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Elliptic Gauss Sums and Hecke L-values at $s = 1$

Dedicated to Professor Tomio Kubota

By

Tetsuya ASAI*

Abstract

The rationality of the elliptic Gauss sum coefficient will be proved. The elliptic Gauss sum appears in the central value of a certain Hecke L-series, and the proof is based on the functional equation of Hecke L and the Cassels-Matthews formula of the classical Gauss sum. A new view on Hecke L-values and a better understanding of the latter formula will be given.

Introduction

It seems that some retro-fashioned but still fascinating formulas lead us to consider the *elliptic Gauss sum.* The classical formulas concerned are the following :

$$\frac{1}{2}\sum_{k=1}^{p-1}\left(\frac{k}{p}\right)\cot\frac{\pi k}{p} = h(-p)\ \sqrt{p} \quad \text{and} \quad \frac{1}{2}\sum_{k=1}^{p-1}\left(\frac{k}{p}\right)\sec\frac{2\pi k}{p} = h(-p)\ \sqrt{p},$$

where p (> 3) is a rational prime such that $p \equiv 3 \pmod 4$ and $p \equiv 1 \pmod 4$, respectively ; $(\frac{k}{p})$ is the Legendre symbol and $h(-p)$ is the class number of the quadratic field $\mathbf{Q}(\sqrt{-p})$. The formulas are apparently related to the Dirichlet L-values at $s = 1$.

To get a typical elliptic Gauss sum, we have only to replace the Legendre symbol by the cubic or the quartic residue character, and the trigonometric function by a suitable elliptic function. The notion of elliptic Gauss sum was first introduced by Eisenstein for a concern of higher reciprocity laws, but since then it has been regarded seemingly as a minor object of study. (cf. [2, Chap. 9, p. 311])

We shall here try to reconsider it. Especially, we treat the problem of rationality of the *coefficient*, so we call, of the elliptic Gauss sum, which is an analogy of the coefficient $h(-p)$ in the above classical case. A typical example is as follows.

2000 Mathematics Subject Classification(s): 11L05, 11R42, 11G15
*658-19 Sangamyo, Yaizu 425-0071, Japan / e-mail : `apeiros@thn.ne.jp`

Let $\mathrm{sl}(u)$ be the lemniscatic sine of Gauss so that $\mathrm{sl}((1-i)\varpi u)$ is an elliptic function with the period lattice $\mathbf{Z}[i]$, where $\varpi = 2\int_0^1 \frac{dx}{\sqrt{1-x^4}}$. Let π be a primary prime in $\mathbf{Z}[i]$; $\pi \equiv 1 \pmod{(1+i)^3}$ and assume $p = \pi\overline{\pi} \equiv 13 \pmod{16}$. We consider the sum

$$\mathcal{G}_\pi = \frac{1}{4}\sum_{\nu\in(\mathcal{O}/(\pi))^\times} \left(\frac{\nu}{\pi}\right)_4 \mathrm{sl}((1-i)\varpi\,\nu/\pi).$$

It is not difficult to see that it is expressible as $\mathcal{G}_\pi = \alpha_\pi(\sqrt[4]{-\pi})^3$ by an integer α_π in $\mathbf{Z}[i]$. We can, furthermore, find remarkable facts by some experimental observation of the exact value α_π after having chosen the *canonical quartic root* $\widetilde{\pi} = \sqrt[4]{-\pi}$. Namely,

$\mathcal{G}_\pi = \alpha_\pi\,\widetilde{\pi}^3$ *with a rational integer* α_π ; *the magnitude of which is rather small.*

This is not so trivial, but we can now prove the rationality. It proceeds as follows.

Let $\widetilde{\chi}_\pi$ be the Hecke character of weight one induced by the quartic residue character to the modulus π. As is well known, the associated Hecke L-series $L(s, \widetilde{\chi}_\pi)$ has the functional equation. We have in particular the *central value equation* $L(1, \widetilde{\chi}_\pi) = C(\widetilde{\chi}_\pi)\,\overline{L(1, \widetilde{\chi}_\pi)}$ at $s = 1$; the constant $C(\widetilde{\chi}_\pi)$ is the so-called *root number.* Then it will be first shown that the value of $L(1, \widetilde{\chi}_\pi)$ is expressed by the elliptic Gauss sum $\mathcal{G}_\pi$. Secondly, it will be seen the root number $C(\widetilde{\chi}_\pi)$ coincides with the classical quartic Gauss sum $G_4(\pi)$ in this case, and fortunately the explicit formula of the value is known owing to Cassels-Matthews. Finally, the accordant expression of $\widetilde{\pi}$ and $G_4(\pi)$ combined with the central value equation proves immediatly the fact $\alpha_\pi = \overline{\alpha}_\pi$, that is, the coefficient α_π of $\mathcal{G}_\pi$ is a rational integer. As a corollary we shall obtain a new formula on the value $L(1, \widetilde{\chi}_\pi)$. It also should be remarked that we shall bring out a better understanding of Matthews' formula by considering together with the elliptic Gauss sum.

On the other hand, Mr. Naruo Kanou has observed these coefficients α_π for many primes by computer. According to his result, it holds $-49 \le \alpha_\pi \le 49$ for 35,432 primes in the interval $13 \le p \le 3999949$. At present, however, the reason is not completely clarified, so we shall not touch on the topic of small magnitude, while several persons say that α_π^2 might closely relate to the order of a certain Tate-Shafarevich group.

In this paper, we describe detail of the proof, mainly of rationality of the coefficient. There are two cases : the cubic character case and the quartic one. Although the idea is common and most of discussion goes in parallel, we would like to treat the two cases separately in the sections §1 and §2 to avoid possible confusion and complication. Since the same notations appear with each different meaning, we hope the reader would read carefully and also he would tolerate some redundant and overlapped description.

Main statements on the rationality of the elliptic Gauss sum coefficient are Theorem 1.19 in the cubic character case, and Theorem 2.22 together with subsequent two corollaries in the quartic character case. The central value of the associated Hecke L-series is expressed by the elliptic Gauss sum in Theorem 1.16 or Theorem 2.19, respectively. A new expession of the value $L(s, \widetilde{\chi}_\pi)$ by Kummer sum is given in Theorem 1.21, and the corresponding one of quartic case is in Theorem 2.25. The Cassels-Matthews formulas are a bit elegantly interpreted in Lemma 1.17 (Kummer sum case) and Lemma 2.20 (The quartic sum case).

Acknowledgements

I am deeply grateful to Prof. Yoshikazu Baba, who called my attention to a theorem[1] of regular heptagon, which had made a beginning of the study.

Special thanks are due to my younger friends for their constant support, and the task has never been completed without the collaboration with them. Especially, Naruo Kanou has first made and provided a big table of the coefficients, which still waits for better elucidation; Yoshihiro Ônishi himself has discovered a congruence between the coefficients of elliptic Gauss sums and the generalized Hurwitz numbers, which has brought me a great inspiration and encouragement; Yoshinori Mizuno has suggested accurately to use the doubly periodic but non-analytic function in computation of Hecke L-values, and I followed him fully in this point.

While it was not till a week after the meeting at Kyoto that I completed the proof by using the functional equation, Mr. Seidai Yasuda became independently aware of the same idea just after my meeting talk. He informed me of this after my realization. I thank him for this and permitting my own publication.

Finally I must thank the referee, who pointed out a gap in my first manuscript, and so I could change my argument so as to use a rather weak lemma. Also the referee remarked that our main results are fairly compatible with the strong form of Birch and Swinnerton-Dyer conjecture formulated by Gross ([6, p. 227, Conjecture 3.6]), and in particular our ideal (α_π) is closely related to the order ideal of Tate-Shafarevich group of an appropriate elliptic curve. Though the author could not make use of this suggestion enough, I added Gross' to the references for the readers.

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[1] *Heptagon harmony : Let $a < b < c$ be the three lengths of the side and the dia heptagon. Then $1/a = 1/b + 1/c$ holds.*

§ 1. The Cubic Character Case

Let ρ be the cubic root of unity $e^{2\pi i/3}$. Throughout §1, the field $\mathbf{Q}(\rho)$ and the ring $\mathbf{Z}[\rho]$ are abbreviated to F and $\mathcal{O}$, respectively. The set $\mathcal{O}$ or its constant multiple appears also as a period lattice for elliptic functions. We use the notations as well : $W = \{\pm 1, \pm\rho, \pm\overline{\rho}\}$, $W' = \{1, \rho, \overline{\rho}\}$.

§ 1.1. Special elliptic functions with complex multiplication

1.1.1. We shall define some special functions which play the leading role in our argument. We denote by ϖ_1 the real period given by

$$\varpi_1 \doteq \int_0^1 \frac{dx}{\sqrt[3]{(1-x^3)^2}} = \sqrt[3]{2}\int_0^1 \frac{dx}{1-x^3} = 1.76663875\cdots,$$

and let $\wp(u)$ denote Weierstrass' $\wp$ with the period lattice $\varpi_1\mathcal{O}$, so that $\wp'^2 = 4\wp^3 - 27$. Then it is obvious $\wp(\varpi_1 u)$ and $\wp'(\varpi_1 u)$ are elliptic functions of the period $\mathcal{O}$. Further, we can get another doubly periodic function by a slight modification of Weierstrass' ζ :

Definition 1.1. The non-analytic doubly periodic function $\mathsf{Z}(u)$ is defined by

$$\mathsf{Z}(u) \doteq \zeta(\varpi_1 u) - \frac{2\pi}{3\varpi_1}\overline{u}, \tag{1.1}$$

where $\zeta(u)$ is Weierstrass' ζ relative to the period $\varpi_1\mathcal{O}$.

Double periodicity of $\mathsf{Z}(u)$ relative to $\mathcal{O}$ is easily verified by a usual formula of ζ.

The addition formula of Z, that follows immediately from one of ζ, is useful :

$$\mathsf{Z}(u+v) = \mathsf{Z}(u) + \mathsf{Z}(v) + \frac{1}{2}\frac{\wp'(\varpi_1 u) - \wp'(\varpi_1 v)}{\wp(\varpi_1 u) - \wp(\varpi_1 v)}. \tag{1.2}$$

The significance of the following two functions will be clear later when we see that they are closely related to some Hecke L-values at $s = 1$. In fact, they are the corresponding functions with the lemniscatic sine and cosine of the quartic case. Anyway we shall find that these functions are automatically introduced by the associated Hecke L-series.

Definition 1.2. The elliptic functions $\varphi(u)$ and $\psi(u)$ relative to the period $\mathcal{O}$ are defined by

$$\varphi(u) \doteq \frac{1}{3}\left\{\mathsf{Z}\left(u - \frac{1}{3}\right) + \overline{\rho}\,\mathsf{Z}\left(u - \frac{\rho}{3}\right) + \rho\,\mathsf{Z}\left(u - \frac{\overline{\rho}}{3}\right)\right\}, \tag{1.3}$$

$$\psi(u) \doteq -\frac{1}{3}\left\{\mathsf{Z}\left(u - \frac{1}{3}\right) + \rho\,\mathsf{Z}\left(u - \frac{\rho}{3}\right) + \overline{\rho}\,\mathsf{Z}\left(u - \frac{\overline{\rho}}{3}\right)\right\}. \tag{1.4}$$

From the addition formula (1.2) we can easily derive the following expressions.

$$\varphi(u) = \frac{6\,\wp(\varpi_1 u)}{9+\wp'(\varpi_1 u)}, \quad \psi(u) = \frac{9-\wp'(\varpi_1 u)}{9+\wp'(\varpi_1 u)}. \tag{1.5}$$

We need also the following formula.

$$\varphi(u)^{-1}+\varphi(-u)^{-1} = \frac{3}{\wp(\varpi_1 u)} = \frac{1}{\sqrt{-3}}\left\{\mathsf{Z}\left(u-\frac{1}{\sqrt{-3}}\right)-\mathsf{Z}\left(u+\frac{1}{\sqrt{-3}}\right)\right\}. \tag{1.6}$$

It seems that these functions are highly basic in the theory of elliptic functions relative to the lattice $\mathcal{O}$, especially in the theory of complex multiplication. We here list some fundamental properties of these functions, which are easily derived from the definitions and by usual theory of elliptic functions. Sometimes we need further properties, including the addition and multiplication formulas, which, however, we shall collect at the end of the section (Appendix) for descriptive simplicity.

$$\begin{gathered}
\operatorname{div}(\varphi) = (0)+((1-\rho)/3)+((\rho-1)/3)-(1/3)-(\rho/3)-(\overline{\rho}/3),\\
\operatorname{div}(\psi) = (-1/3)+(-\rho/3)+(-\overline{\rho}/3)-(1/3)-(\rho/3)-(\overline{\rho}/3),\\
\mathsf{Z}(\rho u) = \overline{\rho}\,\mathsf{Z}(u), \quad \varphi(\rho u) = \rho\,\varphi(u), \quad \psi(\rho u) = \psi(u),\\
\mathsf{Z}(-u) = -\mathsf{Z}(u), \quad \varphi(-u) = -\varphi(u)\,\psi(u)^{-1}, \quad \psi(-u) = \psi(u)^{-1},\\
\psi(u) = \varphi(-u-1/3), \quad \varphi(u)^{-1} = \varphi(-u+1/3), \quad \psi(0) = \varphi(-1/3) = 1,\\
\varphi'(u) = -3\varpi_1\,\psi(u)^2, \quad \psi'(u) = 3\varpi_1\,\varphi(u)^2, \quad \varphi(u)^3+\psi(u)^3 = 1.
\end{gathered}$$

1.1.2. Let π be a complex prime in $\mathcal{O} = \mathbf{Z}[\rho]$ so that $p = \pi\overline{\pi} \equiv 1 \pmod 3$, and we assume also $\pi \equiv 1 \pmod 3$. Then we have $(\mathcal{O}/(\pi))^\times \cong (\mathbf{Z}/p\mathbf{Z})^\times$. We often abbreviate as $\nu \pmod{\pi}$ in such a case when ν runs over $(\mathcal{O}/(\pi))^\times$.

