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Kronecker limit formula for real quadratic fields and Shintani invariant

By

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Abstract

We report on a result on Shintani's ray class invariant obtained in [5].

§ 1. Shintani invariant

Let K be a real quadratic field. We denote by $Cl_K(\mathfrak{f})$ the narrow ray class group of K modulo an ideal $\mathfrak{f} \subset O_K$, and associate the partial zeta function

$$\zeta(s, \mathfrak{C}) = \sum_{\mathfrak{a} \in \mathfrak{C}, \mathfrak{a} \subset O_K} \mathrm{N}(\mathfrak{a})^{-s}$$

with each ray class $\mathfrak{C} \in Cl_K(\mathfrak{f})$, where $\mathrm{N}(\mathfrak{a})$ is the norm of $\mathfrak{a}$. T. Shintani [2, 4] studied analytic expressions of the values $\zeta(0, \mathfrak{C})$ and $\zeta'(0, \mathfrak{C})$, and recognized the number theoretic significance of the values

$$X(\mathfrak{C}) := \exp\bigl(-\zeta'(0, \mathfrak{C}) + \zeta'(0, \mathfrak{C}\mathfrak{C}_1\mathfrak{C}_2)\bigr),$$

now called Shintani invariants. Here $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are the ray classes 'representing the signatures', defined by

$$\begin{aligned} \mathfrak{C}_1 &= [(\mu_1)] \in Cl_K(\mathfrak{f}), \quad \mu_1 \in 1 + \mathfrak{f},\ \mu_1 < 0,\ \mu_1' > 0, \\ \mathfrak{C}_2 &= [(\mu_2)] \in Cl_K(\mathfrak{f}), \quad \mu_2 \in 1 + \mathfrak{f},\ \mu_2 > 0,\ \mu_2' < 0 \end{aligned}$$

(we regard K as a subfield of $\mathbb{R}$, and denote the conjugate of $x \in K$ by x'). The great importance of Shintani invariants in the arithmetic of K is expressed by the Stark-Shintani conjecture that, roughly speaking, states that the invariants $X(\mathfrak{C})$ are algebraic units and generate the class fields of K.

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Shintani also gave a formula which can be regarded as an analogue of the Kronecker limit formula for real quadratic fields. His formula expresses $X(\mathfrak{C})$ by means of a certain special function. That function, now called the double sine function (see [1]), is defined by

$$\mathcal{S}(\omega, z) = \exp\bigl(-\zeta_2'(0, \omega, z) + \zeta_2'(0, \omega, 1 + \omega - z)\bigr) \quad (\omega, z > 0),$$

where

$$\zeta_2(s, \omega, z) = \sum_{p,q=0}^{\infty} (z + p\omega + q)^{-s}$$

is Barnes' double zeta function.

Theorem 1.1 (Shintani's formula). *If ε denotes the totally positive fundamental unit of K, we have*

$$X(\mathfrak{C}) = \prod_{z \in R} \mathcal{S}(\varepsilon, z)\mathcal{S}(\varepsilon', z') \tag{1.1}$$

with a finite subset R of K.

§ 2. An example

The following example was given in [2]:

If $K = \mathbb{Q}(\sqrt{5})$, $\mathfrak{f} = (4 - \sqrt{5})$ and $\mathfrak{C} = [O_K] \in Cl_K(\mathfrak{f})$, then the totally positive fundamental unit is $\varepsilon = \dfrac{3 + \sqrt{5}}{2}$, and we have

$$\begin{aligned} &X(\mathfrak{C}) \\ (2.1)\quad &= \frac{1}{2}\left(\frac{3 + \sqrt{5}}{2} + \sqrt{\frac{3\sqrt{5} - 1}{2}}\right) \\ &= \mathcal{S}\Bigl(\varepsilon, \frac{7\varepsilon + 9}{11}\Bigr)\mathcal{S}\Bigl(\varepsilon, \frac{8\varepsilon + 4}{11}\Bigr)\mathcal{S}\Bigl(\varepsilon, \frac{6\varepsilon + 3}{11}\Bigr)\mathcal{S}\Bigl(\varepsilon, \frac{10\varepsilon + 5}{11}\Bigr)\mathcal{S}\Bigl(\varepsilon, \frac{2\varepsilon + 1}{11}\Bigr) \\ (2.2)\quad &\times \mathcal{S}\Bigl(\varepsilon', \frac{7\varepsilon' + 9}{11}\Bigr)\mathcal{S}\Bigl(\varepsilon', \frac{8\varepsilon' + 4}{11}\Bigr)\mathcal{S}\Bigl(\varepsilon', \frac{6\varepsilon' + 3}{11}\Bigr)\mathcal{S}\Bigl(\varepsilon', \frac{10\varepsilon' + 5}{11}\Bigr)\mathcal{S}\Bigl(\varepsilon', \frac{2\varepsilon' + 1}{11}\Bigr). \end{aligned}$$

The equation (2.1) was obtained from the relative class number formula for a quadratic extension over K, which works only in rather special cases, while (2.2) is a consequence of the general formula (1.1). This situation suggests the following problem:

> Deduce the equation (2.1) from the expression (2.2) by exploiting the properties of the double sine function.

