_{\lambda_1} \boxtimes L_{\lambda_2})/\!/_\mu T)^{\otimes k}$.

(3) Even if $\lambda_1, \lambda_2 \in P_+$ does not satisfy (a2), $n\lambda_1$ and $n\lambda_2$ does satisfy (a2) for some positive integer n. Hence, as far as the symplectic volume $\mathrm{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})/\!/_\mu T)$ is concerned, we can assume (a2) without loss of generality.

On the other hand, we can regard $\chi(M, L)$ as a representation of T. Then we obtain the weight decomposition

$$\chi(M, L) = \sum_{\mu \in P} m_\mu W_\mu,$$

where W_μ denotes the weight space associated with $\mu \in P$ and

$$m_\mu = [\chi(M, L) : W_\mu] := \dim_{\mathbb{C}}(\chi(M, L) \otimes W_\mu^*)^T$$

is the weight multiplicity of W_μ. Besides, for a representation V of T,

$$V^T := \{v \in V | t \cdot v = v \ (\forall t \in T)\}$$

is the set of invariants in V.

The Guillemin-Sternberg theorem and its generalization (see, e.g., [8] and [17]) tell us the following.

Theorem 2.2. *Assume that the action of T on $\Phi^{-1}(\mu)$ is free. Then we have*

$$\dim_{\mathbb{C}} \chi(M//_\mu T, L//_\mu T) = \dim_{\mathbb{C}}(\chi(M, L) \otimes W_\mu^*)^T = [\chi(M, L) : W_\mu].$$

In our situation, we obtain the following.

Proposition 2.3. *Suppose that $\lambda_1, \lambda_2 \in P_+$ and $\mu \in P$ satisfy the assumptions* (a0), (a1) *and* (a2)*. Then we have*

(1) $\dim_{\mathbb{C}} \chi((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_\mu T, (L_{\lambda_1} \boxtimes L_{\lambda_2})//_\mu T) = [V_{\lambda_1} \otimes V_{\lambda_2} : W_\mu]$,

(2) $\mathrm{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_\mu T) = \lim_{k \to \infty} \frac{1}{k^d} \cdot [V_{k\lambda_1} \otimes V_{k\lambda_2} : W_{k\mu}]$.

Proof. (1) It follows from Theorem 2.2, Remark 2 (1), and the Borel-Weil theorem.
(2) By Lemma 2.1, Remark 2 (2), and the assertion (1), we obtain

$$\begin{aligned}
&\mathrm{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_\mu T) \\
&= \lim_{k \to \infty} \frac{1}{k^d} \cdot \dim_{\mathbb{C}} \chi((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_\mu T, ((L_{\lambda_1} \boxtimes L_{\lambda_2})//_\mu T)^{\otimes k}) \\
&= \lim_{k \to \infty} \frac{1}{k^d} \cdot \dim_{\mathbb{C}} \chi((\mathcal{O}_{k\lambda_1} \times \mathcal{O}_{k\lambda_2})//_{k\mu} T, (L_{k\lambda_1} \boxtimes L_{k\lambda_2})//_{k\mu} T) \\
&= \lim_{k \to \infty} \frac{1}{k^d} \cdot [V_{k\lambda_1} \otimes V_{k\lambda_2} : W_{k\mu}].
\end{aligned}$$

□

In the next section, we will compute the Rimann-Roch number $\dim_{\mathbb{C}} \chi((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_\mu T, (L_{\lambda_1} \boxtimes L_{\lambda_2})//_\mu T)$ and the volume $\mathrm{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_\mu T)$ for $\lambda_1, \lambda_2 \in P_+$ and $\mu \in P$.

§ 3. Main results

§ 3.1. Combinatorial expression

For $\lambda \in P_+$, let us set

$$\chi_\lambda := \frac{\sum_{w \in W} \varepsilon(w) e^{w(\lambda+\rho)}}{e^\rho \prod_{\alpha \in \triangle_+}(1 - e^{-\alpha})} \in \mathbb{C}[e^{\Lambda_1}, e^{\Lambda_2}][[e^{-\Lambda_1}, e^{-\Lambda_2}]]$$

and define

$$F^{\mu}_{\lambda_1,\lambda_2} := \chi_{\lambda_1} \cdot \chi_{\lambda_2} \cdot e^{-\mu}$$

for $\lambda_1, \lambda_2 \in P_+$ and $\mu \in P$. As in the Weyl character formula, we also regard them as functions on T.