Class field or complex multiplication theory tells us that such a division value $\varphi(1/\pi)$ or $\psi(1/\pi)$ generates an abelian extension field of $F = \mathbf{Q}(\rho)$. Namely, we have

Lemma 1.3. *Let $L = F(\varphi(1/\pi))$ and $L_1 = F(\psi(1/\pi))$. One has*

(i) *L/F is a cyclic extension of degree $p-1$, and L_1 is a subfield ; $[L : L_1] = 3$.*

(ii) *$\mathrm{Gal}(L/F) \cong (\mathcal{O}/(\pi))^\times$ by corresponding σ_μ to μ, and it holds $\varphi(\nu/\pi)^{\sigma_\mu} = \varphi(\mu\nu/\pi)$, $\psi(\nu/\pi)^{\sigma_\mu} = \psi(\mu\nu/\pi)$ for arbitrary $\mu, \nu \in (\mathcal{O}/(\pi))^\times$.*

(iii) *$\varphi(1/\pi)$, $\psi(1/\pi)$ are algebraic integers, and particularly $\psi(1/\pi)$ is a unit.*

(iv) *The prime ideal (π) splits completely in L : $(\pi) = \mathfrak{P}^{p-1}$ where $\mathfrak{P} = (\varphi(1/\pi))$.*

In fact, L and L_1 are nothing but the fields called ray class fields of the conductors (3π) and $(\sqrt{-3}\pi)$. We omit a proof, but in Appendix we shall give a brief comment on the complex multiplication formula which is crucial for the theory of division values. For a general consultation and a background we can refer to [2, Chap. 8]. We here only give some numerical examples.

Example 1.4. In each case tabulated below, we have π-multiplication formula :

$$\varphi(\pi u) = \varphi(u) \cdot \frac{U(\varphi(u))}{R(\varphi(u))}, \quad R(x) = x^{p-1}\, U(x^{-1}).$$

We can also verify

$$U(x) = \prod_{\nu \,(\mathrm{mod}\, \pi)} (x - \varphi(\nu/\pi)), \quad V(x)^3 = \prod_{\nu \,(\mathrm{mod}\, \pi)} (x - \psi(\nu/\pi)),$$

$$x\, U(x) - R(x) = (x-1)\, V(x)^3.$$

Furthermore it is easy to check

$$U(x),\ V(x) \in \mathcal{O}[x], \quad U(x) \equiv x^{p-1} \ (\mathrm{mod}\ (\pi)), \quad U(0) = \pi, \quad V(0) = 1.$$

In fact, $U(x)$ and $V(x)$ are the minimal polynomials of $\varphi(1/\pi)$ and $\psi(1/\pi)$, respectively.

p	π	$U(x)$
7	$1+3\rho$	$x^6 - \pi\, x^3 + \pi$
13	$4+3\rho$	$x^{12} + (1+3\rho)\,\pi\, x^9 - 3\rho\,\pi\, x^6 - 2\pi\, x^3 + \pi$
19	$-2+3\rho$	$x^{18} - 3\rho\,\pi\, x^{15} - (3-9\rho)\,\pi\, x^{12} + (5-12\rho)\,\pi\, x^9 + 6\rho\,\pi\, x^6 - 3\pi\, x^3 + \pi$

p	π	$V(x)$
7	$1+3\rho$	$x^2 + \rho\, x + 1$
13	$4+3\rho$	$x^4 - (1+\rho)\, x^3 - (1+2\rho)\, x^2 - (1+\rho)\, x + 1$
19	$-2+3\rho$	$x^6 + (1-\rho)\, x^5 + (1+2\rho)\, x^4 - x^3 + (1+2\rho)\, x^2 + (1-\rho)\, x + 1$

Remark. As for the division value $\mathsf{Z}(1/\pi)$, we can prove the following. (cf. §2.1)

$$L = F(\varphi(1/\pi)) = F(\mathsf{Z}(1/\pi)), \quad \mathsf{Z}(\nu/\pi)^{\sigma_\mu} = \mathsf{Z}(\mu\nu/\pi) \quad (\mu, \nu \in (\mathcal{O}/(\pi))^\times).$$

§ 1.2. L-series for Hecke characters of weight one

1.2.1. In this section we recall some fundamental facts about the topic, which will give a basis and a framework of our whole discussion.

Let $\widetilde{\chi}$ denote a Hecke character of weight one relative to the modulus $(\beta) \subset \mathcal{O}$, so that it is a multiplicative function on the ideal group of $\mathcal{O}$ of the following form :

$$\widetilde{\chi}((\nu)) = \chi_1(\nu)\overline{\nu}, \quad \chi_1 : (\mathcal{O}/(\beta))^\times \to \mathbf{C}^\times, \quad \chi_1(\varepsilon) = \varepsilon \quad (\varepsilon \in W),$$

where χ_1 is an ordinary residue class character to the modulus (β), and (β) is called the coductor of $\widetilde{\chi}$ if χ_1 is a primitive character to the modulus (β).

It is well known the associated L-series has the analytic continuation and satisfies the functional equation. We here follow Weil's argument and his notation.

$$\begin{aligned} L(s,\ \widetilde{\chi}) &= \sum_{\mathfrak{a}} \widetilde{\chi}(\mathfrak{a})\, N\mathfrak{a}^{-s} = \frac{1}{6} \sum_{\nu \in \mathcal{O}} \chi_1(\nu)\, \overline{\nu}\, |\nu|^{-2s} \\ &= \frac{1}{6} \sum_{\lambda \,(\mathrm{mod}\, \beta)} \chi_1(\lambda) \sum_{\mu \in \mathcal{O}} (\overline{\lambda} + \overline{\mu}\overline{\beta})\, |\lambda + \mu\beta|^{-2s}. \end{aligned}$$

Therefore we have

$$L(s,\ \widetilde{\chi}) = \beta^{-1}\, |\beta|^{2-2s} \cdot \frac{1}{6} \sum_{\lambda \,(\mathrm{mod}\, \beta)} \chi_1(\lambda)\, K_1(\lambda/\beta,\ 0,\ s). \tag{1.7}$$

Here the function K_1 is defined for $\mathrm{Re}\, s > 3/2$ as follows, and it is analytically continued to the whole s-plane and satisfies the own functional equation (cf. [5, Chap. VIII]) :

$$K_1(u, u_0, s) = \sum_{\mu \in \mathcal{O}} e^{\frac{2\pi}{\sqrt{3}}(\overline{u}_0\mu - u_0\overline{\mu})}\, (\overline{u} + \overline{\mu})\, |u + \mu|^{-2s},$$

$$\left(\frac{2\pi}{\sqrt{3}}\right)^{-s} \Gamma(s)\, K_1(u, u_0, s) = e^{\frac{2\pi}{\sqrt{3}}(\overline{u}_0 u - u_0\overline{u})} \left(\frac{2\pi}{\sqrt{3}}\right)^{s-2} \Gamma(2-s)\, K_1(u_0, u, 2-s).$$

If (β) is the conductor of $\widetilde{\chi}$, we can apply a usual computation of Gauss sum, and thus we obtain the functional equation of Hecke L-series in this case :

$$\Lambda(s,\ \widetilde{\chi}) = C(\widetilde{\chi})\, \Lambda(2-s,\ \overline{\widetilde{\chi}}),$$

where

$$\Lambda(s,\ \widetilde{\chi}) = \left(\frac{2\pi}{\sqrt{3 \cdot N(\beta)}}\right)^{-s} \Gamma(s)\, L(s,\ \widetilde{\chi}),$$

and

$$C(\widetilde{\chi}) = -\rho\, \beta^{-1} \sum_{\lambda \,(\mathrm{mod}\, \beta)} \chi_1(\lambda)\, e^{2\pi i\, S(\lambda/\beta)}.$$

In the above we use the following abbreviation :

$$S(\lambda) = a \quad \text{for} \quad \lambda = a + b\overline{\rho} \quad (a,\ b \in \mathbf{Q}), \quad \text{i.e.} \quad 2\pi i\, S(\lambda) = \tfrac{2}{\sqrt{3}}\, \pi\, (\lambda\rho - \overline{\lambda}\overline{\rho}).$$

In particular we have a simple equality of Hecke L-values at $s = 1$, which we call *the central value equation*, and the constant $C(\widetilde{\chi})$ is called *the root number* :

Lemma 1.5. *Let $\widetilde{\chi}$ be a Hecke character of weight* 1 *with the conductor* (β). *Then we have*

$$L(1,\ \widetilde{\chi}) = C(\widetilde{\chi})\,\overline{L(1,\ \widetilde{\chi})}, \tag{1.8}$$

$$C(\widetilde{\chi}) = -\rho\,\beta^{-1} \sum_{\lambda\,(\mathrm{mod}\,\beta)} \chi_1(\lambda)\,e^{2\pi i\,S(\lambda/\beta)}. \tag{1.9}$$

Remark. $L(1,\ \overline{\widetilde{\chi}}) = \overline{L(1,\ \widetilde{\chi})}$.

1.2.2. On the other hand, we can notice that the value $L(1,\ \widetilde{\chi})$ relates to some elliptic functions. As is well known (e.g. [5, Chap. VIII, §14]), the following is valid.

$$E_1^*(u) \doteq K_1(u,\ 0,\ 1) = \varpi_1\,\zeta(\varpi_1 u) - \frac{2\pi}{\sqrt{3}}\,\overline{u}.$$

By the definition (1.1) the right-hand side is nothing but our function $\varpi_1\,\mathsf{Z}(u)$. Combining this and the equation (1.7) at $s = 1$, we obtain the following formula.

Lemma 1.6. *Under the same condition of the preceding Lemma, it holds*

$$L(1,\ \widetilde{\chi}) = \frac{\varpi_1}{6\,\beta} \sum_{\lambda\,(\mathrm{mod}\,\beta)} \chi_1(\lambda)\,\mathsf{Z}(\lambda/\beta). \tag{1.10}$$

As we shall discuss later, the sum appeared in the right-hand of (1.10) is a prototype of *elliptic Gauss sum.* When a Hecke character $\widetilde{\chi}$ is given in a suitably explicit form, we may evaluate both the elliptic Gauss sum and the root number more explicitly, and the central value equation will give some relation between the two. In particular, from the value of the elliptic Gauss sum, if non-vanishing, we can know the value of the root number. This is the case of our cubic characters, that is the point of this report.

Example 1.7. The following is probably the simplest case and the derived formula $L(1,\ \widetilde{\chi}_0) = \frac{\varpi_1}{3}$ may be compared with the classical formula : $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots = \frac{\pi}{4}$.

The conductor is the ideal (3). The Hecke character $\widetilde{\chi}_0$ mod (3) is given as follows :

$\widetilde{\chi}_0((\nu)) \doteq \chi_0(\nu)\,\overline{\nu}$, *where* $\chi_0\ :\ (\mathcal{O}/(3))^\times \cong W$ *is the natural isomorphism.*

Then we can evaluate the L-value at $s = 1$ directly by (1.10) :

$$L(1,\ \widetilde{\chi}_0) = \frac{\varpi_1}{18} \sum_{\varepsilon \in W} \varepsilon\,\mathsf{Z}(\varepsilon/3) = \frac{\varpi_1}{3}\,\mathsf{Z}(1/3) = \frac{\varpi_1}{3},$$

because $\mathsf{Z}(1/3) = 1$ by usual theory of elliptic functions. Also we can easily check $C(\widetilde{\chi}_0) = -\frac{\rho}{3} \sum_{\varepsilon \in W} \varepsilon\,e^{2\pi i\,S(\varepsilon/3)} = 1$ and hence the expected central value equation holds.

§ 1.3. Elliptic Gauss sums for cubic characters

1.3.1. Let π be a primary prime in $\mathcal{O}$; namely $\pi \equiv 1 \pmod 3$. Let χ_π be the cubic residue character to the modulus π ; the notation will be fixed throughout §1 :

$$\chi_\pi(\nu) = \left(\frac{\nu}{\pi}\right)_3 \ : \ \chi_\pi(\nu)^3 = 1 \ \text{ and } \ \chi_\pi(\nu) \equiv \nu^{(p-1)/3} \pmod{\pi} \quad (\nu \in (\mathcal{O}/(\pi))^\times).$$

Let $f(u)$ be a certain elliptic function with the periods $\mathcal{O}$, which we specify below.

