A determinant formula for the quotient of the relative class numbers of imaginary abelian number fields of relative degree 2

By

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Abstract

We give a determinant formula for the quotient of the relative class numbers of imaginary abelian number fields of relative degree 2, which is a generalization of Endō's formulas for the mth cyclotomic field, m an odd integer, and its quadratic extension.

§1. Introduction

Let p be an odd prime. For an integer u let \( R_p(u) \) and \( R'_p(u) \) be the integers such that

\[
R_p(u) \equiv u \pmod{p}, \quad 0 \leq R_p(u) < p
\]

and

\[
R'_p(u) \equiv u \pmod{p}, \quad -\frac{p}{2} < R'_p(u) < \frac{p}{2},
\]

respectively. For an integer u coprime to p, let \( u^{-1} \) be an integer with \( uu^{-1} \equiv 1 \pmod{p} \). We have already obtained a lot of determinant formulas for the pth cyclotomic

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field \( \mathbb{Q}(\zeta_{p}) \), \( \zeta_{p} \) a primitive \( p \)th root of unity. For example,

\[
\det \left( R_{p}(uv^{-1}) \right)_{1 \leq u, v \leq (p-1)/2} = (-1)^{\frac{p-3}{2}} p^{\frac{p-3}{2}} h_{p}^{*},
\]

(1.1)

\[
\det \left( R'_{p}(uv^{-1}) \right)_{1 \leq u, v \leq (p-1)/2} = \begin{cases} 
2^{\frac{p-1}{\ord_{p}(2)}-1} p^{\frac{p-3}{2}} h_{p}^{*} & \text{if } 2 \mid \ord_{p}(2), \\
0 & \text{otherwise},
\end{cases}
\]

(1.2)

\[
\det \left( (-1)^{R_{p}(uv^{-1})} \right)_{1 \leq u, v \leq (p-1)/2} = (-1)^{\frac{p-3}{2}} 2^{\frac{p-2}{2}} \prod_{\chi \in X^{-}} (1 - \chi(2)) \cdot h_{p}^{*},
\]

(1.3)

where \( \ord_{p}(2) \) is the order of 2 modulo \( p \), \( X^{-} \) the set of odd characters of the field \( \mathbb{Q}(\zeta_{p}) \) and \( h_{p}^{*} \) the relative class number of the field \( \mathbb{Q}(\zeta_{p}) \).

The determinant in the formula (1.1) is called Maillet determinant (See [1]) and the one in (1.3) could be called Dem'janenko determinant. The formulas (1.1) and (1.3) are special ones of the generalized formulas in [3], [14], [17] and [18]; and the formula (1.2) a special one of [3], [15] and [17]. Funakura [7] gave, up to sign, a generalized formula of (1.3) for the \( m \)th cyclotomic field, \( m \) an odd integer.

As a corresponding formula to (1.3), we shall obtain by the formula (2.5) in Corollary 2.3

\[
\det \left( (-1)^{R'_{p}(uv^{-1})} \right)_{1 \leq u, v \leq (p-1)/2} = \begin{cases} 
-2^{\frac{p-2}{2}} \frac{h_{4p}^{*}}{h_{p}^{*}} & \text{if } p \equiv 3 \pmod{4}, \\
0 & \text{otherwise},
\end{cases}
\]

(1.4)

where \( h_{4p}^{*} \) is the relative class number of the \( 4p \)th cyclotomic field. Kanemitsu and Kuzumaki [15, Corollary 4] have already obtained the formula (1.4), up to sign, under some condition.

The aim of this paper is to give a determinant formula for the quotient of the relative class numbers of imaginary abelian number fields with relative degree 2, which is a generalization not only of the formula (1.4) but also of Endō's formulas in [2] and [4]. As does the formula in [14], our determinant formula has a parameter \( b \). By taking

\[
b = fm + 1 \quad \text{(} fm : \text{the conductor of "the larger field")}
\]

we obtain the formula in [2, Theorem 1 for \( k = 1 \)]; by taking \( b = 2 \), the one in [4, Theorem 1]; by taking \( b = g \) (\( g : \) a primitive root modulo \( p \)), the one in Corollary 2.5, in which the elements of determinant are coefficients of some digit expression as in [11].