Lemma 3.1. *For $\lambda_1, \lambda_2 \in P_+$ and $\mu \in P$, $[V_{\lambda_1} \otimes V_{\lambda_2} : W_\mu]$ is equal to the coefficient of e^0 (i.e., the constant term) in $F^{\mu}_{\lambda_1,\lambda_2}$.*

Proof. Let $d\mu_T$ be the normalized invariant measure on T. Then we have

$$[V_{\lambda_1} \otimes V_{\lambda_2} : W_\mu] = \int_T \chi_{\lambda_1}(t)\chi_{\lambda_2}(t)e^{-\mu}(t)\ d\mu_T. \tag{3.1}$$

Let H_1, H_2 be the basis of the integral lattice in $\mathfrak{t}$ as (2.1). we write an element $t \in T$ as $t = \exp(x_1H_1 + x_2H_2)$ with $x_1, x_2 \in [0,1]$. Let us set $u_i = e^{2\pi\sqrt{-1}x_i}$ $(i = 1,2)$ and define an isomorphism $T \cong U(1)^2$ by $t \mapsto (u_1, u_2)$. Then we have

$$d\mu_T = dx_1dx_2 = \frac{du_1}{2\pi\sqrt{-1}u_1}\frac{du_2}{2\pi\sqrt{-1}u_2}.$$

Hence (3.1) is equal to the coefficient of ${u_1}^0{u_2}^0$ in $F^{\mu}_{\lambda_1,\lambda_2}(t)$, which is regarded as a Laurent series of (u_1, u_2). If we write $F^{\mu}_{\lambda_1,\lambda_2}$ as

$$F^{\mu}_{\lambda_1,\lambda_2} = \sum C_{m_1,m_2}e^{m_1\Lambda_1+m_2\Lambda_2},$$

then we have

$$F^{\mu}_{\lambda_1,\lambda_2}(u_1,u_2) = \sum C_{m_1,m_2}{u_1}^{m_1}{u_2}^{m_2}.$$

Therefore, (3.1) is equal to the coefficient of e^0 in $F^{\mu}_{\lambda_1,\lambda_2}$. □

Proposition 3.2. *Let $\lambda_1, \lambda_2 \in P_+$ and $\mu \in P$. Then we have*

$$[V_{\lambda_1} \otimes V_{\lambda_2} : W_\mu] = \sum_{w_1,w_2\in W} \varepsilon(w_1)\varepsilon(w_2)E(\lambda_1,\lambda_2,\mu;w_1,w_2),$$

where for $w_1, w_2 \in W$, we define

$$E(\lambda_1,\lambda_2,\mu;w_1,w_2) = \sum_{(j_1,j_2,j_3)} (j_1+1)(j_2+1)(j_3+1)$$

and the sum is taken over all $(j_1,j_2,j_3) \in (\mathbb{Z}_{\geq 0})^3$ which satisfy the condition

$$w_1(\lambda_1+\rho) + w_2(\lambda_2+\rho) - \mu - 2\rho - j_1\alpha_1 - j_2\alpha_2 - j_3(\alpha_1+\alpha_2) = 0. \tag{3.2}$$

Proof. Applying the generalized binomial theorem to $(e^{\rho}\prod_{\alpha\in\triangle_+}(1-e^{-\alpha}))^{-2}$ in $F^{\mu}_{\lambda_1,\lambda_2}$, we have a power series expansion