Definition 1.8. The following is called an *elliptic Gauss sum.*

$$\mathcal{G}_\pi(\chi_\pi,\, f) \doteq \frac{1}{3} \sum_{\nu \,(\mathrm{mod}\, \pi)} \chi_\pi(\nu)\, f(\nu/\pi). \tag{1.11}$$

In §1, we deal with the elliptic Gauss sums $\mathcal{G}_\pi(\chi_\pi,\, \varphi)$, $\mathcal{G}_\pi(\chi_\pi,\, \varphi^{-1})$ and $\mathcal{G}_\pi(\chi_\pi,\, \psi)$ only, where $\varphi(u)$ and $\psi(u)$ are the special elliptic functions defined by (1.3), (1.4). So we understand that $f(u)$ denotes an arbitrary one of these functions $\varphi(u), \varphi(u)^{-1}$ and $\psi(u)$ in the subsequence. In these cases, unless "the parity condition", so we call, $\chi_\pi(\rho\nu)\, f(\rho\nu/\pi) = \chi_\pi(\nu)\, f(\nu/\pi)$ is satisfied, $\mathcal{G}_\pi(\chi_\pi,\, f)$ vanishes trivially. Since $\varphi(\rho u) = \rho\varphi(u)$, $\psi(\rho u) = \psi(u)$ and $\chi_\pi(\rho) = \rho^{(p-1)/3}$, we can easily check that $\mathcal{G}_\pi(\chi_\pi,\, f)$ is not trivial in the following only three cases. The parity condition, however, is not sufficient for non-vanishing of the elliptic Gauss sum as we shall see later.

The elliptic Gauss sums which we shall consider in §1 are the following :

(a) $\mathcal{G}_\pi(\chi_\pi,\, \varphi)$ *in the case* $p = \pi\overline{\pi} \equiv 7 \pmod 9$,

(b) $\mathcal{G}_\pi(\chi_\pi,\, \varphi^{-1})$ *in the case* $p = \pi\overline{\pi} \equiv 4 \pmod 9$,

(c) $\mathcal{G}_\pi(\chi_\pi,\, \psi)$ *in the case* $p = \pi\overline{\pi} \equiv 1 \pmod 9$.

So in these cases we have the following expression of the elliptic Gauss sum.

$$\mathcal{G}_\pi(\chi_\pi,\, f) = \sum_{\nu \in S} \chi_\pi(\nu)\, f(\nu/\pi), \tag{1.12}$$

where S is an an arbitrary third set of $(\mathcal{O}/(\pi))^\times$, namely, $(\mathcal{O}/(\pi))^\times = S \cup \rho S \cup \overline{\rho} S$.

As noted in Lemma 1.3, the division values $\varphi(\nu/\pi), \psi(\nu/\pi)$ are algebraic integers in $L = F(\varphi(1/\pi))$, and $\mathrm{Gal}(L/F) \cong (\mathcal{O}/(\pi))^\times$, hence we can immediately see the following.

$$\mathcal{G}_\pi(\chi_\pi,\, f)^{\sigma_\mu} = \overline{\chi}_\pi(\mu)\, \mathcal{G}_\pi(\chi_\pi,\, f) \quad (\mu \in (\mathcal{O}/(\pi))^\times) \tag{1.13}$$

In particular, $\mathcal{G}_\pi(\chi_\pi,\, f)^3$ is an element of F, and furthermore we have

Lemma 1.9. *$\mathcal{G}_\pi(\chi_\pi, f)^3$ is an algebraic integer in $\mathcal{O}$.*

Proof. We show only the integrality of $\mathcal{G}_\pi(\chi_\pi, \varphi^{-1})^3$. Since $\varphi(1/\pi)\,\varphi(\nu/\pi)^{-1}$ is a unit, $\mathfrak{P}^3\,(\mathcal{G}_\pi(\chi_\pi, \varphi^{-1})^3)$ is an integral ideal, where $\mathfrak{P} = (\varphi(1/\pi))$. On the other hand, $(\pi) = \mathfrak{P}^{p-1}$ and $p-1 \geq 12$, hence $(\mathcal{G}_\pi(\chi_\pi, \varphi^{-1})^3)$ must be integral by itself.

Lemma 1.10. $\mathcal{G}_\pi(\chi_\pi, \varphi)^3 \equiv 1 \pmod{\sqrt{-3}}$, $\mathcal{G}_\pi(\chi_\pi, \varphi^{-1})^3 \equiv -1 \pmod{\sqrt{-3}}$ *and* $\mathcal{G}_\pi(\chi_\pi, \psi)^3 \equiv 0 \pmod{\sqrt{-3}}$, *if* $p = \pi\overline{\pi} \equiv 7 \pmod 9$, $p = \pi\overline{\pi} \equiv 4 \pmod 9$ *and* $p = \pi\overline{\pi} \equiv 1 \pmod 9$, *respectively.*

Proof. First, we quote the $\sqrt{-3}$ multiplication formula of $\varphi(u)$ (cf. Appendix) :

$$\varphi(\sqrt{-3}\,u) = \frac{\sqrt{-3}\,\varphi(u)\,\psi(u)}{1+\overline{\rho}\,\varphi(u)^3}.$$

Put $u = \nu/\pi$, then we know $\mathfrak{P} = (\varphi(\nu/\pi)) = (\varphi(\sqrt{-3}\,\nu/\pi))$ and $\psi(\nu/\pi)$ is a unit. Hence an ideal equality $(\varphi(\nu/\pi)^3 + \rho) = (\sqrt{-3})$ holds. Therefore we have

$$\varphi(\nu/\pi)^3 \equiv -1 \pmod{\sqrt{-3}}. \tag{1.14}$$

Next, let $p = \pi\overline{\pi} \equiv 7 \pmod 9$ and S be an arbitrary third set, then we obtain

$$\mathcal{G}_\pi(\chi_\pi, \varphi)^3 \equiv \sum_{\nu\in S} \varphi(\nu/\pi)^3 \equiv \frac{p-1}{3}\cdot(-1) \equiv 1 \pmod{\sqrt{-3}}.$$

For the cases of $\mathcal{G}_\pi(\chi_\pi, \varphi^{-1})$ and $\mathcal{G}_\pi(\chi_\pi, \psi)$, the same argument holds by using

$$\pi\cdot\varphi(\nu/\pi)^{-3} \equiv -1 \pmod{\sqrt{-3}} \quad \text{and} \quad \psi(\nu/\pi)^3 \equiv -1 \pmod{\sqrt{-3}},$$

instead of (1.14), respectively. Thus we complete the proof of Lemma 1.10.

1.3.2. Obviously from (1.13), the value $\mathcal{G}_\pi(\chi_\pi, f)$ belongs to the cubic extension over F. So it is convenient to give an suitable cubic root of π for the precise investigation of the value of $\mathcal{G}_\pi(\chi_\pi, f)$. Thus we introduce an idea of canonical cubic root of π.

Let S be an arbitrary third set of $(\mathcal{O}/(\pi))^\times$. Let $\gamma(S)$ be the cubic root of unity such that $\gamma(S) \equiv -\prod_{\nu\in S} \nu \pmod{(\pi)}$. (cf. [3, (1.6)])

Definition 1.11. The following is called the *canonical cubic root* of π.

$$\tilde{\pi} \doteq \gamma(S)^{-1} \prod_{\nu\in S} \varphi(\nu/\pi). \tag{1.15}$$

Because of the property $\varphi(\rho u) = \rho\varphi(u)$, $\tilde{\pi}$ is independent of the choice of S, and also we can easily show the following by the theory of complex multiplication. (cf. Appendix)

$$\tilde{\pi}^3 = \prod_{\nu \pmod{\pi}} \varphi(\nu/\pi) = \pi.$$

The following is a fundamental property of the cubic residue symbol :

$$\tilde{\pi}^{\sigma_\mu} = \chi_\pi(\mu)\,\tilde{\pi}, \quad (\mu \in (\mathcal{O}/(\pi))^\times) \tag{1.16}$$

which is also easily verified in view of $\gamma(\mu S) = \chi_\pi(\mu)\,\gamma(S)$.

Definition 1.12. The following is called the *coefficient* of the elliptic Gauss sum $\mathcal{G}_\pi(\chi_\pi, f)$, or the elliptic Gauss sum *coefficient.*

$$\alpha_\pi \doteq \tilde{\pi}^{-2}\,\mathcal{G}_\pi(\chi_\pi, f). \tag{1.17}$$

Theorem 1.13. *Let $\tilde{\pi}$ be the canonical cubic root of π. Then we have*

$$\mathcal{G}_\pi(\chi_\pi, f) = \alpha_\pi\,\tilde{\pi}^2,$$

where the coefficient α_π is an algebraic integer in $\mathcal{O}$. Further, it holds

$$\alpha_\pi \equiv \begin{cases} 1 \pmod{\sqrt{-3}} & \text{if } p = \pi\overline{\pi} \equiv 7 \pmod{9}, \\ -1 \pmod{\sqrt{-3}} & \text{if } p = \pi\overline{\pi} \equiv 4 \pmod{9}, \\ 0 \pmod{\sqrt{-3}} & \text{if } p = \pi\overline{\pi} \equiv 1 \pmod{9}. \end{cases} \tag{1.18}$$

Proof. By the definition of the coefficient α_π and by virtue of the properties (1.13) and (1.16), it is valid $\alpha_\pi^{\sigma_\mu} = \alpha_\pi$ $(\mu \in (\mathcal{O}/(\pi))^\times)$, and hence $\alpha_\pi \in F$. For the integrality, we can check it similarly to the proof of Lemma 1.9 : $\alpha_\pi\,\tilde{\pi}^2$ is an algebraic integer, while $\pi = \tilde{\pi}^3$ is a prime in $\mathcal{O}$; it means α_π itself is already an integer. The latter part of Theorem 1.13 is immediately observed by Lemma 1.10.

Remark. When $p \equiv 7$ or $4 \pmod{9}$, we can take $S = \ker\chi_\pi$ as a typical third set of $(\mathcal{O}/(\pi))^\times$; namely, S is the subgroup consisting of all cubic residues mod π. This choice has some advantages. Particularly, it is valid

$$\gamma(S) = 1, \quad \tilde{\pi} = \prod_{\nu\in S} \varphi(\nu/\pi).$$

Example 1.14. Consider the case of $\pi = 4 + 3\rho$ ($p = \pi\overline{\pi} = 13$), and we shall show $\alpha_\pi = -\overline{\rho}$. Take $S = \ker \chi_\pi$. Since $S = \{\pm 1, \pm 5\} = \{1, -1, 1-\overline{\rho}, -2\overline{\rho}\}$, we have

$$\widetilde{\pi} = \prod_{\nu \in S} \varphi(\nu/\pi) = \varphi(1/\pi)\,\varphi(-1/\pi)\,\varphi((1-\overline{\rho})/\pi)\,\varphi(-2\overline{\rho}/\pi).$$

By using suitable multiplication formulas (cf. Appendix) we can compute the right-hand to the following form :

$$\widetilde{\pi} = \frac{(\rho - \overline{\rho})\,\varphi(1/\pi)^4\,(\varphi(1/\pi)^3 - 1)}{(1 + \rho\,\varphi(1/\pi)^3)(1 - 2\,\varphi(1/\pi)^3)}.$$

Therefore $\varphi(1/\pi)$ is a solution of the following equaition.

$$(1+2\rho)\,x^7 + 2\rho\,\widetilde{\pi}\,x^6 - 2(1+2\rho)\,x^4 + (2-\rho)\,\widetilde{\pi}\,x^3 - \widetilde{\pi} = 0.$$

The equation is decomposed as follows.

$$((1+2\rho)\,x^3 + 2\,\widetilde{\pi}\,x^2 + \overline{\rho}\,\widetilde{\pi}^2\,x - 1)(x^4 - 2\overline{\rho}\,\widetilde{\pi}\,x^3 - (1-\rho)\,\widetilde{\pi}^2\,x^2 + \overline{\rho}\,\widetilde{\pi}^3\,x + \widetilde{\pi}) = 0.$$

The second factor must be the minimal polynomial of $\varphi(1/\pi)$ over $F(\widetilde{\pi}) = \mathbf{Q}(\rho, \sqrt[3]{\pi})$. So $\varphi(\nu/\pi)^{-1}$ $(\nu \in S)$ are the four roots of the reciprocal equation.

$$x^4 + \overline{\rho}\,\widetilde{\pi}^2\,x^3 - (1-\rho)\,\widetilde{\pi}\,x^2 - 2\overline{\rho}\,x + \widetilde{\pi}^{-1} = 0.$$

Comparing the second coefficient, we have

$$\mathcal{G}_\pi(\chi_\pi, \varphi^{-1}) = \sum_{\nu \in S} \varphi(\nu/\pi)^{-1} = -\overline{\rho}\,\widetilde{\pi}^2$$

Namely $\alpha_\pi = -\overline{\rho}$, which satisfies certainly $\alpha_\pi \equiv -1 \pmod{\sqrt{-3}}$.

In general, it seems pretty hard to compute the value of the coefficient α_π by hand. Some examples by computor are given in Table 1 at the end.

§ 1.4. The cubic Hecke characters and L-values at $s = 1$

1.4.1. We introduce a Hecke character $\widetilde{\chi}_\pi$ induced by the cubic residue character χ_π. As mentioned before, it is of the form $\widetilde{\chi}_\pi((\nu)) = \chi_1(\nu)\,\overline{\nu}$ with a residue class character χ_1. For the purpose we first modify the character χ_π into χ_1 satisfying $\chi_1(-\rho) = -\rho$. After the preparation of supplementary characters χ_0 and χ_0', we shall treat the three cases separately in view of $\chi_\pi(-\rho) = \rho^{(p-1)/3}$.

Let χ_0 be the character of $(\mathcal{O}/(3))^\times$ defined by

$$\chi_0(\nu) \doteq \varepsilon \quad \text{for} \quad \nu \equiv \varepsilon \pmod 3, \quad \varepsilon \in W = \{\pm 1, \pm\rho, \pm\overline{\rho}\}.$$

We here should notice that χ_0 gives the natural isomorphism $(\mathcal{O}/(3))^\times \cong W$.