To the author's knowledge, any solution to this problem is unknown. On the other hand, we can show that

$$\mathcal{S}\Big(\varepsilon', \frac{7\varepsilon'+9}{11}\Big)\mathcal{S}\Big(\varepsilon', \frac{8\varepsilon'+4}{11}\Big)\mathcal{S}\Big(\varepsilon', \frac{6\varepsilon'+3}{11}\Big)\mathcal{S}\Big(\varepsilon', \frac{10\varepsilon'+5}{11}\Big)\mathcal{S}\Big(\varepsilon', \frac{2\varepsilon'+1}{11}\Big) = 1 \tag{2.3}$$

by using only the following properties of $\mathcal{S}$ (see [5, Proposition 3.3.1]):

$$\mathcal{S}(\omega, z) = \mathcal{S}(1/\omega, z/\omega), \tag{2.4}$$

$$\mathcal{S}(\omega, z) = 2\sin(\pi z)\mathcal{S}(\omega, z+\omega) = 2\sin(\pi z/\omega)\mathcal{S}(\omega, z+1), \tag{2.5}$$

$$\mathcal{S}(\omega, z) = 2\sin(\pi z)\frac{\mathcal{S}(1/\omega - 1, z/\omega)}{\mathcal{S}(1-\omega, z)} \quad (\text{if } 0<\omega<1). \tag{2.6}$$

To prove (2.3), we rewrite the values $\mathcal{S}(\varepsilon', x\varepsilon'+y)$ for $(x,y) = \Big(\frac{7}{11}, \frac{9}{11}\Big), \ldots, \Big(\frac{2}{11}, \frac{1}{11}\Big)$ as

$$\mathcal{S}(\varepsilon', x\varepsilon'+y) = \begin{cases} \dfrac{\mathcal{S}\big(\varepsilon-1, y(\varepsilon-1)+x+y\big)}{\mathcal{S}\big(1-\varepsilon', (1-x)(1-\varepsilon')+x+y\big)} & (x+y\le 1), \\[2ex] \dfrac{\mathcal{S}\big(\varepsilon-1, y(\varepsilon-1)+x+y-1\big)}{\mathcal{S}\big(1-\varepsilon', (1-x)(1-\varepsilon')+x+y-1\big)} & (x+y>1) \end{cases}$$

by using (2.5) and (2.6) (note that $1/\varepsilon' = \varepsilon$). For example, we have

$$\mathcal{S}\Big(\varepsilon', \frac{7\varepsilon'+9}{11}\Big) = \frac{\mathcal{S}\big(\varepsilon-1, \frac{9}{11}(\varepsilon-1)+\frac{5}{11}\big)}{\mathcal{S}\big(1-\varepsilon', \frac{4}{11}(1-\varepsilon')+\frac{5}{11}\big)}, \tag{2.7}$$

$$\mathcal{S}\Big(\varepsilon', \frac{6\varepsilon'+3}{11}\Big) = \frac{\mathcal{S}\big(\varepsilon-1, \frac{3}{11}(\varepsilon-1)+\frac{9}{11}\big)}{\mathcal{S}\big(1-\varepsilon', \frac{5}{11}(1-\varepsilon')+\frac{9}{11}\big)}. \tag{2.8}$$

Then, since $(\varepsilon-1)(1-\varepsilon') = 1$, the formula (2.4) implies that the numerator of (2.7) and the denominator of (2.8) are equal. All factors of the product in (2.3) cancel out similarly.

§ 3. General result

To state a result in [5] of which the cancellation (2.3) is a consequence, we have to introduce some notations.

Let a ray class $\mathfrak{C} \in Cl_K(\mathfrak{f})$ be given. We choose the following data:

- $\mathfrak{a} \subset O_K$ is a representative of $\mathfrak{C}$.
- $\mathfrak{b}$ is a fractional ideal which satisfies that $\mathfrak{b} \cap \mathbb{Q} = \mathbb{Z}$ and belongs to the narrow ideal class $[\mathfrak{a}^{-1}\mathfrak{f}] \in Cl_K(O_K)$ of $\mathfrak{a}^{-1}\mathfrak{f}$.

- $\omega \in K$ satisfies that $\mathfrak{b} = \mathbb{Z} + \mathbb{Z}\omega$ and $0 < \omega' < 1 < \omega$.
- $z \in K$ is a totally positive element such that $\mathfrak{b} = (z)\mathfrak{a}^{-1}\mathfrak{f}$.