$$\begin{aligned} F^{\mu}_{\lambda_1,\lambda_2} &= \sum_{w_1,w_2\in W}\sum_{(j_1,j_2,j_3)} \varepsilon(w_1)\varepsilon(w_2)(-1)^{j_1+j_2+j_3}\binom{-2}{j_1}\binom{-2}{j_2}\binom{-2}{j_3} \\ &\qquad\qquad \cdot e^{w_1(\lambda_1+\rho)+w_2(\lambda_2+\rho)-\mu-j_1\alpha_1-j_2\alpha_2-j_3(\alpha_1+\alpha_2)-2\rho} \\ &= \sum_{w_1,w_2\in W}\sum_{(j_1,j_2,j_3)} \varepsilon(w_1)\varepsilon(w_2)\binom{j_1+1}{1}\binom{j_2+1}{1}\binom{j_3+1}{1} \\ &\qquad\qquad \cdot e^{w_1(\lambda_1+\rho)+w_2(\lambda_2+\rho)-\mu-j_1\alpha_1-j_2\alpha_2-j_3(\alpha_1+\alpha_2)-2\rho} \\ &= \sum_{w_1,w_2\in W}\sum_{(j_1,j_2,j_3)} \varepsilon(w_1)\varepsilon(w_2)(j_1+1)(j_2+1)(j_3+1) \\ &\qquad\qquad \cdot e^{w_1(\lambda_1+\rho)+w_2(\lambda_2+\rho)-\mu-j_1\alpha_1-j_2\alpha_2-j_3(\alpha_1+\alpha_2)-2\rho}. \end{aligned}$$

Hence our claim follows from Proposition 3.1. □

§ 3.2. Formulas for the Rimann-Roch number and the volume

In the following, we assume that $\lambda_1, \lambda_2 \in P_+$ and $\mu \in P$ satisfy the following three conditions.

(A0) $\mu \in \text{conv.}\{w_1\lambda_1 + w_2\lambda_2 | w_1, w_2 \in W\}$,

(A1) For any $w_1, w_2 \in W$,

$$\begin{aligned} &\langle w_1\lambda_1 + w_2\lambda_2 - \mu, \Lambda_1\rangle \neq 0, \\ &\langle w_1\lambda_1 + w_2\lambda_2 - \mu, \Lambda_2\rangle \neq 0, \\ &\langle w_1\lambda_1 + w_2\lambda_2 - \mu, \Lambda_1\rangle \neq \langle w_1\lambda_1 + w_2\lambda_2 - \mu, \Lambda_2\rangle, \end{aligned}$$

(A2) $\lambda_1, \lambda_2, \mu \in Q$,

where $\text{conv.}\{w_1\lambda_1+w_2\lambda_2|w_1, w_2 \in W\}$ denotes the convex full of $\{w_1\lambda_1+w_2\lambda_2|w_1, w_2 \in W\}$.

Remark 3. We mention the relation between the conditions (a0)–(a2) and the conditions (A0)–(A2). By the convexity theorem of Hamiltonian torus actions on symplectic manifolds (see, e.g., [1], [6] and [7]), (A0) implies (a0), and (A1) implies the former part of (a1). Moreover, for $G = SU(3)$, the former part of (a1) implies the latter part of (a1). Besides, by Remark 1 (2), (A2) implies (a2). As we noted in Remark 2 (3), as far as the symplectic volume $\text{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_{\mu}T)$ is concerned, we may assume (A2) without loss of generality.

Under the conditions above, let us concretely compute $E(\lambda_1, \lambda_2, \mu; w_1, w_2)$ in Proposition 3.2. The condition (3.2) in Proposition 3.2 means that $(j_1, j_2, j_3) \in \mathbb{Z}^3$ satisfies

$$\begin{cases} j_1 = \langle w_1(\lambda_1 + \rho) + w_2(\lambda_2 + \rho) - \mu - 2\rho,\ \Lambda_1 \rangle - j_3 \geq 0, \\ j_2 = \langle w_1(\lambda_1 + \rho) + w_2(\lambda_2 + \rho) - \mu - 2\rho,\ \Lambda_2 \rangle - j_3 \geq 0, \\ j_3 \geq 0. \end{cases}$$

We write

$$A = \langle w_1\lambda_1 + w_2\lambda_2 - \mu,\ \Lambda_1 \rangle\ ,\ B = \langle w_1\lambda_1 + w_2\lambda_2 - \mu,\ \Lambda_2 \rangle,$$
$$C = \langle w_1\rho + w_2\rho,\ \Lambda_1 \rangle\ ,\ D = \langle w_1\rho + w_2\rho,\ \Lambda_2 \rangle$$

for brevity. Note that $\langle \rho, \Lambda_1 \rangle = \langle \rho, \Lambda_2 \rangle = 1$. Then the condition (3.2) means that $(j_1, j_2, j_3) \in \mathbb{Z}^3$ satisfies

$$\begin{cases} j_1 = A + C - 2 - j_3 \geq 0, \\ j_2 = B + D - 2 - j_3 \geq 0, \\ j_3 \geq 0. \end{cases}$$