Our result would be an answer to the inquiry of Kanemitsu and Kuzumaki [15, p.285] about the relation between Tsumura's and the author's generalized Dem'janenko determinants and (generalization of) Endō's determinants \( S_{p}, T_{p} \) and \( U_{p} \) in [5]. The determinants \( S_{p}, T_{p} \) and \( U_{p} \) are special ones of the left-hand sides of (2.5), (2.6) and (2.7) for the \( p \)th cyclotomic field \( \mathbb{Q}(\zeta_{p}) \), respectively.
§ 2. Results

Let $m$ be an integer with $m \geq 3$ and $m \not\equiv 2 \pmod{4}$. For an integer $u$ let $R_m(u)$ and $R'_m(u)$ be the integers such that

$$R_m(u) \equiv u \pmod{m}, \quad 0 \leq R_m(u) < m$$

and

$$R'_m(u) \equiv u \pmod{m}, \quad -\frac{m}{2} \leq R'_m(u) < \frac{m}{2},$$

respectively. For an integer $u$ coprime to $m$, let $u^{-1}$ be an integer such that $uu^{-1} \equiv 1 \pmod{m}$.

Let $K$ be an imaginary abelian number field of degree $2n = [K : \mathbb{Q}]$ and with conductor $m$. Let $h_K$, $Q_K$ and $w_K$ be the relative class number of $K$, the unit index of $K$ and the number of roots of unity in $K$, respectively.

Let $G_m$ be the multiplicative group $(\mathbb{Z}/m\mathbb{Z})^\times$, $\mathbb{Z}$ the ring of integers, and $H$ the subgroup of $G_m$ corresponding to $K$. For an integer $t$ coprime to $m$ let $\bar{t} = t + m\mathbb{Z} \in G_m$.

Since $H$ does not contain $-1$, we can take classes $C_1, C_2, \ldots, C_n$ of $G_m/H$ satisfying

$$G_m/H = \{C_1, -C_1, C_2, -C_2, \ldots, C_n, -C_n\}.$$
where \([x]\) means the integral part of a rational number \(x\). When \(H = \{1\}\), we use \(T_{c,\psi}^{(b)}\) instead of \(T_{c,\psi}^{(b)}\) for \(C = \overline{c}H = \{\overline{c}\}\).

Let \(h_{\overline{K}}^{*}, Q_{\overline{K}}, w_{\overline{K}}\) and \(\overline{c}_{\chi}(b)\) be defined for \(\widetilde{K}\) as above. Note that we define \(\overline{c}_{\chi}(b)\) by using \(fm\) instead of \(m\).

**Theorem 2.1.** Let \(K\) be an imaginary abelian number field of degree \(2n\) and with conductor \(m\). Let \(\widetilde{K}, \psi\) and \(f\) be as above. Take an integer \(b\) with \(b \geq 2\) and \(fm \not| b\). Then we have

\[
\text{det} \left( T_{C_{i}C_{j}^{-1}, \psi}^{(b)} \right)_{C_{i}, C_{j} \in \mathcal{R}} = \prod_{\chi \in X^{*}} \overline{c}_{\chi\psi}(b) \cdot \prod_{\chi \in X^{*}} \frac{1}{2} B_{1, \chi\psi} 
\]

where \(X^{*}\) is \(X^{+}\) or \(X^{-}\) according as \(\psi(-1) = -1\) or \(\psi(-1) = +1\).

When \(b = fm + 1\), we have the following formula, which is obtained by taking \(k = 1\) in [2, Theorem 1]:

**Corollary 2.2** (cf. [2, Theorem 1]). Let \(K\) be an imaginary abelian number field of degree \(2n\) and with conductor \(m\). Let \(\widetilde{K}, \psi\) and \(f\) be as above. Then we have

\[
\text{det} \left( \sum_{\overline{t} \in H} \sum_{d=0}^{f-1} \psi(R_{rn}(c_{i}c_{j}^{-1}t) + dm) \left( \frac{R_{m}(c_{i}c_{j}^{-1}t) + dm}{fm} - \frac{1}{2} \right) \right)_{C_{i}, C_{j} \in \mathcal{R}} 
\]

where \(C_{i} = \overline{c_{i}}H, C_{j} = \overline{c_{j}}H\).