By the condition $0 < \omega' < 1 < \omega$, ω has the purely periodic continued fraction expansion

$$\omega = b_0 - \cfrac{1}{b_1 - \cdots \cfrac{1}{b_{m-1} - \cfrac{1}{b_0 - \cdots}}}$$

where $b_0, \dots, b_{m-1}$ are integers greater than or equal to 2. We define b_k for $k \in \mathbb{Z}$ by the periodicity $b_k = b_{k+m}$, and put

$$\omega_k = b_k - \cfrac{1}{b_{k+1} - \cdots \cfrac{1}{b_{k+m-1} - \cfrac{1}{b_k - \cdots}}}.$$

Then, if we define a sequence $\{A_k\}_{k\in\mathbb{Z}}$ by $A_0 = 1$ and $A_k\omega_k = A_{k-1}$, it is easy to show by induction that A_{k-1} and A_k spans $\mathfrak{b}$ over $\mathbb{Z}$ for each $k \in \mathbb{Z}$. Hence there is a unique pair (x_k, y_k) of rational numbers such that

$$0 < x_k \leq 1, \quad 0 \leq y_k < 1, \quad x_k A_{k-1} + y_k A_k \equiv z \pmod{\mathfrak{b}}.$$

We put $z_k = x_k\omega_k + y_k$. We call the sequence $\{(\omega_k, z_k)\}$ the decomposition data associated with $\mathfrak{C}$, because of the following formula [5, Proposition 2.1.4].

Proposition 3.1. *We have*

$$\zeta(s, \mathfrak{C}) = \sum_{k=1}^{m} \left(\frac{\omega_k - \omega_k'}{\sqrt{D}\,\mathrm{N}(\mathfrak{f})} \right)^s \sum_{p,q=0}^{\infty} \mathrm{N}_{K/\mathbb{Q}}(z_k + p\omega_k + q)^{-s},$$

where D is the discriminant of K.

Remark 3.2.

(1) The sequence $\{\omega_k\}$ has the period m, while $\{z_k\}$ has the period rm where r is the index of the subgroup

$$\{u \in O_K^\times \mid u \gg 0, \quad u \equiv 1 \pmod{\mathfrak{f}}\}$$

in the group of totally positive units.

(2) When we begin with another choice of the data $\mathfrak{a}$, $\mathfrak{b}$, ω and z, the indices of $\{(\omega_k, z_k)\}$ are shifted, i.e. the new data $\{(\omega_k^{(1)}, z_k^{(1)})\}$ can be written as $(\omega_k^{(1)}, z_k^{(1)}) = (\omega_{k+k_0}, z_{k+k_0})$ with some constant k_0. In other words, the decomposition data is determined by $\mathfrak{C}$ up to shift of indices.

Now we can state the generalization of the cancellation (2.3).

Theorem 3.3. *Let $\{(\omega_k, z_k)\}$ be the decomposition data associated with $\mathfrak{C} \in Cl_K(\mathfrak{f})$, and m and rm the periods of the sequences $\{\omega_k\}$ and $\{z_k\}$ respectively (see Remark 3.2 (1)).*

(1) We have a product expression $X(\mathfrak{C}) = X_1(\mathfrak{C})X_2(\mathfrak{C})$, where

$$X_1(\mathfrak{C}) = \prod_{k=1}^{rm} \mathcal{S}(\omega_k, z_k), \quad X_2(\mathfrak{C}) = \prod_{k=1}^{rm} \mathcal{S}(\omega_k', z_k').$$

(2) We have the relations

$$X_1(\mathfrak{C}) = X_1(\mathfrak{C}\mathfrak{C}_1) = X_1(\mathfrak{C}\mathfrak{C}_2)^{-1},$$
$$X_2(\mathfrak{C}) = X_2(\mathfrak{C}\mathfrak{C}_1)^{-1} = X_2(\mathfrak{C}\mathfrak{C}_2).$$

In the case considered in §2, i.e. when $K = \mathbb{Q}(\sqrt{5})$ and $\mathfrak{f} = (4 - \sqrt{5})$, the ray class $\mathfrak{C}_1 \in Cl_K(\mathfrak{f})$ is trivial because the unit $\mu_1 = -\left(\frac{1+\sqrt{5}}{2}\right)^5$ satisfies that $\mu_1 \in 1 + \mathfrak{f}$, $\mu_1 < 0$ and $\mu_1' > 0$. Hence, by applying Theorem 3.3(2), we obtain $X_2(\mathfrak{C}) = 1$ for any $\mathfrak{C} \in Cl_K(\mathfrak{f})$.

Although Theorem 3.3(1) is a variant of Shintani's formula (1.1), it seems more convenient to use our decomposition data constructed from the continued fraction. In fact, the proof of Theorem 3.3(2) is based on a beautiful relation of the data associated with classes $\mathfrak{C}$ and $\mathfrak{C}\mathfrak{C}_2$, which is a result of the continued fraction theory. We remark that such a relation was obtained and used by Zagier [6] when $\mathfrak{f} = O_K$, the case in which the quantities z_k don't appear explicitly since $z_k = \omega_k$ for all k. Once such a relation is established, the relations (2) can be shown by rewriting the expressions (1) in a way similar to that given in §2. See [5] for details.

References

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