Therefore we consider the following cases. Recall that

$$\bar{\gamma_1} = \mathbb{R}_{\geq 0}\alpha_1 + \mathbb{R}_{\geq 0}(\alpha_1 + \alpha_2)\ ,\ \bar{\gamma_2} = \mathbb{R}_{\geq 0}(\alpha_1 + \alpha_2) + \mathbb{R}_{\geq 0}\alpha_2.$$

Case 1. Suppose that $A + C - 2 \geq 0$ and $B + D - 2 \geq 0$.

(1-1) If $A + C \geq B + D$, that is, $w_1(\lambda_1 + \rho) + w_2(\lambda_2 + \rho) - \mu - 2\rho \in \bar{\gamma_1}$, then we have

$$\begin{aligned} &E(\lambda_1, \lambda_2, \mu; w_1, w_2) \\ &= \sum_{j_3=0}^{B+D-2} (A + C - 1 - j_3)(B + D - 1 - j_3)(j_3 + 1) \\ &= \frac{1}{12}(B + D - 1)(B + D)(B + D + 1)(2(A + C) - (B + D)). \end{aligned}$$

(1-2) If $A + C \leq B + D$, that is, $w_1(\lambda_1 + \rho) + w_2(\lambda_2 + \rho) - \mu - 2\rho \in \bar{\gamma_2}$, then we have

$$\begin{aligned} &E(\lambda_1, \lambda_2, \mu; w_1, w_2) \\ &= \sum_{j_3=0}^{A+C-2} (A + C - 1 - j_3)(B + D - 1 - j_3)(j_3 + 1) \\ &= \frac{1}{12}(A + C - 1)(A + C)(A + C + 1)(-(A + C) + 2(B + D)). \end{aligned}$$

Case 2. Suppose that $A+C-2<0$ or $B+D-2<0$.
We have $E(\lambda_1,\lambda_2,\mu;w_1,w_2)=0$.

Combining all the results above, we obtain the following explicit formula for the Rimann-Roch number $\dim_{\mathbb{C}}\chi((\mathcal{O}_{\lambda_1}\times\mathcal{O}_{\lambda_2})//_\mu T,(L_{\lambda_1}\boxtimes L_{\lambda_2})//_\mu T)$.

Proposition 3.3. *Let $\lambda_1,\lambda_2\in P_+$ and $\mu\in P$ satisfy the assumptions* (A0), (A1) *and* (A2). *Then we have*

$$\dim_{\mathbb{C}}\chi((\mathcal{O}_{\lambda_1}\times\mathcal{O}_{\lambda_2})//_\mu T,(L_{\lambda_1}\boxtimes L_{\lambda_2})//_\mu T)=\sum_{w_1,w_2\in W}\varepsilon(w_1)\varepsilon(w_2)E(\lambda_1,\lambda_2,\mu;w_1,w_2),$$

where $E(\lambda_1,\lambda_2,\mu;w_1,w_2)$ is given as follows. We write

$$A=\langle w_1\lambda_1+w_2\lambda_2-\mu,\Lambda_1\rangle\ ,\ B=\langle w_1\lambda_1+w_2\lambda_2-\mu,\Lambda_2\rangle,$$
$$C=\langle w_1\rho+w_2\rho,\Lambda_1\rangle\ ,\ D=\langle w_1\rho+w_2\rho,\Lambda_2\rangle$$

for brevity.

(1) *If $w_1(\lambda_1+\rho)+w_2(\lambda_2+\rho)-\mu-2\rho\in\bar{\gamma_1}$,*

$$\begin{aligned}&E(\lambda_1,\lambda_2,\mu;w_1,w_2)\\&=\frac{1}{12}(B+D-1)(B+D)(B+D+1)(2(A+C)-(B+D)).\end{aligned}$$

(2) *If $w_1(\lambda_1+\rho)+w_2(\lambda_2+\rho)-\mu-2\rho\in\bar{\gamma_2}$,*

$$\begin{aligned}&E(\lambda_1,\lambda_2,\mu;w_1,w_2)\\&=\frac{1}{12}(A+C-1)(A+C)(A+C+1)(-(A+C)+2(B+D)).\end{aligned}$$

(3) *Otherwise,*

$$E(\lambda_1,\lambda_2,\mu;w_1,w_2)=0.$$

In the following, we assume that both of two coadjoint orbits $\mathcal{O}_{\lambda_1}$ and $\mathcal{O}_{\lambda_2}$ are flag manifolds of $SU(3)$. Namely, we assume $\lambda_1,\lambda_2\in P_{++}$. Combining Propositions 2.3 and 3.3, we obtain the following formula for the volume $\mathrm{vol}((\mathcal{O}_{\lambda_1}\times\mathcal{O}_{\lambda_2})//_\mu T)$.