Let \(\psi_{0}\) be the principal character modulo \(f\) and let

\[
T_{C,\psi_{0}}^{(b)} = \sum_{\overline{t} \in H} \sum_{d=0}^{f-1} \psi_{0}(R_{m}(ct) + dm) \left( b \cdot \frac{R_{m}(ct) + dm}{fm} - \frac{b - 1}{2} \right)
\]

for \(C = \overline{c}H\).
We remark that we have already obtained

\begin{equation}
\det \left( T_{C_iC_j^{-1},\psi_0}^{(b)} \right)_{C_i, C_j \in \mathcal{R}} = \prod_{\chi \in \tilde{X}^-} \tilde{c}_\chi(b) \cdot \prod_{\chi \in X^-} \frac{1}{2} B_{1,\chi}
\end{equation}

(See [12, (24)]) and

\begin{equation}
\det \left( T_{C_iC_j^{-1},\psi_0}^{(b)} \right)_{C_i, C_j \in \mathcal{R}} \det \left( T_{C_iC_j^{-1},\psi}^{(b)} \right)_{C_i, C_j \in \mathcal{R}} = \prod_{\chi \in \tilde{X}^-} \tilde{c}_\chi(b) \cdot \prod_{\chi \in X^-} \frac{1}{2} B_{1,\chi}
\end{equation}

where $\tilde{X}^-$ is the set of odd characters of $\tilde{K}$. Kučera [16] gave a determinant formula generalizing the formula (2.4) but he did not refer to the formula (2.3).

Endô [6, Theorem] gave the formula (2.4), up to sign, in the case where $K = \mathbb{Q}(\zeta_{p^\mu})$, a $p$-power-th cyclotomic field, $\tilde{K} = \mathbb{Q}(\zeta_{p^\mu}, \zeta)$, $(f, p) = 1$ and $b = fp^\mu + 1$.

We can not get our result (2.1) "directly" from (2.3) and (2.4), because

\begin{equation}
\det \left( T_{C_iC_j^{-1},\psi_0}^{(b)} \right)_{C_i, C_j \in \mathcal{R}}
\end{equation}

is equal to zero under some conditions.

In Theorem 2.1, taking $b = fm + 1$; $\psi = \chi_4$, $\psi = \chi_4 \psi_8$ and $\psi = \psi_8$, we have generalizations of Endô’s formulas for the $m$th cyclotomic field $\mathbb{Q}(\zeta_m)$, $m$ an odd integer, in [2, Theorem 2], where $\chi_4$ is the odd character with conductor 4 and $\psi_8$ the even character with conductor 8:

**Corollary 2.3 (cf. [2, Theorem 2]).** Let $K$ be an imaginary abelian number field of degree $2n$ and with odd conductor $m$. Then we have

\begin{equation}
\det \left( \sum_{\tilde{t} \in H} (-1)^{R_{n\mathrm{z}}'(c_{i}c_{j}^{-1}\tilde{t})} \right)_{C_i, C_j \in \mathcal{R}}
\end{equation}

\begin{equation}
= \chi_4(m)^n \prod_{\chi \in X^+} \chi(2) \cdot \prod_{\chi \in X^+ \ l|\ m} (1 - \chi \chi_4(l)) \cdot \frac{Q_{K} w_{K}}{Q_{K(\sqrt{-1})} w_{K(\sqrt{-1})}} \cdot \frac{h_{K(\sqrt{-1})}^*}{h_{K}^*},
\end{equation}

\begin{equation}
\det \left( \sum_{\tilde{t} \in H} T_{m}^\prime(c_{i}c_{j}^{-1}\tilde{t}) \right)_{C_i, C_j \in \mathcal{R}}
\end{equation}

\begin{equation}
= \chi_4 \psi_8(m)^n \prod_{\chi \in X^+} \chi(2) \cdot \prod_{\chi \in X^+ \ l|\ m} (1 - \chi \chi_4 \psi_8(l)) \cdot \frac{Q_{K} w_{K}}{Q_{K(\sqrt{-2})} w_{K(\sqrt{-2})}} \cdot \frac{h_{K(\sqrt{-2})}^*}{h_{K}^*}.
\end{equation}
and

\[
\det \left( \sum_{t \in H} U'_m(c_i c_j^{-1} t) \right)_{C_i, C_j \in \mathcal{R}}
\]