Let χ_0' be the character of $(\mathcal{O}/(\sqrt{-3}))^\times$ defined by

$$\chi_0'(\nu) \doteq \delta \quad \text{for} \quad \nu \equiv \delta \pmod{\sqrt{-3}}, \quad \delta \in \{\pm 1\}.$$

We also should notice that χ_0' gives the natural isomorphism $(\mathcal{O}/(\sqrt{-3}))^\times \cong \{\pm 1\}$.

Definition 1.15. For each primary prime π in $\mathcal{O}$, the Hecke character $\widetilde{\chi}_\pi$ for π is defined and fixed throughout §1 as follows :

$$\widetilde{\chi}_\pi((\nu)) \doteq \chi_1(\nu)\,\overline{\nu}, \quad \chi_1 \doteq \begin{cases} \chi_\pi \cdot \overline{\chi}_0 & \text{for} \quad p = \pi\overline{\pi} \equiv 7 \pmod 9, \\ \chi_\pi \cdot \chi_0' & \text{for} \quad p = \pi\overline{\pi} \equiv 4 \pmod 9, \\ \chi_\pi \cdot \chi_0 & \text{for} \quad p = \pi\overline{\pi} \equiv 1 \pmod 9. \end{cases} \tag{1.19}$$

For later use, we present a list of the circumstance of each case.

(a) The case $p = \pi\overline{\pi} \equiv 7 \pmod 9$.

$$(\mathcal{O}/(\beta))^\times \cong (\mathcal{O}/(\pi))^\times \times W \quad \text{by} \quad \lambda \quad \text{to} \quad (\kappa,\, \varepsilon) : \ \lambda \equiv 3\kappa + \pi\varepsilon \pmod{\beta}. \tag{1.20}$$

The conductor of $\widetilde{\chi}_\pi$ is (β) where $\beta = 3\pi$, and we have $\chi_1(\lambda) = \chi_\pi(3)\,\chi_\pi(\kappa)\,\overline{\varepsilon}$.

(b) The case $p = \pi\overline{\pi} \equiv 4 \pmod 9$.

$$(\mathcal{O}/(\beta))^\times \cong (\mathcal{O}/(\pi))^\times \times \{\pm 1\} \quad \text{by} \quad \lambda \quad \text{to} \quad (\kappa,\, \delta) : \ \lambda \equiv \sqrt{-3}\,\kappa + \pi\delta \pmod{\beta}. \tag{1.21}$$

The conductor of $\widetilde{\chi}_\pi$ is (β) where $\beta = \sqrt{-3}\,\pi$, and we have $\chi_1(\lambda) = \overline{\chi}_\pi(3)\,\chi_\pi(\kappa)\,\delta$.

(c) The case $p = \pi\overline{\pi} \equiv 1 \pmod 9$.

$$(\mathcal{O}/(\beta))^\times \cong (\mathcal{O}/(\pi))^\times \times W \quad \text{by} \quad \lambda \quad \text{to} \quad (\kappa,\, \varepsilon) : \ \lambda \equiv 3\kappa + \pi\varepsilon \pmod{\beta}. \tag{1.22}$$

The conductor of $\widetilde{\chi}_\pi$ is (β) where $\beta = 3\pi$, and we have $\chi_1(\lambda) = \chi_\pi(3)\,\chi_\pi(\kappa)\,\varepsilon$.

1.4.2. We are now ready to evaluate the value of the associated L-series at $s = 1$, and we show that $L(1,\, \widetilde{\chi}_\pi)$ is expressed by the corresponding elliptic Gauss sum.

Theorem 1.16. *Let $\widetilde{\chi}_\pi$ be the Hecke character for π. Then*

$$\varpi_1^{-1}\, L(1,\, \widetilde{\chi}_\pi) = \begin{cases} -\chi_\pi(3)\,\pi^{-1}\,\mathcal{G}_\pi(\chi_\pi,\, \varphi) & \text{if} \quad p = \pi\overline{\pi} \equiv 7 \pmod 9, \\ -\overline{\chi}_\pi(3)\,\pi^{-1}\,\mathcal{G}_\pi(\chi_\pi,\, \varphi^{-1}) & \text{if} \quad p = \pi\overline{\pi} \equiv 4 \pmod 9, \\ \chi_\pi(3)\,\pi^{-1}\,\mathcal{G}_\pi(\chi_\pi,\, \psi) & \text{if} \quad p = \pi\overline{\pi} \equiv 1 \pmod 9. \end{cases} \tag{1.23}$$

Proof. We follow the formula (1.10) and refer to (1.20), (1.21) and (1.22).

(a) The case $p=\pi\overline{\pi}\equiv 7 \pmod 9$. In view of (1.20), we have

$$
\begin{aligned}
L(1,\ \tilde{\chi}_\pi) &= \frac{\varpi_1}{6\beta}\cdot \sum_{\lambda \,(\mathrm{mod}\, \beta)} \chi_1(\lambda)\, \mathrm{Z}(\lambda/\beta) \\
&= \frac{\varpi_1}{18\pi}\chi_\pi(3)\cdot \sum_{\kappa \,(\mathrm{mod}\, \pi)} \chi_\pi(\kappa) \sum_{\varepsilon \in W} \bar{\varepsilon}\, \mathrm{Z}(\kappa/\pi+\varepsilon/3) \\
&= -\frac{\chi_\pi(3)\,\varpi_1}{6\cdot\pi}\cdot \sum_{\kappa \,(\mathrm{mod}\, \pi)} \chi_\pi(\kappa)\,\{\varphi(\kappa/\pi)+\varphi(-\kappa/\pi)\} \\
&= -\frac{\chi_\pi(3)\,\varpi_1}{\pi}\cdot\frac{1}{3} \sum_{\kappa \,(\mathrm{mod}\, \pi)} \chi_\pi(\kappa)\,\varphi(\kappa/\pi),
\end{aligned}
$$

by the definition (1.3).

(b) The case $p=\pi\overline{\pi}\equiv 4 \pmod 9$. In view of (1.21), we have

$$
\begin{aligned}
L(1,\ \tilde{\chi}_\pi) &= \frac{\varpi_1}{6\sqrt{-3}\,\pi}\overline{\chi}_\pi(3)\cdot \sum_{\kappa \,(\mathrm{mod}\, \pi)} \chi_\pi(\kappa) \sum_{\delta=\pm 1} \delta\, \mathrm{Z}(\kappa/\pi+\delta/\sqrt{-3}) \\
&= -\frac{\overline{\chi}_\pi(3)\,\varpi_1}{6\cdot\pi}\cdot \sum_{\kappa \,(\mathrm{mod}\, \pi)} \chi_\pi(\kappa)\,\{\varphi(\kappa/\pi)^{-1}+\varphi(-\kappa/\pi)^{-1}\} \\
&= -\frac{\overline{\chi}_\pi(3)\,\varpi_1}{\pi}\cdot\frac{1}{3} \sum_{\kappa \,(\mathrm{mod}\, \pi)} \chi_\pi(\kappa)\,\varphi(\kappa/\pi)^{-1},
\end{aligned}
$$

by the formula (1.6).

(c) The case $p=\pi\overline{\pi}\equiv 1 \pmod 9$. In view of (1.22), we have

$$
\begin{aligned}
L(1,\ \tilde{\chi}_\pi) &= \frac{\varpi_1}{18\pi}\chi_\pi(3)\cdot \sum_{\kappa \,(\mathrm{mod}\, \pi)} \chi_\pi(\kappa) \sum_{\varepsilon \in W} \varepsilon\, \mathrm{Z}(\kappa/\pi+\varepsilon/3) \\
&= \frac{\chi_\pi(3)\,\varpi_1}{6\cdot\pi}\cdot \sum_{\kappa \,(\mathrm{mod}\, \pi)} \chi_\pi(\kappa)\,\{\psi(\kappa/\pi)+\psi(-\kappa/\pi)\} \\
&= \frac{\chi_\pi(3)\,\varpi_1}{\pi}\cdot\frac{1}{3} \sum_{\kappa \,(\mathrm{mod}\, \pi)} \chi_\pi(\kappa)\,\psi(\kappa/\pi),
\end{aligned}
$$

by the definition (1.4). Thus the proof of Theorem 1.16 is finished.

It may be noteworthy that the special elliptic functions $\varphi(u), \varphi(u)^{-1}$ and $\psi(u)$ appear naturally and automatically in these L-series ; consequently the associated L-series themselves would introduce those elliptic functions the division values of which generate some abelian extensions of the field F.

§ 1.5. An explicit formula of the root number $C(\widetilde{\chi}_\pi)$

1.5.1. We require an important formula about the classical cubic Gauss sum. Let π be a primary prime in $\mathcal{O}$; $\pi \equiv 1 \pmod 3$, and set $p = \pi\overline{\pi}$ as before. The cubic Gauss sum, often called the Kummer sum, is defined and denoted here by

$$G_3(\pi) \doteq \sum_{r \,(\mathrm{mod}\, p)} \chi_\pi(r)\, e^{2\pi i r/p} = \sum_{r=1}^{p-1} \left(\frac{r}{\pi}\right)_3 e^{2\pi i r/p}. \tag{1.24}$$

Also we should recall our definition of the canonical cubic root $\widetilde{\pi}$ of π in §1.3.2 :

$$\widetilde{\pi} = \gamma(S)^{-1} \prod_{\nu \in S} \varphi(\nu/\pi) \quad \text{where} \quad \gamma(S)^3 = 1 \quad \text{and} \quad \gamma(S) \equiv -\prod_{\nu \in S} \nu \pmod{\pi},$$

where S is an arbitrary third set of modulus (π), i.e. $(\mathcal{O}/(\pi))^\times = S \cup \rho S \cup \overline{\rho} S$.

Lemma 1.17 (**G_3-formula**).

$$G_3(\pi) = -\chi_\pi(3)\, \widetilde{\pi}^2\, \overline{\widetilde{\pi}}. \tag{1.25}$$

Proof. This is only a slight modification of the celebrated Cassels-Matthews formula. They use the lattice $\theta\,\mathcal{O}$ instead of our $\varpi_1\,\mathcal{O}$, where $\theta = \sqrt{3}\,\varpi_1 = 3.05990807\cdots$. Let $\wp_1(u)$ denote Weierstrass' $\wp$ with the period lattice $\theta\,\mathcal{O}$. Hence the relation $\wp(\varpi_1 u) = 3\,\wp_1(\theta\, u)$ holds, so that $\wp_1'(u)^2 = 4\,\wp_1(u)^3 - 1$. Their formula states

Formula ([3, Theorem 1])

$$G_3(\pi) = -\gamma(S)^{-1}\, \pi\, p^{1/3} \prod_{\nu \in S} \wp_1(\theta\,\nu/\pi). \tag{1.26}$$

We shall show the formula (1.25) from (1.26). Indeed it will be seen they are equivalent.

First, by using the following two identities : the latter being the $\sqrt{-3}$ multiplication,

$$\varphi(u) = \frac{6\,\wp_1(\theta\, u)}{\sqrt{3}(\wp_1'(\theta\, u) + \sqrt{3})} \quad \textit{and} \quad \wp_1(\sqrt{-3}\, u) = -\frac{\wp_1'(u)^2 - 3}{12\,\wp_1(u)^2},$$

we have

$$\varphi(u)^{-1}\varphi(-u)^{-1} = \wp_1(\sqrt{-3}\,\theta\, u).$$

Next, substitute $u = \nu/\pi$ and make the product over $\nu \in S$, then we have

$$\prod_{\nu \in S} \varphi(\nu/\pi)^{-1} \cdot \prod_{\nu \in -S} \varphi(\nu/\pi)^{-1} = \prod_{\nu \in \sqrt{-3}\, S} \wp_1(\theta\,\nu/\pi).$$

Finally, multiply the factor $\gamma(S)^{-1}$ to the both sides and notice such properties as below, then we can see that Cassels-Matthews' formula easily turns to our formula (1.25).

In fact, on the one hand $\gamma(S)^{-1}=\gamma(S)^2=\gamma(S)\cdot\gamma(-S)$, and on the other hand $\gamma(S)^{-1}=\chi_\pi(\sqrt{-3})\,\gamma(\sqrt{-3}\,S)^{-1}=\overline{\chi}_\pi(3)\,\gamma(\sqrt{-3}\,S)^{-1}$.

Remark. As is immediately observed by G_3-formula, $G_3(\pi)^3=-\pi^2\,\overline{\pi}$ holds.