Theorem 3.4. *Let $\lambda_1,\lambda_2\in P_{++}$ and $\mu\in P$ satisfy the assumptions* (A0), (A1) *and* (A2). *Then we have*

$$\mathrm{vol}((\mathcal{O}_{\lambda_1}\times\mathcal{O}_{\lambda_2})//_\mu T)=\sum_{w_1,w_2\in W}\varepsilon(w_1)\varepsilon(w_2)F(\lambda_1,\lambda_2,\mu;w_1,w_2),\tag{3.3}$$

where $F(\lambda_1,\lambda_2,\mu;w_1,w_2)$ is given as follows.

(1) *If* $w_1\lambda_1 + w_2\lambda_2 - \mu \in \gamma_1$,

$$F(\lambda_1, \lambda_2, \mu; w_1, w_2) = \frac{1}{12}\langle w_1\lambda_1 + w_2\lambda_2 - \mu, \Lambda_2\rangle^3 \langle w_1\lambda_1 + w_2\lambda_2 - \mu, 2\Lambda_1 - \Lambda_2\rangle.$$

(2) *If* $w_1\lambda_1 + w_2\lambda_2 - \mu \in \gamma_2$,

$$F(\lambda_1, \lambda_2, \mu; w_1, w_2) = \frac{1}{12}\langle w_1\lambda_1 + w_2\lambda_2 - \mu, \Lambda_1\rangle^3 \langle w_1\lambda_1 + w_2\lambda_2 - \mu, -\Lambda_1 + 2\Lambda_2\rangle.$$

(3) *Otherwise,*

$$F(\lambda_1, \lambda_2, \mu; w_1, w_2) = 0.$$

Proof. We first note that for $\lambda_1, \lambda_2 \in P_{++}$ and $\mu \in P$ satisfying the assumptions (A0), (A1) and (A2), we have

$$d = \dim_{\mathbb{C}}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_{\mu}T) = 4.$$

According to Propositions 2.3 and 3.3, we must compute $E(k\lambda_1, k\lambda_2, k\mu; w_1, w_2)$. In this case, the condition in (1) of Proposition 3.3 means that $kB + D - 2 \geq 0$ and

$$\begin{aligned} 0 &\leq \langle w_1(k\lambda_1 + \rho) + w_2(k\lambda_2 + \rho) - 2\rho - k\mu, \Lambda_1 - \Lambda_2\rangle \\ &= k(A - B) + (C - D). \end{aligned}$$

Let us take k large enough. By the assumption (A1), these inequalities above means that $A > B > 0$, that is, $w_1\lambda_1 + w_2\lambda_2 - \mu \in \gamma_1$. Then we have

$$\begin{aligned} &E(k\lambda_1, k\lambda_2, k\mu; w_1, w_2) \\ &= \frac{1}{12}(kB + D - 1)(kB + D)(kB + D + 1)(2(kA + C) - (kB + D)) \\ &= \frac{k^4}{12}B^3(2A - B) + (\text{lower terms of } k). \end{aligned}$$

Similarly, when $k \gg 0$, the condition in (2) of Proposition 3.3 means that $B > A > 0$, that is, $w_1\lambda_1 + w_2\lambda_2 - \mu \in \gamma_2$. Then we have

$$\begin{aligned} &E(k\lambda_1, k\lambda_2, k\mu; w_1, w_2) \\ &= \frac{1}{12}(kA + C - 1)(kA + C)(kA + C + 1)(-(kA + C) + 2(kB + D)) \\ &= \frac{k^4}{12}A^3(-A + 2B) + (\text{lower terms of } k). \end{aligned}$$

Finally, when $k \gg 0$, the condition in (3) of Proposition 3.3 means that $A < 0$ or $B < 0$. Then we have

$$E(k\lambda_1, k\lambda_2, k\mu; w_1, w_2) = 0.$$

Combining all the results above and Proposition 2.3, we obtain our claim. $\square$

Remark 4. (1) In the case where λ_1 or λ_2 is in $P_+ - P_{++}$, although $F(\lambda_1, \lambda_2, \mu; w_1, w_2) \neq 0$ in general, the right hand side of the formula (3.3) is always equal to 0. We need another consideration for this. We will discuss it in the forthcoming paper [19].