\[
= (-1)^{m+1} \prod_{\chi \in X^+} \chi(2) \cdot \prod_{\chi \in X^-} \prod_{l|m} (1 - \chi \psi_8(l)) \cdot \frac{Q_K w_K}{Q_K(\sqrt{2}) w_K(\sqrt{2})} \cdot \frac{h^*_K}{h^*_K},
\]

where \( C_i = \overline{c_i} H, C_j = \overline{c_j} H \) and for an integer \( c \)

\[
T'_m(c) = \begin{cases} (-1)^{R'_m(c)/2} & \text{if } R'_m(c) \equiv 0 \pmod{2}, \\ 0 & \text{if } R'_m(c) \equiv 1 \pmod{2} \end{cases}
\]

and

\[
U'_m(c) = \begin{cases} 0 & \text{if } R'_m(c) \equiv 0 \pmod{2}, \\ (-1)^{R'_m(c)-1/2} & \text{if } R'_m(c) \equiv 1 \pmod{2}. \end{cases}
\]

Endô’s formula in [2, Theorem 2] are represented, up to sign, by the form of product of first generalized Bernoulli numbers. Endô [5] has already given such determinants in (2.5), (2.6) and (2.7) for the \( p \)th cyclotomic field \( \mathbb{Q}(\zeta_p) \).

As introduced in §1, the formula (1.4) is a special case of (2.5) for the \( p \)th cyclotomic field \( \mathbb{Q}(\zeta_p) \). Here we note that if \( p \equiv 3 \pmod{4} \), then \( \prod_{\chi \in X} \chi(2) = \chi_p(2)^{(p-1)\frac{p-3}{4}} = +1 \), where \( \chi_p \) is a Dirichlet character with conductor \( p \) and of degree \( p-1 \).

In Theorem 2.1 taking \( K = \mathbb{Q}(\zeta_p) \) and \( b = 2 \), we have formulas in [4]:

**Corollary 2.4 ([4, Theorem 1]).** Let \( K \) be the \( p \)th cyclotomic field \( \mathbb{Q}(\zeta_p) \), \( p \) an odd prime. Let \( D \) be a square-free integer such that \( (D,p) = 1 \) and \( D \equiv 1 \pmod{4} \). Let \( \overline{K} \) be the composite of \( K \) and the quadratic field \( \mathbb{Q}(\sqrt{D}) \). Let \( \psi(u) \) be the quadratic character corresponding to the field \( \mathbb{Q}(\sqrt{D}) \). For an integer \( u \) with \( (u,p) = 1 \) put

\[
S_u(\psi) = \sum_{(k,pD) = 1, k \equiv u \pmod{p}} \psi(k).
\]

Then we have: If \( D > 0 \), then

\[
(2.8) \quad \det (S_{uv}(\psi))_{1 \leq u, v \leq (p-1)/2} = \pm \prod_{i=1}^{(p-1)/2} (2 - \chi_p^{2i-1}(\psi(2))) \frac{B_{1, \chi_p^{2i-1}(\psi)}}{Q_K w_K} \cdot \frac{h^*_K}{h^*_K},
\]

where

\[
\frac{Q_K w_K}{Q_K(\sqrt{2}) w_K(\sqrt{2})} \cdot \frac{h^*_K}{h^*_K}.
\]
If $D < 0$, then

$$(2.9) \det (S_{uv} (\psi))_{1 \leq u, v \leq (p-1)/2} = \pm (1 - \psi (p)) \prod_{i=1}^{(p-1)/2} (2 - \chi_p^{2i} \psi (2)) \frac{1}{2} B_1 \chi_p^{2i} \psi$$

$$= \pm (1 - \psi (p)) \frac{2p}{Q_{K} w_{\overline{K}}} \prod_{i=1}^{(p-1)/2} (2 - \psi \chi_p^{2i} \psi) \cdot \frac{h_{\overline{K}}^*}{h_K^*}.$$

In Theorem 2.1 taking $K = \mathbb{Q} (\zeta_p)$ and $b = g$, $g$ a primitive root modulo $p$, we have a formula corresponding to the one in [11, Corollary 2]:

**Corollary 2.5.** Let $K$ be the $p$th cyclotomic field $\mathbb{Q} (\zeta_p)$, $p$ an odd prime. Let $g$ be a primitive root modulo $p$ with $g \geq 2$ and $g \equiv 1 \pmod{4}$. Let $\overline{K}$ be the composite of $K$ and the quadratic field $\mathbb{Q} (-1)$. Then for an integer $i$ we have

$$T_{R_p (g^i \chi_4)} = \sum_{0 \leq k \leq p-2} \chi_4 (R_{4p} (g^k)) \left[ \frac{gR_{4p}(g^k)}{4p} \right]$$

$$+ \sum_{0 \leq k \leq p-2} \chi_4 (R_{4p} (-g^k)) \left[ \frac{gR_{4p}(-g^k)}{4p} \right]$$

and

$$\det (T_{R_p (g^{i-j} \chi_4)})_{0 \leq i, j \leq (p-3)/2} = (-1)^{\frac{p-1}{2}} \frac{1}{4} (1 - \chi_4 (p)) \prod_{\chi \in X^+} (g - \chi (g)) \cdot \frac{h_{\overline{K}}^*}{h_K^*}.$$

**Remark.** With the notation in Corollary 2.5 expand $1/(4p)$ to the basis $1/g$:

$$\frac{1}{4p} = \sum_{k=1}^{\infty} \frac{x(k)}{g^k}, \quad x(k) \in \{0, 1, \ldots, g-1\}.$$

Then we have

$$x(k) = \left[ \frac{gR_{4p}(g^{k-1})}{4p} \right] \quad \text{for } k = 1, 2, \ldots$$

(See [11, Theorem 10]).

**Example 2.6.** We give here an example of Corollary 2.5. Let $K$, $\overline{K}$ and $g$ be
as in Corollary 2.5. We take $p = 7$, $g = 5$. Then $K = \mathbb{Q}(\zeta_7)$, $\overline{K} = \mathbb{Q}(\zeta_{28})$ and

\[
T_{R_7(5^0),\chi_4}^{(5)} = \left[\frac{5 \cdot 1}{28}\right] - \left[\frac{5 \cdot 15}{28}\right] = 0 - 2 = -2,
\]

\[
T_{R_7(5^1),\chi_4}^{(5)} = \left[\frac{5 \cdot 5}{28}\right] - \left[\frac{5 \cdot 19}{28}\right] = 0 - 3 = -3,
\]

\[
T_{R_7(5^2),\chi_4}^{(5)} = \left[\frac{5 \cdot 25}{28}\right] - \left[\frac{5 \cdot 11}{28}\right] = 4 - 1 = 3,
\]

\[
T_{R_7(5^3),\chi_4}^{(5)} = \left[\frac{5 \cdot 13}{28}\right] - \left[\frac{5 \cdot 27}{28}\right] = 2 - 4 = -2,
\]

\[
T_{R_7(5^4),\chi_4}^{(5)} = \left[\frac{5 \cdot 9}{28}\right] - \left[\frac{5 \cdot 23}{28}\right] = 1 - 4 = -3,
\]

\[
T_{R_7(5^5),\chi_4}^{(5)} = \left[\frac{5 \cdot 17}{28}\right] - \left[\frac{5 \cdot 3}{28}\right] = 3 - 0 = 3.
\]

Hence

\[
\det \left( T_{R_7(5^{i-j}),\chi_4}^{(5)} \right)_{0 \leq i,j \leq 2} = \det \begin{pmatrix} -2 & 3 & -3 \\ -3 & -2 & 3 \\ 3 & -3 & -2 \end{pmatrix} = -62.
\]

On the other hand, letting $\zeta_u$ be a primitive $u$th root of unity, we have

\[
(-1)^{\frac{7-1}{2}} \frac{1}{4} (1 - \chi_4(7)) \prod_{\chi \in \chi^+} (5 - \chi(5)) \cdot \frac{h_{\overline{K}}^*}{h_K^*} = -\frac{1}{4} \cdot 2 \cdot (5 - 1)(5 - \zeta_3)(5 - \zeta_3^2) \cdot \frac{h_{\overline{K}}^*}{h_K^*}
\]

\[
= -62 \cdot \frac{h_{\overline{K}}^*}{h_K^*}.
\]

Therefore $h_{\overline{K}}^* = h_K^*$. Actually, we have already known that $h_{\overline{K}}^* = h_K^* = 1$.