1.5.2. We are now ready to give the value of the root number $C(\widetilde{\chi}_\pi)$ explicitly.

Theorem 1.18. *Let $\widetilde{\chi}_\pi$ be the Hecke character for π. Then*

$$C(\widetilde{\chi}_\pi)=\begin{cases}\chi_\pi(3)\,\widetilde{\pi}^{-1}\overline{\widetilde{\pi}} & \text{if } p=\pi\overline{\pi}\equiv 7 \pmod 9,\\ \overline{\chi}_\pi(3)\,\widetilde{\pi}^{-1}\overline{\widetilde{\pi}} & \text{if } p=\pi\overline{\pi}\equiv 4 \pmod 9,\\ -\chi_\pi(3)\,\widetilde{\pi}^{-1}\overline{\widetilde{\pi}} & \text{if } p=\pi\overline{\pi}\equiv 1 \pmod 9.\end{cases}\tag{1.27}$$

Proof. As a preparation we shall evaluate some simple Gauss sums. The first three are easily verified by direct calculation :

$$\begin{aligned}&g(\chi_0)\doteq\sum_{\varepsilon\in W}\varepsilon\,e^{2\pi iS(\varepsilon/3)}=-3\overline{\rho},\quad g(\overline{\chi}_0)\doteq\sum_{\varepsilon\in W}\overline{\varepsilon}\,e^{2\pi iS(\varepsilon/3)}=3\,\rho\\ &\text{and}\quad g(\chi_0')\doteq\sum_{\delta=\pm1}\delta\,e^{2\pi iS(\delta/\sqrt{-3})}=\sqrt{-3}.\end{aligned}\tag{1.28}$$

The next sum is nothing but the cubic Gauss sum :

$$g(\chi_\pi)\doteq\sum_{\kappa\,(\mathrm{mod}\,\pi)}\chi_\pi(\kappa)\,e^{2\pi iS(\kappa/\pi)}=-\overline{\chi}_\pi(\rho)\,\widetilde{\pi}^2\,\overline{\widetilde{\pi}}.\tag{1.29}$$

In fact, we first replace the sum over $\kappa\,(\mathrm{mod}\,\pi)$ by one over $r\,(\mathrm{mod}\,p)$, and then, by using $S(r\,\overline{\pi}/p)=ar/p$ where $\pi=a+b\rho$ $(a,\,b\in\mathbf{Z})$, we can calculate as follows :

$$g(\chi_\pi)=\sum_{r\,(\mathrm{mod}\,p)}\chi_\pi(r)\,e^{2\pi iS(r\,\overline{\pi}/p)}=\sum_{r\,(\mathrm{mod}\,p)}\chi_\pi(r)\,e^{2\pi iar/p}=\overline{\chi}_\pi(a)\,G_3(\pi).$$

Further, since we know $\overline{\chi}_\pi(a)=\chi_\pi(1-\rho)$ (cf. [1, Chap. 9, Exerc. 24, 26.]), it follows $\overline{\chi}_\pi(a)=\overline{\chi}_\pi(\rho)\,\overline{\chi}_\pi(3)$. Finally, by applying G_3-formula, we can obtain (1.29).

We now follow the formula (1.9) of the root number in Lemma 1.5, and we treat the three cases separately as in §1.4.1, especially in view of (1.20), (1.21) and (1.22).

(a) The case $p=\pi\,\overline{\pi}\equiv 7 \pmod 9$.

Since $\beta=3\,\pi$, $\lambda\equiv 3\,\kappa+\pi\,\varepsilon \pmod\beta$ and $\chi_1(\lambda)=\chi_\pi(3)\,\chi_\pi(\kappa)\,\overline{\varepsilon}$, we have

$$\begin{aligned}C(\widetilde{\chi}_\pi)&=-\rho\,\beta^{-1}\sum_{\lambda\,(\mathrm{mod}\,\beta)}\chi_1(\lambda)\,e^{2\pi i\,S(\lambda/\beta)}\\ &=-\rho\,(\pi\cdot 3)^{-1}\chi_\pi(3)\,g(\overline{\chi}_0)\,g(\chi_\pi)=\chi_\pi(3)\,\widetilde{\pi}^{-1}\overline{\widetilde{\pi}}.\end{aligned}$$

(b) The case $p = \pi\overline{\pi} \equiv 4 \pmod 9$.

Since $\beta = \sqrt{-3}\,\pi$, $\lambda \equiv \sqrt{-3}\,\kappa + \pi\,\delta \pmod{\beta}$ and $\chi_1(\lambda) = \overline{\chi}_\pi(3)\,\chi_\pi(\kappa)\,\delta$, we have

$$C(\widetilde{\chi}_\pi) = -\rho(\sqrt{-3}\,\pi)^{-1} \cdot \chi_\pi(\sqrt{-3}) \cdot g(\chi_0')\,g(\chi_\pi) = \overline{\chi}_\pi(3)\,\widetilde{\pi}^{-1}\widetilde{\overline{\pi}}.$$

(c) The case $p = \pi\overline{\pi} \equiv 1 \pmod 9$.

Since $\beta = 3\,\pi$, $\lambda \equiv 3\,\kappa + \pi\,\varepsilon \pmod{\beta}$ and $\chi_1(\lambda) = \chi_\pi(3)\,\chi_\pi(\kappa)\,\varepsilon$, we have

$$C(\widetilde{\chi}_\pi) = -\rho\,(\pi \cdot 3)^{-1} \cdot \chi_\pi(3) \cdot g(\chi_0)\,g(\chi_\pi) = -\chi_\pi(3)\,\widetilde{\pi}^{-1}\,\widetilde{\overline{\pi}}.$$

These complete the proof of Theorem 1.18.

§ 1.6. Rationality of the elliptic Gauss sum coefficient

1.6.1. In Theorem 1.13 we have seen that each coefficient α_π is an algebraic integer in $\mathcal{O}$. Now we can mention about their $\mathbf{Q}$-rationality. More precisely, the coefficient itself is not always rational, but it will be seen that the essential factor of this is certainly a rational integer. The next is our main theorem of §1.

Theorem 1.19. *For a primary prime π in $\mathcal{O}$ there exists a rational integer a_π, and the coefficient α_π of the elliptic Gauss sum is expressed by a_π as follows.*

$$\alpha_\pi = \begin{cases} \chi_\pi(3)\,a_\pi \quad \text{and} \quad a_\pi \equiv 1 \pmod 3 & \text{if} \quad p = \pi\overline{\pi} \equiv 7 \pmod 9, \\ \overline{\chi}_\pi(3)\,a_\pi \quad \text{and} \quad a_\pi \equiv -1 \pmod 3 & \text{if} \quad p = \pi\overline{\pi} \equiv 4 \pmod 9, \\ \chi_\pi(3)\,a_\pi\,\sqrt{-3} & \text{if} \quad p = \pi\overline{\pi} \equiv 1 \pmod 9. \end{cases} \tag{1.30}$$

Proof. By the theorems 1.13, 1.16 and 1.18 we know already both the explicit values of $L(1, \widetilde{\chi}_\pi)$ and $C(\widetilde{\chi}_\pi)$. To prove Theorem 1.19, we have only to substitute them for the both sides of the central value equation (1.8) of Lemma 1.5. There are three cases :

(a) The case $p = \pi\overline{\pi} \equiv 7 \pmod 9$. In this case we have

$$\varpi_1^{-1}\,L(1, \widetilde{\chi}_\pi) = -\chi_\pi(3)\,\widetilde{\pi}^{-1}\,\alpha_\pi \quad \text{and} \quad C(\widetilde{\chi}_\pi) = \chi_\pi(3)\,\widetilde{\pi}^{-1}\,\widetilde{\overline{\pi}}.$$

Hence from the central value equation $L(1, \widetilde{\chi}_\pi) = C(\widetilde{\chi}_\pi)\,\overline{L(1, \widetilde{\chi}_\pi)}$, we can deduce

$$-\chi_\pi(3)\,\widetilde{\pi}^{-1}\,\alpha_\pi = \chi_\pi(3)\,\widetilde{\pi}^{-1}\,\widetilde{\overline{\pi}} \cdot (-1)\,\overline{\chi}_\pi(3)\,\widetilde{\overline{\pi}}^{-1}\,\overline{\alpha_\pi} \quad \therefore \quad \alpha_\pi = \overline{\chi}_\pi(3)\,\overline{\alpha_\pi}.$$

This means $\overline{\chi}_\pi(3)\,\alpha_\pi = \chi_\pi(3)\,\overline{\alpha_\pi} \in \mathcal{O} \cap \mathbf{R}$, which we may denote by a_π, so that

$$\alpha_\pi = \chi_\pi(3)\,a_\pi \quad \text{where } a_\pi \in \mathbf{Z} \text{ and } a_\pi \equiv 1 \pmod 3.$$

The last congruence follows from Theorem 1.13.

(b) The case $p = \pi\overline{\pi} \equiv 4 \pmod 9$. Since we have

$$\varpi_1^{-1} L(1, \widetilde{\chi}_\pi) = -\overline{\chi}_\pi(3)\,\widetilde{\pi}^{-1}\,\alpha_\pi \quad \text{and} \quad C(\widetilde{\chi}_\pi) = \overline{\chi}_\pi(3)\,\widetilde{\pi}^{-1}\,\overline{\widetilde{\pi}},$$

we can deduce quite similarly to the above

$$\alpha_\pi = \overline{\chi}_\pi(3)\,a_\pi \quad \text{where} \quad a_\pi \in \mathbf{Z} \quad \text{and} \quad a_\pi \equiv -1 \pmod 3.$$

(c) The case $p = \pi\overline{\pi} \equiv 1 \pmod 9$. We know in this case

$$\varpi_1^{-1} L(1, \widetilde{\chi}_\pi) = \chi_\pi(3)\,\widetilde{\pi}^{-1}\,\alpha_\pi \quad \text{and} \quad C(\widetilde{\chi}_\pi) = -\chi_\pi(3)\,\widetilde{\pi}^{-1}\,\overline{\widetilde{\pi}},$$

so that we have $\overline{\chi}_\pi(3)\,\alpha_\pi = -\chi_\pi(3)\,\overline{\alpha_\pi}$, which means

$$\alpha_\pi = \chi_\pi(3)\,a_\pi\,\sqrt{-3} \quad \text{with some} \quad a_\pi \in \mathbf{Z}.$$

Thus the proof is completed.

Example 1.20. We follow Example 1.14, where we evaluated the coefficient of the elliptic Gauss sum $\mathcal{G}_\pi(\chi_\pi, \varphi^{-1}) : \alpha_\pi = -\overline{\rho}$ in the case $\pi = 4 + 3\rho$, $p = \pi\overline{\pi} = 13$. Since we find $\chi_\pi(3) = \rho$ in this case, we can represent this as $\alpha_\pi = -\overline{\rho} = \overline{\chi}_\pi(3)\cdot(-1)$, thus we get $a_\pi = -1$, which satisfies obviously the expected congruence $a_\pi \equiv -1 \pmod 3$. Other examples by computor are given in Table 1 at the end.

Remark. By tracing the process of the proof we can observe a remarkable fact. Under the theorems 1.13, 1.16 and 1.18, the assertions of Theorem 1.19 and Lemma 1.17 (G_3-formula) are equivalent to each other. Therefore if the rationality of the elliptic Gauss sum coefficient would be independently verified beforehand, we can get Cassels-Matthews' formula as a corollary. It might be a natural proof of G_3-formula.

1.6.2. The substance of Theorem 1.19 can be stated by the language of Hecke L-values. The following may be simply regarded as a precise form of Damerell's general result in a very special case. Also it shows that there is a direct relation between the values $L(1, \widetilde{\chi}_\pi)$ and $G_3(\pi)$, especially between their arguments.

Theorem 1.21. *Let a_π be a rational integer as given in Theorem 1.19.*

$$\varpi_1^{-1} L(1, \widetilde{\chi}_\pi) = \begin{cases} p^{1/3}\,G_3(\pi)^{-1}\,a_\pi & \text{if } p = \pi\overline{\pi} \equiv 7 \text{ or } 4 \pmod 9, \\ -p^{1/3}\,\sqrt{-3}\,G_3(\pi)^{-1}\,a_\pi & \text{if } p = \pi\overline{\pi} \equiv 1 \pmod 9. \end{cases} \tag{1.31}$$

Proof. Combining the four theorems and G_3-formula, it is easily verified.

Corollary 1.22. $L(1, \widetilde{\chi}_\pi) \neq 0$ *if* $p = \pi\overline{\pi} \equiv 7$ or $4 \pmod 9$.

Proof. Because $a_\pi \equiv \pm 1 \pmod 3$ in these cases.

Remark. We can observe that $L(1, \widetilde{\chi}_\pi)$ happens to vanish often in the case $p = \pi\overline{\pi} \equiv 1 \pmod 9$. For examples it is the case for each prime as follows :

$$p = 73,\ 271,\ 307,\ 523,\ 577,\ 919,\ 1531,\ 1549,\ 1783,\ 2179,\ 2287,\ 2971,\ 3079,\ 3529,\ \ldots,$$

while any reason or any rule is not known yet. (cf. Table 1)

Appendix. Formulas of special elliptic functions ($\mathbf{Z}[\rho]$-case)

Addition and multiplication formulas of the functions $\varphi(u)$ and $\psi(u)$ are selected. For definitions of these functions cf. §1.1.1. Those formulas listed might be less familiar in comparison with the lemniscatic function case ($\mathbf{Z}[i]$-case). It, however, is not difficult to obtain them. For example, we first deduce the expressions of $\wp(\varpi_1 u)$ and $\wp'(\varpi_1 u)$ in $\varphi(u), \psi(u)$ from (1.5), and substitute them into an ordinary addition formula of $\wp$, $\wp'$, e.g. the determinant formula, to derive the addition formula of φ, ψ, and so forth. Detail of the proof is omitted, while the following may be useful in calculation process.

$$\varphi(u-(1-\rho)/3) = \overline{\rho}\,\varphi(u), \quad \varphi(u+(1-\rho)/3) = \rho\,\varphi(u)$$
$$\psi(u-(1-\rho)/3) = \rho\,\psi(u), \quad \psi(u+(1-\rho)/3) = \overline{\rho}\,\psi(u)$$

For a general survey, one may refer to the book [2, Chap. 8]. While the lemniscatic case is mainly treated there, the cubic case proceeds quite analogously.