(2) In general, for an element μ in the weight lattice P, there exist $w \in W$ and $\mu' \in P_+$ such that $\mu = w\mu'$. Since $[V_{\lambda_1} \otimes V_{\lambda_2} : W_\mu] = [V_{\lambda_1} \otimes V_{\lambda_2} : W_{\mu'}]$, we have $\mathrm{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_\mu T) = \mathrm{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_{\mu'} T)$. But the computation of the right hand side of the formula (3.3) for $\mu' \in P_+$ becomes more simpler than that for $\mu \in P$.

Now, let $\lambda_1, \lambda_2 \in (P \otimes \mathbb{Q}) \cap \mathfrak{t}^*_{++}$ and $\mu \in P \otimes \mathbb{Q}$ satisfy the conditions (A0) and (A1). Then there exists $n \in \mathbb{Z}_{>0}$ such that $n\lambda_1, n\lambda_2 \in P_{++}$ and $n\mu \in P$. Therefore it follows from Remark 2 (2) and Proposition 2.3 that the volume $\mathrm{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_\mu T)$ is given by the right hand side of the formula (3.3). Furthermore, by continuity of the symplectic volume of symplectic quotients (see, e.g., [3] and [9]), we obtain the following.

Corollary 3.5. *Suppose that* $\lambda_1, \lambda_2 \in \mathfrak{t}^*_{++}$ *and* $\mu \in \mathfrak{t}^*$ *satisfy the conditions* (A0) *and* (A1) *in Section 3.2. Then the volume* $\mathrm{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_\mu T)$ *is given by the right hand side of the formula* (3.3).

§ 4. An example

As a simple example of Theorem 3.4 or Corollary 3.5, let us consider the case where μ is sufficiently close to $\lambda_1 + \lambda_2$. In this case, we have $F(\lambda_1, \lambda_2, \mu; w_1, w_2) = 0$ unless the case $w_1 = w_2 = e$. Hence we have

$$\begin{aligned}
&\mathrm{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_\mu T) = F(\lambda_1, \lambda_2, \mu; w_1 = e, w_2 = e) \\
&= \begin{cases} \dfrac{1}{12}\langle \lambda_1 + \lambda_2 - \mu, \Lambda_2\rangle^3 \langle \lambda_1 + \lambda_2 - \mu, 2\Lambda_1 - \Lambda_2\rangle & (\text{if } \lambda_1 + \lambda_2 - \mu \in \gamma_1), \\ \dfrac{1}{12}\langle \lambda_1 + \lambda_2 - \mu, \Lambda_1\rangle^3 \langle \lambda_1 + \lambda_2 - \mu, -\Lambda_1 + 2\Lambda_2\rangle & (\text{if } \lambda_1 + \lambda_2 - \mu \in \gamma_2). \end{cases}
\end{aligned}$$

Now if we write $\lambda_1 = p\alpha_1 + q\alpha_2, \lambda_2 = r\alpha_1 + s\alpha_2 \in P_{++}$ and $\mu = x\alpha_1 + y\alpha_2 \in P$, we can express the volume $\mathrm{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_\mu T)$ as a polynomial of p, q, r, s, x, y as follows.

$$\begin{aligned}
&\mathrm{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_\mu T) \\
&= \begin{cases} \dfrac{1}{12}(q + s - y)^3 (2p - q + 2r - s - 2x + y) & (\text{if } \lambda_1 + \lambda_2 - \mu \in \gamma_1), \\ \dfrac{1}{12}(p + r - x)^3 (-p + 2q - r + 2s + x - 2y) & (\text{if } \lambda_1 + \lambda_2 - \mu \in \gamma_2). \end{cases}
\end{aligned}$$

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