\section*{§ 3. Proofs of Theorem and Corollaries}

To prove Theorem 2.1 we need the following lemma originating from [8].

\textbf{Lemma 3.1 ([14, Lemma 1]).} Let $K$ be an imaginary abelian number field with conductor $m$ and $b$ an integer with $b \geq 2$ and $m \not| b$. Then, for an odd character $\chi$ of $K$ and for $C = \overline{c}H$ we have

\[
\frac{1}{2} c_{\chi}(b) B_1,\overline{\chi} = \sum_{C \in \mathcal{R}_K} \overline{\chi}(c) T_C^{(b)}
\]

\[
= \frac{1}{2} \sum_{k=1}^{m} \overline{\chi}(k) \left( \left[ b \cdot \frac{R_m(k)}{m} \right] - \frac{b - 1}{2} \right),
\]
where

\[ T_{C}^{(b)} = \sum_{\overline{t} \in H} \left( b \cdot \frac{R_{m}(ct)}{m} - \frac{b-1}{2} \right). \]

**Proof of Theorem 2.1.** Recall that for a class \( C = \overline{c}H \) in \( G_{m}/H \) we define

\[ T_{C, \psi}^{(b)} = \sum_{\overline{t} \in H} \sum_{d=0}^{f-1} \psi(R_{m}(ct)+dm) \left( b \cdot \frac{R_{m}(ct)+dm}{fm} - \frac{b-1}{2} \right). \]

Let \( \psi(-1) = (-1)^{k'} \), \( k' = 0 \) or \( 1 \). Then we have

\[ T_{-C, \psi}^{(b)} = (-1)^{k'+1} T_{C, \psi}^{(b)}. \]

First we consider the case where \( \psi(-1) = -1 \). Since \( T_{-C, \psi}^{(b)} = T_{C, \psi}^{(b)} \), it follows from the group determinant (cf. for example, [19, p.71]) that

\[ \det(T_{C, C_{j}^{-1}, \psi}^{(b)})_{C_{i}, C_{j} \in R} = \prod_{\chi \in \chi_{X^{+}}} \sum_{\overline{c}H \in R} \chi(c)T_{\overline{c}H, \psi}^{(b)}. \]

Since

\[ \sum_{\overline{c}H \in R} \chi(c)T_{\overline{c}H, \psi}^{(b)} = \frac{1}{2} \sum_{k=1}^{fm} \chi\psi(k) \left( \frac{bk}{fm} - \frac{b-1}{2} \right), \]

noting that the assumption of \( b \), we obtain by Lemma 3.1

\[ \sum_{\overline{c}H \in R} \chi(c)T_{\overline{c}H, \psi}^{(b)} = \frac{1}{2} \tilde{c}_{\chi\psi}(b)B_{1, \chi\psi}. \]

Therefore we have

\[ \det(T_{C_{i}, C_{j}^{-1}, \psi}^{(b)})_{C_{i}, C_{j} \in R} = \prod_{\chi \in \chi_{X^{+}}} \tilde{c}_{\chi\psi}(b) \cdot \prod_{\chi \in \chi_{X^{+}}} \frac{1}{2} B_{1, \chi\psi} \]

\[ = (-1)^{n} \prod_{\chi \in \chi_{X^{+}}} \tilde{c}_{\chi\psi}(b) \cdot \frac{Q_{K} w_{K}}{Q_{\overline{K}} w_{\overline{K}}} \cdot \frac{h_{\overline{K}}^{*}}{h_{K}^{*}}. \]

Next we consider the case where \( \psi(-1) = +1 \). Taking some odd character \( \chi_{1} \) of
K, we have, letting \( C_i = \overline{c}_i H, C_j = \overline{c}_j H \),

\[
\det(T^{(b)}_{C_iC_j^{-1},\psi})_{C_i,C_j \in \mathcal{R}} = \det(\chi_1(c_i^{-1}c_j)T^{(b)}_{C_iC_j^{-1},\psi})_{C_i,C_j \in \mathcal{R}}
\]

\[
= \prod_{\chi \in \mathcal{R}^+} \sum_{\overline{a}H \in \mathcal{R}} \chi_1(a)T^{(b)}_{\overline{a}H,\psi}
\]

\[
= \prod_{\chi \in \mathcal{R}^-} \sum_{\overline{a}H \in \mathcal{R}} \chi(a)T^{(b)}_{\overline{a}H,\psi}
\]

\[
= \prod_{\chi \in \mathcal{R}^-} \overline{c}_{\chi \psi}(b) \cdot \prod_{\chi \in \mathcal{R}^+} \frac{1}{2}B_{1,\chi}\psi
\]

\[
= (-1)^n \prod_{\chi \in \mathcal{R}^+} \overline{c}_{\chi \psi}(b) \cdot \frac{Q_K w_K}{Q_{\overline{K}} w_{\overline{K}}} \cdot \frac{h^*_K}{h^*_\overline{K}}.
\]