1. Addition Formula

(i) $$\varphi(u+v) = \frac{\varphi(u)^2\,\psi(v) - \varphi(v)^2\,\psi(u)}{\varphi(u)\,\psi(v)^2 - \varphi(v)\,\psi(u)^2} = \frac{\varphi(v) + \varphi(u)\,\psi(u)\,\psi(v)^2}{\psi(u) + \varphi(u)^2\,\varphi(v)\,\psi(v)}$$

(ii) $$\psi(u+v) = \frac{\varphi(u)\,\psi(u) - \varphi(v)\,\psi(v)}{\varphi(u)\,\psi(v)^2 - \varphi(v)\,\psi(u)^2} = \frac{\psi(u)^2\,\psi(v) - \varphi(u)\,\varphi(v)^2}{\psi(u) + \varphi(u)^2\,\varphi(v)\,\psi(v)}$$

(iii) $$\varphi(u-v) = \frac{\varphi(u)^2\,\psi(v) - \varphi(v)^2\,\psi(u)}{\varphi(u) + \varphi(v)\,\psi(v)\,\psi(u)^2} = \frac{\varphi(u)\,\psi(u) - \varphi(v)\,\psi(v)}{\psi(u)\,\psi(v)^2 - \varphi(u)^2\,\varphi(v)}$$

(iv) $$\psi(u-v) = \frac{\psi(u)^2\,\psi(v) - \varphi(u)\,\varphi(v)^2}{\psi(u)\,\psi(v)^2 - \varphi(u)^2\,\varphi(v)} = \frac{\varphi(v) + \varphi(u)\,\psi(u)\,\psi(v)^2}{\varphi(u) + \varphi(v)\,\psi(v)\,\psi(u)^2}$$

2. Multiplication Formula

(v) $$\varphi(\rho u) = \rho\varphi(u), \quad \psi(\rho u) = \psi(u), \quad \mathsf{Z}(\rho u) = \overline{\rho}\,\mathsf{Z}(u)$$

(vi) $$\varphi(-u) = -\varphi(u)\,\psi(u)^{-1}, \quad \psi(-u) = \psi(u)^{-1}, \quad \mathsf{Z}(-u) = -\mathsf{Z}(u)$$

(vii) $$\varphi(-2u) = \varphi(u)\cdot\frac{\varphi(u)^3-2}{1-2\,\varphi(u)^3}, \quad \psi(-2u) = \psi(u)\cdot\frac{\psi(u)^3-2}{1-2\,\psi(u)^3}$$

(viii) $$\varphi(\sqrt{-3}\,u) = \frac{\sqrt{-3}\,\varphi(u)\psi(u)}{1+\overline{\rho}\,\varphi(u)^3}, \quad \psi(\sqrt{-3}\,u) = \frac{\rho+\psi(u)^3}{1+\rho\,\psi(u)^3}$$

(ix) $$\mathsf{Z}((1-\rho)\,u) = (1-\overline{\rho})\,\mathsf{Z}(u) + (1-\overline{\rho})\{\varphi(u)^{-1} - \varphi(-u)^{-1}\}$$

3. Primary Prime Multiplication : $p = \pi\overline{\pi}, \quad \pi \equiv 1 \pmod 3$

(x) $$\varphi(\pi\,u) = \varphi(u)\prod_{\nu \,(\mathrm{mod}\,\pi)}\varphi(u+\nu/\pi), \quad \psi(\pi\,u) = \psi(u)\prod_{\nu\,(\mathrm{mod}\,\pi)}\psi(u+\nu/\pi)$$

(xi) $$\prod_{\nu\,(\mathrm{mod}\,\pi)}\varphi(\nu/\pi) = \pi, \quad \prod_{\nu\,(\mathrm{mod}\,\pi)}\psi(\nu/\pi) = 1$$

(xii) $$\varphi(\pi\,u) = \varphi(u)\cdot\frac{U(\varphi(u))}{R(\varphi(u))}, \quad \psi(\pi\,u) = \psi(u)\cdot\frac{U(\psi(u))}{R(\psi(u))}$$

(xiii) $$U(x) = \prod_{\nu\,(\mathrm{mod}\,\pi)}(x-\varphi(\nu/\pi)), \quad R(x) = x^{p-1}\,U(x^{-1})$$

(xiv) $$U(x)\in\mathcal{O}[x], \quad U(x)\equiv x^{p-1} \pmod{\pi}, \quad U(0) = \pi$$

Proof of (x) (sketch) : Comparing the divisors and the values at u=1/3.
Proof of (xii) (sketch) : By using the first form of (i),

$$\prod_{\varepsilon\in W'}\varphi(u+\varepsilon\,v) = \frac{\psi(v)^3\,(\varphi(u)^3-\varphi(-v)^3)}{1-\varphi(u)^3\,\varphi(v)^3}, \quad \prod_{\varepsilon\in W'}\varphi(u-\varepsilon\,v) = \frac{\varphi(u)^3-\varphi(v)^3}{\psi(v)^3\,(1-\varphi(u)^3\,\varphi(-v)^3)},$$

where $W = \{\pm1, \pm\rho, \pm\overline{\rho}\}$ and $W' = \{1, \rho, \overline{\rho}\}$, and therefore

$$\prod_{\varepsilon\in W}\varphi(u+\varepsilon\,v) = \frac{(\varphi(u)^3-\varphi(-v)^3)\,(\varphi(u)^3-\varphi(v)^3)}{(1-\varphi(u)^3\,\varphi(v)^3)\,(1-\varphi(u)^3\,\varphi(-v)^3)} = \prod_{\varepsilon\in W}\frac{\varphi(u)-\varphi(\varepsilon\,v)}{1-\varphi(u)\,\varphi(\varepsilon\,v)}.$$

This combined with (x) leads to (xii) ; here U is an arbitrary $\frac{1}{6}$-set mod π :

$$\frac{\varphi(\pi\,u)}{\varphi(u)} = \prod_{\nu\in U}\prod_{\varepsilon\in W}\varphi(u+\varepsilon\,\nu/\pi)$$
$$= \prod_{\nu\in U}\prod_{\varepsilon\in W}\frac{\varphi(u)-\varphi(\varepsilon\nu/\pi)}{1-\varphi(\varepsilon\nu/\pi)\,\varphi(u)} = \prod_{\nu\,(\mathrm{mod}\,\pi)}\frac{\varphi(u)-\varphi(\nu/\pi)}{1-\varphi(\nu/\pi)\,\varphi(u)}.$$

§ 2. The Quartic Character Case

Throughout §2, the field $\mathbf{Q}(i)$ and the ring $\mathbf{Z}[i]$ are abbreviated to F and $\mathcal{O}$, respectively. The unit group is denoted by $W = \{\pm 1, \pm i\}$. The set $\mathcal{O}$ or its constant multiple appears also as a period lattice for elliptic functions. Though we don't treat the octic case, we shall come on a scene to need the eighth root of unity $\zeta_8 = e^{2\pi i/8}$ and so it is not strange to meet $\sqrt{2} = (1-i)\,\zeta_8$ or $i\sqrt{2} = (1+i)\,\zeta_8$ in some formulas.

§ 2.1. Special elliptic functions with complex multiplication

2.1.1. We shall define some special functions which play the leading role in our argument. Let $\wp(u)$ denote the Weierstrass function respect to the period lattice $\varpi\mathcal{O}$ so that $\wp'(u)^2 = 4\wp(u)^3 - 4\wp(u)$; the real period ϖ is given by

$$\varpi \doteq 2\int_0^1 \frac{dx}{\sqrt{1-x^4}} = 2.62205755\cdots.$$

It is obvious $\wp(\varpi u)$ and $\wp'(\varpi u)$ are elliptic functions of periods $\mathcal{O}$. Further, we can get another doubly periodic function by a slight modification of Weierstrass' ζ of $\varpi\mathcal{O}$:

Definition 2.1. The non-analytic doubly periodic function $\mathsf{Z}(u)$ is defined by

$$\mathsf{Z}(u) \doteq \zeta(\varpi u) - \frac{\pi}{\varpi}\,\overline{u}. \tag{2.1}$$

Double periodicity of Z relative to $\mathcal{O}$ is easily verified by a usual formula of ζ. The addition formula of Z, that follows immediately from one of ζ, is useful :

$$\mathsf{Z}(u+v) = \mathsf{Z}(u) + \mathsf{Z}(v) + \frac{1}{2}\,\frac{\wp'(\varpi u) - \wp'(\varpi v)}{\wp(\varpi u) - \wp(\varpi v)}. \tag{2.2}$$

In particular, such a function $\sum_{k=1}^{r} c_k\,\mathsf{Z}(u+\gamma_k)$ is an elliptic function if $\sum_{k=1}^{r} c_k = 0$.

The following two functions are specially important, really which are nothing but the old lemniscatic sine and cosine functions of Gauss.

Definition 2.2. The elliptic functions $\varphi(u)$ and $\psi(u)$ relative to the period lattice $\mathcal{O}$ are defined by

$$\varphi(u) \doteq -\frac{1-i}{2}\left\{\mathsf{Z}\left(u-\frac{1}{2}\right) - \mathsf{Z}\left(u-\frac{i}{2}\right)\right\}, \tag{2.3}$$

$$\psi(u) \doteq -\frac{1-i}{2}\left\{\mathsf{Z}\left(u-\frac{1-i}{4}\right) - \mathsf{Z}\left(u+\frac{1-i}{4}\right)\right\}. \tag{2.4}$$

From (2.2) we can easily derive the other expressions :

$$\varphi(u) = -2(1-i)\cdot\frac{\wp(\varpi u)}{\wp'(\varpi u)},\quad \psi(u) = \frac{\wp(\varpi u)+i}{\wp(\varpi u)-i}. \tag{2.5}$$

We need also the following formula.

$$\varphi(u)^{-1} = \frac{1+i}{2}\left\{\mathsf{Z}(u) - \mathsf{Z}\left(u - \frac{1+i}{2}\right)\right\}. \tag{2.6}$$

Here are some basic properties of these functions, which are all classical or easily deduced from the definitions and by usual theory of elliptic functions.

$$\operatorname{div}(\varphi) = (0) + ((1+i)/2) - (1/2) - (i/2),$$

$$\operatorname{div}(\psi) = ((1+i)/4) + (-(1+i)/4) - ((1-i)/4) - (-(1-i)/4),$$

$$\mathsf{Z}(iu) = -i\,\mathsf{Z}(u),\quad \varphi(iu) = i\,\varphi(u),\quad \psi(iu) = \psi(u)^{-1},$$

$$\mathsf{Z}(-u) = -\mathsf{Z}(u),\quad \varphi(-u) = -\varphi(u)\,\psi(u)^{-1},\quad \psi(-u) = \psi(u)^{-1},$$

$$\psi(u) = \varphi(u+(1+i)/4),\quad \varphi(u)^{-1} = -i\,\varphi(u+1/2),\quad \psi(0) = \varphi((1+i)/4) = 1,$$

$$\varphi'(u) = (1-i)\varpi\,\psi(u)\,(1+\varphi(u)^2),\quad \psi'(u) = -(1-i)\varpi\,\varphi(u)\,(1+\psi(u)^2),$$

$$\varphi(u)^2\psi(u)^2 + \varphi(u)^2 + \psi(u)^2 = 1.$$

In particular $(((1-i)\varpi)^{-1}\varphi'(u))^2 = 1-\varphi(u)^4$ holds, and so we can ascertain that $\varphi(u) = \mathrm{sl}((1-i)\varpi u)$ and $\psi(u) = \mathrm{cl}((1-i)\varpi u)$ by Gauss' lemniscatic sine and cosine.

We can refer to the survey monograph [2, esp. Chap.8] by F. Lemmermeyer for further general facts and some background of these elliptic functions. (cf. Appendix)

2.1.2. Let π be a complex prime in $\mathcal{O}$ so that $p = \pi\overline{\pi} \equiv 1 \pmod 4$, and we assume also π is primary : $\pi \equiv 1 \pmod{(1+i)^3}$. Then we have $(\mathcal{O}/(\pi))^\times \cong (\mathbf{Z}/p\mathbf{Z})^\times$. We often abbreviate like as $\nu \,(\mathrm{mod}\,\pi)$ in such a case when ν runs over $(\mathcal{O}/(\pi))^\times$.

It is well known that such a division value $\varphi(1/\pi)$ or $\psi(1/\pi)$ generates an abelian extension of the imaginary quadratic field $F = \mathbf{Q}(i)$. In fact,

Lemma 2.3. *Let* $L = F(\varphi(1/\pi))$ *and* $L_1 = F(\psi(1/\pi))$. *One has*

(i) *L/F is a cyclic extension of degree $p-1$, and L_1 is a subfield of $[L : L_1] = 2$.*

(ii) *$\mathrm{Gal}(L/F) \cong (\mathcal{O}/(\pi))^\times$ by corresponding σ_μ to μ, and it holds $\varphi(\nu/\pi)^{\sigma_\mu} = \varphi(\mu\nu/\pi)$, $\psi(\nu/\pi)^{\sigma_\mu} = \psi(\mu\nu/\pi)$ for arbitrary $\mu, \nu \in (\mathcal{O}/(\pi))^\times$.*

(iii) *$\varphi(1/\pi)$, $\psi(1/\pi)$ are algebraic integers, and particularly $\psi(1/\pi)$ is a unit.*

(iv) *The prime ideal (π) splits completely in L : $(\pi) = \mathfrak{P}^{p-1}$ where $\mathfrak{P} = (\varphi(1/\pi))$.*

L and L_1 are nothing but the ray class fields of the conductors $((1+i)^3\pi)$ and $((1+i)^2\pi)$, respectively. The proof is omitted, but it may be found mostly in [2, Chap. 8]. We only give some numerical examples below.

The following is also very classical and originated from Eisenstein, but we state it as a lemma because of the special utility.

Lemma 2.4. *For arbitrary $\nu \in (\mathcal{O}/(\pi))^\times$,*

$$\varphi(\nu/\pi)^2 \equiv 1 \pmod{(1+i)}. \tag{2.7}$$

Proof. Recall the $(1+i)$-multiplication formula of $\varphi(u)$: (cf. Appendix)

$$\varphi((1+i)\,u) = \frac{(1+i)\,\varphi(u)\,\psi(u)}{1-\varphi(u)^2}.$$

Substituting $u = \nu/\pi$ and cancelling by $\mathfrak{P} = (\varphi(\nu/\pi)) = (\varphi((1+i)\nu/\pi))$, we obtain an ideal equality $(1-\varphi(\nu/\pi)^2) = (1+i)$, which implies (2.7).

As for the division value $\mathsf{Z}(1/\pi)$, we have

Lemma 2.5. *The value $\mathsf{Z}(1/\pi)$ generates the same extension L. Namely,*

$$L = F(\varphi(1/\pi)) = F(\mathsf{Z}(1/\pi)) \quad \text{and} \quad \mathsf{Z}(\nu/\pi)^{\sigma_\mu} = \mathsf{Z}(\mu\nu/\pi) \quad (\,\mu,\nu \in (\mathcal{O}/(\pi))^\times\,).$$

Proof. By substituting $v = iu$ in the addition formula (2.2), we have

$$\mathsf{Z}((1+i)u) = (1-i)\,\mathsf{Z}(u) + i\,\varphi(u)^{-1}. \tag{2.8}$$

Using the formula (2.8) repeatedly and in view of $(1+i)^{p-1} \equiv 1 \pmod{\pi}$, we have

$$(1-(-4)^{(p-1)/4})\,\mathsf{Z}(\nu/\pi) = i\sum_{k=1}^{p-1}(1-i)^{p-1-k}\,\varphi((1+i)^{k-1}\nu/\pi)^{-1}.$$

Now it is easy to see the assertion of Lemma 2.5.

Example 2.6. In each case tabulated below, π-multiplication formula holds:

$$\varphi(\pi\,u) = \varphi(u)\cdot\frac{U(\varphi(u))}{R(\varphi(u))}, \quad R(x) = x^{p-1}\,U(x^{-1}),$$

$$U(x) = \prod_{\nu\,(\mathrm{mod}\,\pi)}(x-\varphi(\nu/\pi)), \quad V(x)^2 = \prod_{\nu\,(\mathrm{mod}\,\pi)}(x-\psi(\nu/\pi)),$$

$$x\,U(x) - R(x) = (x-1)\,V(x)^2.$$

Furthermore it is easy to check

$$U(x),\ V(x) \in \mathcal{O}[x], \quad U(x) \equiv x^{p-1} \pmod{(\pi)}, \quad U(0) = \pi, \quad V(0) = 1.$$

In fact, $U(x)$ and $V(x)$ are the minimal polynomials of $\varphi(1/\pi)$ and $\psi(1/\pi)$, respectively.

p	π	$U(x)$
5	$-1+2i$	$x^4+\pi$
13	$3+2i$	$x^{12}-(1-4i)\,\pi\,x^8+(1-2i)\,\pi\,x^4+\pi$
17	$1+4i$	$x^{16}-(4+4i)\,\pi\,x^{12}+(6+4i)\,\pi\,x^8-(4-4i)\,\pi\,x^4+\pi$

p	π	$V(x)$
5	$-1+2i$	$x^2+(1-i)\,x+1$
13	$3+2i$	$x^6-(1+i)\,x^5-(1+2i)\,x^4-4i\,x^3-(1+2i)\,x^2-(1+i)\,x+1$
17	$1+4i$	$x^8-2i\,x^7+(2-2i)\,x^6+(4+2i)\,x^5+2\,x^4+(4+2i)\,x^3+(2-2i)\,x^2-2i\,x+1$

§ 2.2. *L*-series for Hecke characters of weight one

2.2.1. Let $\widetilde{\chi}$ denote a Hecke character of weight one relative to a modulus (β), namely it is a multiplicative function on the ideal group of $\mathcal{O}$ of the following form :

$$\widetilde{\chi}((\nu)) = \chi_1(\nu)\overline{\nu}, \quad \chi_1 : (\mathcal{O}/(\beta))^\times \to \mathbf{C}^\times, \quad \chi_1(\varepsilon) = \varepsilon \quad (\varepsilon \in W),$$

where χ_1 is an ordinary residue class character to the modulus (β), and (β) is called the coductor of $\widetilde{\chi}$ if χ_1 is a primitive character of the modulus (β).

It is well known the associated *L*-series has the analytic continuation and satisfies a functional equation. We follow Weil's argument and his notation in [5].

$$\begin{aligned} L(s,\ \widetilde{\chi}) &= \sum_{\mathfrak{a}} \widetilde{\chi}(\mathfrak{a})\,N\mathfrak{a}^{-s} = \frac{1}{4}\sum_{\nu\in\mathcal{O}} \chi_1(\nu)\,\overline{\nu}\,|\nu|^{-2s} \\ &= \frac{1}{4}\sum_{\lambda\,(\mathrm{mod}\,\beta)} \chi_1(\lambda) \sum_{\mu\in\mathcal{O}} (\overline{\lambda}+\overline{\mu\beta})\,|\lambda+\mu\beta|^{-2s}, \end{aligned}$$

$$L(s,\ \widetilde{\chi}) = \beta^{-1}\,|\beta|^{2-2s}\cdot\frac{1}{4}\sum_{\lambda\,(\mathrm{mod}\,\beta)} \chi_1(\lambda)\,K_1(\lambda/\beta,\ 0,\ s). \tag{2.9}$$

Here the function K_1 is defined for $\mathrm{Re}\,s > 3/2$ as follows, and it is analytically continued to the whole s-plane and satisfies the own functional equation : (cf. [5, Chap. VIII])

$$K_1(u, u_0, s) = \sum_{\mu\in\mathcal{O}} e^{\pi\,(\overline{u}_0\mu - u_0\overline{\mu})}\,(\overline{u}+\overline{\mu})\,|u+\mu|^{-2s},$$

$$\pi^{-s}\,\Gamma(s)\,K_1(u, u_0, s) = e^{\pi\,(\overline{u}_0 u - u_0\overline{u})}\,\pi^{s-2}\,\Gamma(2-s)\,K_1(u_0, u, 2-s).$$

When (β) is the conductor of $\widetilde{\chi}$, a usual computation of Gauss sum works. From this combined with the above, the functional equation of Hecke L-series is derived :

$$\Lambda(s,\ \widetilde{\chi}) = C(\widetilde{\chi})\,\Lambda(2-s,\ \overline{\widetilde{\chi}}),$$

where

$$\Lambda(s,\ \widetilde{\chi}) = \left(\frac{2\pi}{\sqrt{4\cdot N(\beta)}}\right)^{-s} \Gamma(s)\,L(s,\ \widetilde{\chi}),$$

and

$$C(\widetilde{\chi}) = -i\,\beta^{-1} \sum_{\lambda\,(\mathrm{mod}\,\beta)} \chi_1(\lambda)\,e^{2\pi i\,\mathrm{Re}(\lambda/\beta)}.$$

In particular, we have a simple equality of Hecke L-values at $s=1$:

Lemma 2.7. *Let $\widetilde{\chi}$ be a Hecke character as above. Then we have*

$$L(1,\ \widetilde{\chi}) = C(\widetilde{\chi})\,\overline{L(1,\ \overline{\widetilde{\chi}})}, \tag{2.10}$$

$$C(\widetilde{\chi}) = -i\,\beta^{-1} \sum_{\lambda\,(\mathrm{mod}\,\beta)} \chi_1(\lambda)\,e^{2\pi i\,\mathrm{Re}(\lambda/\beta)}. \tag{2.11}$$

Remark. $L(1,\ \overline{\widetilde{\chi}}) = \overline{L(1,\ \widetilde{\chi})}$.

We call (2.10) *the central value equation*, and the constant $C(\widetilde{\chi})$ *the root number.*

2.2.2. Now we can notice that the value $L(1,\ \widetilde{\chi})$ relates to some elliptic functions. As is known (e.g. [5, Chap. VIII, §14]), the following is valid.

$$E_1^*(u) \doteq K_1(u,\ 0,\ 1) = \varpi\,\zeta(\varpi u) - \pi\,\overline{u}.$$

By the definition (2.1) the right-hand side is nothing but our function $\varpi\,\mathrm{Z}(u)$. Combining this with the equation (2.9) at $s=1$, we obtain the following formula :

Lemma 2.8. *Let $\widetilde{\chi}$ be a Hecke character as above. Then we have*

$$\varpi^{-1}\,L(1,\ \widetilde{\chi}) = \frac{1}{4\beta} \sum_{\lambda\,(\mathrm{mod}\,\beta)} \chi_1(\lambda)\,\mathrm{Z}(\lambda/\beta). \tag{2.12}$$

Example 2.9. The following is probably the simplest case of Hecke L-value at $s=1$ in this case, and the derived formula $L(1,\ \widetilde{\chi}_0) = \dfrac{\varpi}{4}$ may be compared with the classical formula : $1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \cdots = \dfrac{\pi}{4}$.

The Hecke character $\widetilde{\chi}_0$ of the conductor $(1+i)^3$ (cf. §2.4.1) is given as follows :

$$\widetilde{\chi}_0((\nu)) \doteq \chi_0(\nu)\,\overline{\nu}, \quad \textit{where}\ \chi_0\ :\ (\mathcal{O}/(1+i)^3)^\times \cong W \ \textit{ is the natural isomorphism.}$$

Then we can evaluate the L-value at $s=1$ directly by (2.12) and by using (2.6)

$$\varpi^{-1} L(1,\ \widetilde{\chi}_0) = \frac{1}{4(1+i)^3}\sum_{\varepsilon\in W} \varepsilon\, \mathsf{Z}(-(1+i)\varepsilon/4) = \frac{1+i}{4}\,\mathsf{Z}((1+i)/4) = \frac{1}{4}.$$

Also we can easily check $C(\widetilde{\chi}_0) = -i\,(1+i)^{-3}\sum_{\varepsilon\in W} \varepsilon\, e^{2\pi i\,\mathrm{Re}(-(1+i)\varepsilon/4)} = 1$ and hence the expected central value equation holds.

§ 2.3. Elliptic Gauss sums for quartic characters

2.3.1. Let π be a primary prime in $\mathcal{O}$; $\pi \equiv 1 \pmod{(1+i)^3}$. Let χ_π be the quartic residue character to the modulus (π) and the notation will be fixed throughout :

$$\chi_\pi(\nu) = \left(\frac{\nu}{\pi}\right)_4\ :\ \chi_\pi(\nu)^4 = 1 \ \textit{ and }\ \chi_\pi(\nu) \equiv \nu^{(p-1)/4} \pmod{\pi}\quad (\nu\in(\mathcal{O}/(\pi))^\times).$$

Let $f(u)$ be a doubly periodic function of the period $\mathcal{O}$, which we specify below.

Definition 2.10. The following is called an *elliptic Gauss sum.*

$$\mathcal{G}_\pi(\chi_\pi,\ f) \doteq \frac{1}{r}\sum_{\nu\,(\mathrm{mod}\,\pi)} \chi_\pi(\nu)\, f(\nu/\pi), \quad r = \begin{cases} 4 & \text{for } p\equiv 5 \pmod 8, \\ 2 & \text{for } p\equiv 1 \pmod 8. \end{cases} \tag{2.13}$$

In §2, we deal with the four types of elliptic Gauss sums $\mathcal{G}_\pi(\chi_\pi,\ \varphi)$, $\mathcal{G}_\pi(\chi_\pi,\ \mathsf{Z})$, $\mathcal{G}_\pi(\chi_\pi,\ \psi)$ and $\mathcal{G}_\pi(\chi_\pi,\ \varphi^{-1})$; the last one is for a supplementary use. So we understand that $f(u)$ denotes one of these functions $\varphi(u)$, $\mathsf{Z}(u)$, $\psi(u)$, and $\varphi(u)^{-1}$ in the subsequence. In these cases, if "the parity condition" is not satisfied, $\mathcal{G}_\pi(\chi_\pi,\ f)$ vanishes trivially. Since $\varphi(iu) = i\,\varphi(u)$, $\mathsf{Z}(iu) = -i\,\mathsf{Z}(u)$, $\psi(-u) = \psi(u)$ and $\chi_\pi(i) = i^{(p-1)/4}$, we can easily check that $\mathcal{G}_\pi(\chi_\pi,\ f)$ is not trivial only in the following cases. The parity condition, however, is not sufficient for non-vanishing of the elliptic Gauss sum as we shall see later. (cf. the last remark of §2.6.2)

The elliptic Gauss sums that we shall consider are the following :

(a) $\mathcal{G}_\pi(\chi_\pi,\ \varphi)$ *for the case* $p = \pi\overline{\pi} \equiv 13 \pmod{16}$,

(b) $\mathcal{G}_\pi(\chi_\pi,\ \mathsf{Z})$, $\mathcal{G}_\pi(\chi_\pi,\ \varphi^{-1})$ *for the case* $p = \pi\overline{\pi} \equiv 5 \pmod{16}$ *and* $p > 5$,

(c) $\mathcal{G}_\pi(\chi_\pi,\ \psi)$ *for the case* $p = \pi\overline{\pi} \equiv 1 \pmod 8$.