This completes the proof. \( \square \)

Corollary 2.2 is easily proved by Theorem 2.1, because when \( b = fm + 1 \), it holds that

\[
\overline{c}_{\chi \psi}(b) = fm \prod_{l|m} (1 - \overline{\chi}_\psi(l)) = fm \prod_{l|m} (1 - \overline{\chi}_\psi(l)).
\]

To prove Corollary 2.3 we need the following two lemmas.

**Lemma 3.2 (cf. [13, Lemma 2]).** For an integer \( c \) with \( (c, m) = 1 \), we define the permutation \( \sigma_c \) on \( \mathcal{R} = \{C_1, C_2, \ldots, C_n\} \) up to "±" by

\[
\overline{c}C_i = \pm C_{\sigma_c(i)} \quad \text{for} \quad i = 1, 2, \ldots, n.
\]

Then we have

(3.1)

\[
\text{sgn} \sigma_c = \prod_{\chi \in \mathcal{R}^+} \chi(c),
\]

where \( \text{sgn} \sigma_c = +1 \) or \(-1\) according as \( \sigma_c \) is even or odd.

In [13, Lemma 2] we have shown that

\[
(-1)^{N_c} \text{sgn} \sigma_c = \prod_{\chi \in \mathcal{R}^-} \chi(c),
\]

where \( N_c \) is the number of the "minus cosets" \(-C_{\sigma_c(i)}\) in the set \( \{\overline{c}C_i = \pm C_{\sigma_c(i)}; i = 1, 2, \ldots, n\} \). We can prove the identity (3.1) by taking \( \prod_{\chi \in \mathcal{R}^+} \chi(c) \) instead of \( \prod_{\chi \in \mathcal{R}^-} \chi(c) \) in the proof of [13, Lemma 2]. (The right hand \( \delta_{ij}\zeta_{g_i} \) of the equation on page 22, line 13 from the top of [13] should be \( \zeta_{g_i}^{\delta_{ij}} \).)
Lemma 3.3 ([2, Proof of Theorem 2]). Assume that \( m \) is an odd integer. For an integer \( c \) with \((c, m) = 1\) let \( c' \) an integer such that \((c', m) = 1\) and \(2c' \equiv c \pmod{m}\). Then, for any integers \( c_i, c_j \) coprime to \( m \), we have

\[
\sum_{d=0}^{3} \chi_4(R_m(c_i c_j^{-1}) + dm) \left( \frac{R_m(c_i c_j^{-1}) + dm}{4m} - \frac{1}{2} \right) = -\chi_4(m) \cdot \frac{1}{2} \cdot (-1)^{R_m(c_i c_j^{-1})},
\]

\[
\sum_{d=0}^{7} \chi_4 \psi_8(R_m(c_i c_j^{-1}) + dm) \left( \frac{R_m(c_i c_j^{-1}) + dm}{8m} - \frac{1}{2} \right) = -\chi_4 \psi_8(m) T'_m(c_i c_j^{-1})
\]

and

\[
\sum_{d=0}^{7} \psi_8(R_m(c_i c_j^{-1}) + dm) \left( \frac{R_m(c_i c_j^{-1}) + dm}{8m} - \frac{1}{2} \right) = (-1) \frac{m-1}{2} \psi_8(m) U'_m(c_i c_j^{-1}).
\]

Proof of Corollary 2.3. By Lemmas 3.3 and 3.2 and by Corollary 2.2 we have

\[
(-1)^n \chi_4(m)^n \cdot \frac{1}{2^n} \cdot \det \left( \sum_{\overline{t} \in H} \left( \frac{R_m(c_i c_j^{-1}t) + dm}{4m} - \frac{1}{2} \right) \right)_{C_i, C_j \in \mathcal{R}}
\]

\[
= \det \left( \sum_{\overline{t} \in H} \sum_{d=0}^{3} \chi_4(R_m(c_i c_j^{-1}t) + dm) \left( \frac{R_m(c_i c_j^{-1}t) + dm}{4m} - \frac{1}{2} \right) \right)_{C_i, C_j \in \mathcal{R}}
\]

\[
= \prod_{\chi \in X^+} \chi(2) \cdot \det \left( \sum_{\overline{t} \in H} \sum_{d=0}^{3} \chi_4(R_m(c_i c_j^{-1}t) + dm) \left( \frac{R_m(c_i c_j^{-1}t) + dm}{4m} - \frac{1}{2} \right) \right)_{C_i, C_j \in \mathcal{R}}
\]

\[
= \prod_{\chi \in X^+} \chi(2) \cdot (-1)^n \prod_{\chi \in X^+} \prod_{l|\chi_4(l)} \frac{Q_K w_K}{Q_K(-1) w_K(-1)} \cdot \frac{h^*_K(-1)}{h^*_K},
\]

where \( C_i = \overline{c_i} H, C_j = \overline{c_j} H \). Hence we have obtained the first formula (2.5).

In the same way as above we can prove the second and third formulas (2.6) and (2.7).

\[\square\]

Proof of Corollary 2.4. If \( \psi(-1) = -1 \), then we have \( T_{C_i, \psi}^{(b)} = -T_{C_i, \psi}^{(b)} \) and

\[
\det \left( T_{C_i, C_j^{-1}, \psi}^{(b)} \right)_{C_i, C_j \in \mathcal{R}} = (-1)^{\frac{n-\delta}{2} + \delta'} \det \left( T_{C_i, C_j, \psi}^{(b)} \right)_{C_i, C_j \in \mathcal{R}}.
\]

If \( \psi(-1) = +1 \), then we have \( T_{C_i, \psi}^{(b)} = T_{C_i, \psi}^{(b)} \) and

\[
\det \left( T_{C_i, C_j^{-1}, \psi}^{(b)} \right)_{C_i, C_j \in \mathcal{R}} = (-1)^{\frac{n-\delta}{2}} \det \left( T_{C_i, C_j, \psi}^{(b)} \right)_{C_i, C_j \in \mathcal{R}}.
\]
Here $\delta$ is the number of the cosets $C_i \in \mathcal{R}$ whose square $C_i^2$ are $H$ or $-H$, and $\delta'$ the number of cosets $C_i \in \mathcal{R}$ whose inverses are not contained in $\mathcal{R}$, i.e., $C_i^{-1} = -C_{\sigma(i)}$ for some $\sigma(i) \in \{1, 2, \ldots, n\}$ (As for the signs of the two identities of determinants just above, see [9, Proposition 2]).

Therefore by Theorem 2.1 we obtain, in the both cases where $\psi(-1) = \pm 1$,

$$\det \left( \left( T^{(b)}_{C_iC_j, \psi} \right)_{C_i, C_j \in \mathcal{R}} \right) = \pm \prod_{\chi \in X^*} \tilde{c}_{\chi \psi}(b) \cdot \prod_{\chi \in X^*} \frac{1}{2} B_{1, \chi \psi} \cdot \frac{Q_K w_K}{Q_{\overline{K}} w_{\overline{K}}} \cdot \frac{h_{\tilde{K}}^*}{h_K^*}.$$ 

In our case where $K = \mathbb{Q}(\zeta_p)$, $H = \{1\}$ and $b = 2$, we have $T_{C, \psi}^{(2)} = -S_c(\psi)$ for $C = \tilde{c}H$.

Hence, calculating $\tilde{c}_{x\psi}(2)$ we have the desired formula. 

\[ \square \]

Corollary 2.5 immediately follows from Theorem 2.1 by taking $b = g$. Here we only note that $Q_K = 1$ and $Q_{\overline{K}} = 2$ and that the multiplicative group $\left( \mathbb{Z}/4p\mathbb{Z} \right)^{\times}$ constitutes of $R_{4p}(g^k)$ modulo $4p$ and $R_{4p}(-g^k)$ modulo $4p$ for all $k = 0, 1, \ldots, p-2$.

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References

A DETERMINANT FORMULA