As noted in Lemma 2.3 and Lemma 2.5, $\mathcal{G}_\pi(\chi_\pi, f) \in L$ and $f(\mu\nu/\pi) = f(\nu/\pi)^{\sigma_\mu}$ are valid, and hence we can immediately deduce the property of Lagrange's resolvent :

$$\mathcal{G}_\pi(\chi_\pi, f)^{\sigma_\mu} = \overline{\chi}_\pi(\mu)\, \mathcal{G}_\pi(\chi_\pi, f) \quad (\mu \in (\mathcal{O}/(\pi))^\times) \tag{2.14}$$

In particular, $\mathcal{G}_\pi(\chi_\pi, f)^4$ is an element of F, and furthermore we have

Lemma 2.11. *$\mathcal{G}_\pi(\chi_\pi, \varphi)^4$, $\mathcal{G}_\pi(\chi_\pi, \varphi^{-1})^4$ and $\mathcal{G}_\pi(\chi_\pi, \psi)^4$ are algebraic integers in $\mathcal{O}$.*

Proof. We must show the algebraic integrality of $\mathcal{G}_\pi(\chi_\pi, f)$ for each case. Let S be an arbitrary quarter subset mod (π), namely $(\mathcal{O}/(\pi))^\times = S \cup -S \cup i\,S \cup -i\,S$.

(a) The case $p = \pi\overline{\pi} \equiv 13 \pmod{16}$.

We have $\mathcal{G}_\pi(\chi_\pi, \varphi) = \sum_{\nu\in S} \chi_\pi(\nu)\, \varphi(\nu/\pi)$. Since $\varphi(\nu/\pi)$'s are algebraic integers, the integrality is obvious in this case.

(b) The case $p = \pi\overline{\pi} \equiv 5 \pmod{16}$, $p > 5$.

We have $\mathcal{G}_\pi(\chi_\pi, \varphi^{-1}) = \sum_{\nu\in S} \chi_\pi(\nu)\, \varphi(\nu/\pi)^{-1}$, too. Since $\varphi(1/\pi)\varphi(\nu/\pi)^{-1}$ is a unit, $\mathfrak{P}^4\,(\mathcal{G}_\pi(\chi_\pi, \varphi^{-1})^4)$ is an integral ideal, where $\mathfrak{P} = (\varphi(1/\pi))$. On the other hand, $(\pi) = \mathfrak{P}^{p-1}$ and $p-1 > 4$, and hence $(\mathcal{G}_\pi(\chi_\pi, \varphi^{-1})^4)$ must be already integral ideal. We need the condition $p > 5$ in this case ; in fact we can easily check $\mathcal{G}_\pi(\chi_\pi, \varphi^{-1})^4 = -\pi^{-1} \notin \mathcal{O}$ in the case $\pi = -1 + 2i$ $(p = 5)$.

(c) The case $p = \pi\overline{\pi} \equiv 1 \pmod 8$.

In this case we have $\mathcal{G}_\pi(\chi_\pi, \psi) = \sum_{\nu\in S\cup iS} \chi_\pi(\nu)\psi(\nu/\pi)$, and hence the integrality is obvious. It should be remarked that $\chi_\pi(\nu)\psi(\nu/\pi) \neq \chi_\pi(i\nu)\psi(i\nu/\pi)$ in general, which is the reason why we put $r = 2$ in (2.13) of this case.

Thus the proof of Lemma 2.11 is completed.

For the integrality of $\mathcal{G}_\pi(\chi_\pi, \mathsf{Z})$, as a matter of fact, it is also valid

Claim (Z) : *$\mathcal{G}_\pi(\chi_\pi, \mathsf{Z})^4$ is an algebraic integer in $\mathcal{O}$.*

Although the proof is a bit indirect and will be completed after Theorem 2.22, we shall often assume the claim for convenience' sake. We here only prepare the following.

Lemma 2.12. *One has*

$$((1+i) - i\,\overline{\chi}_\pi(1+i))\, \mathcal{G}_\pi(\chi_\pi, \mathsf{Z}) = \mathcal{G}_\pi(\chi_\pi, \varphi^{-1}). \tag{2.15}$$

Proof. This is immediately derived from the formula (2.8).

Lemma 2.13. $\mathcal{G}_\pi(\chi_\pi, \varphi)^4 \equiv 1 \pmod 2$ *and* $\mathcal{G}_\pi(\chi_\pi, \varphi^{-1})^4 \equiv 1 \pmod 2$, *for* $p = \pi\bar{\pi} \equiv 13 \pmod{16}$ *and* $p = \pi\bar{\pi} \equiv 5 \pmod{16}$, *respectively.*

Proof. By Lemma 2.4 we can deduce $\varphi(\nu/\pi)^4 \equiv 1 \pmod 2$. Hence we obtain

$$\mathcal{G}_\pi(\chi_\pi, \varphi)^4 \equiv \sum_{\nu \in S} \varphi(\nu/\pi)^4 \equiv (p-1)/4 \equiv 1 \pmod 2.$$

Similarly, using $\pi\varphi(\nu/\pi)^{-4} \equiv 1 \pmod 2$, we can prove $\mathcal{G}_\pi(\chi_\pi, \varphi^{-1})^4 \equiv 1 \pmod 2$.

2.3.2. Obviously from (2.14), the value $\mathcal{G}_\pi(\chi_\pi, f)$ belongs to the quartic extension field over F. So it is convenient to give a suitable quartic root of $-\pi$ or π for the precise investigation of the value of $\mathcal{G}_\pi(\chi_\pi, f)$. Thus we shall introduce an idea of the canonical quartic root of $-\pi$.

Let S be an arbitrary quarter subset of $(\mathcal{O}/(\pi))^\times$; hence $(\mathcal{O}/(\pi))^\times = \bigcup_{\varepsilon \in W} \varepsilon S$. First of all, we notice the following two equations :

$$\prod_{\nu \in S} \nu^4 \equiv \chi_\pi(-1) \cdot (p-1)! \equiv -\chi_\pi(-1) \pmod{\pi},$$

$$\prod_{\nu \in S} \varphi(\nu/\pi)^4 = \chi_\pi(-1) \prod_{\nu \in (\mathcal{O}/(\pi))^\times} \varphi(\nu/\pi) = \chi_\pi(-1)\pi.$$

By the first equation we can define a quartic or octic root of unity according to each S as follows. In the case $p \equiv 5 \pmod 8$, i.e. $\chi_\pi(-1) = -1$, we put and denote by $\gamma(S)$ the quartic root of unity determined by the property $\gamma(S) \equiv \prod_{\nu \in S} \nu \pmod{(\pi)}$. In the case $p \equiv 1 \pmod 8$, i.e. $\chi_\pi(-1) = 1$, the prime π is decomposable in $\mathbf{Z}[\zeta_8]$. Let Π be a prime once chosen and fixed such that $\pi = \Pi\Pi'$ in $\mathbf{Z}[\zeta_8]$. We denote by $\gamma(S)$ the quartic root of -1 such that $\gamma(S) \equiv \prod_{\nu \in S} \nu \pmod{(\Pi)}$. Note $\gamma(S) \notin W$, but $\zeta_8\gamma(S) \in W$ in this case. Unfortunately $\gamma(S)$ depends on either choice of (Π), and the sign will be changed when another (Π') is chosen, that is, only $\gamma(S)^2$ is an invariant meaning with respect to π and S. We should also remark that $\gamma(S)^2 \equiv \prod_{\nu \in S} \nu^2 \pmod{(\pi)}$ is valid for both cases.

Definition 2.14. The following is called the *canonical quartic root* of $-\pi$.

$$\tilde{\pi} \doteq \gamma(S)^{-1} \prod_{\nu \in S} \varphi(\nu/\pi). \tag{2.16}$$

We have $\tilde{\pi}^4 = -\pi$, and $\tilde{\pi}$ is independent of the choice of S because of the property $\varphi(iu) = i\varphi(u)$. As is remarked in the above, there is an ambiguity of the sign of $\tilde{\pi}$ in the case $p \equiv 1 \pmod 8$. Also we should remark that $\tilde{\pi} \notin L = F(\varphi(1/\pi))$ but $\zeta_8\tilde{\pi} \in L$, and $(\zeta_8\tilde{\pi})^4 = \pi$ in the case $p \equiv 1 \pmod 8$, while $\tilde{\pi} \in L$ holds in the case $p \equiv 5 \pmod 8$.

Remark. In the case $p \equiv 5 \pmod 8$, we can take $S = \ker \chi_\pi$ as a quarter subset of $(\mathcal{O}/(\pi))^\times$; then S is the subgroup consisting of all quartic residues mod (π). This choice has some advantages. Particularly, it is valid

$$\mathcal{G}_\pi(\chi_\pi, f) = \sum_{\nu \in S} f(\nu/\pi), \quad \gamma(S) = 1, \quad \widetilde{\pi} = \prod_{\nu \in S} \varphi(\nu/\pi).$$

The following is a fundamental property of the quartic residue symbol, which is also directly verified in view of $\gamma(\mu S) = \chi_\pi(\mu)\,\gamma(S)$.

$$\widetilde{\pi}^{\sigma_\mu} = \chi_\pi(\mu)\,\widetilde{\pi}, \quad (\mu \in (\mathcal{O}/(\pi))^\times) \tag{2.17}$$

where we mean $\widetilde{\pi}^{\sigma_\mu} = \zeta_8^{-1}(\zeta_8\widetilde{\pi})^{\sigma_\mu}$ in the strict meaning when $p \equiv 1 \pmod 8$.

Definition 2.15. The following is called the *coefficient* of the elliptic Gauss sum $\mathcal{G}_\pi(\chi_\pi, f)$, or simply the elliptic Gauss sum coefficient.

$$\alpha_\pi \doteq \widetilde{\pi}^{-3}\,\mathcal{G}_\pi(\chi_\pi, f). \tag{2.18}$$

Theorem 2.16. *The elliptic Gauss sum is expressible as follows :*

$$\mathcal{G}_\pi(\chi_\pi, f) = \alpha_\pi\,\widetilde{\pi}^3,$$

where the coefficient α_π is an algebraic integer in $\mathcal{O}$ or in $\zeta_8\mathcal{O}$ for $p \equiv 5 \pmod 8$ or $p \equiv 1 \pmod 8$, respectively. Further, one has

$$\alpha_\pi \equiv 1 \pmod{(1+i)} \quad \textit{in case} \quad p = \pi\overline{\pi} \equiv 5 \pmod 8. \tag{2.19}$$

Proof. By the definition and by virtue of (2.14) and (2.17), we have $\alpha_\pi^{\sigma_\mu} = \alpha_\pi$ for an arbitrary $\mu \in (\mathcal{O}/(\pi))^\times$, and hence α_π or $\zeta_8\alpha_\pi \in F$ for $p \equiv 5 \pmod 8$ or $p \equiv 1 \pmod 8$, respectively. For the integrality, we can check it in similar manner to the proof (b) of Lemma 2.11. We here assume that Claim (Z) is valid. Suppose first $p \equiv 5 \pmod 8$, $p > 5$, then $\alpha_\pi\,\widetilde{\pi}^3$ is an algebraic integer, though $\pi = -\widetilde{\pi}^4$ is a prime in $\mathcal{O}$; it means α_π itself is already an integer. In the case $p \equiv 1 \pmod 8$ we need some modification, but the essence is the very same. The last assertion (2.19) is immediately deduced from Lemma 2.12 and Lemma 2.13.

Example 2.17. This example is based on an idea of Y. Ônishi. Consider the case $\pi = 3 + 2i$ $(p = 13)$, and we shall show $\alpha_\pi = 1$. Take $S = \ker \chi_\pi$. Then $S = \{1, 3, 9\} = \{1, -2i, 1 - i\}$, and so we have

$$\widetilde{\pi} = \prod_{\nu \in S} \varphi(\nu/\pi) = \varphi(1/\pi)\,\varphi(-2i/\pi)\,\varphi((1-i)/\pi) = \frac{(-2-2i)\,\varphi(1/\pi)^3}{1 + \varphi(1/\pi)^4},$$

by using suitable multiplication formulas. Since $-2-2i = 1-\pi = 1+\widetilde{\pi}^4$, $\varphi(1/\pi)$ is a solution of the following equation.

$$\widetilde{\pi}\, x^4 - (1+\widetilde{\pi}^4)\, x^3 + \widetilde{\pi} = 0.$$

The equation is decomposed as follows.

$$(\widetilde{\pi}\, x - 1)(x^3 - \widetilde{\pi}^3\, x^2 - \widetilde{\pi}^2\, x - \widetilde{\pi}) = 0.$$

The second factor must be the minimal polynomial of $\varphi(1/\pi)$ over $F(\widetilde{\pi})$, and hence the sum of the three roots $\varphi(\nu/\pi)$ $(\nu \in S)$ is $\widetilde{\pi}^3$, namely,

$$\mathcal{G}_\pi(\chi_\pi, \varphi) = \sum_{\nu \in S} \varphi(\nu/\pi) = \widetilde{\pi}^3.$$

In general, it seems pretty hard to compute the value of the coefficient α_π by hand. More examples by computor will be given in Table 2 at the end.

§ 2.4. The quartic Hecke characters and L-values at $s=1$

2.4.1. We introduce a Hecke character $\widetilde{\chi}_\pi$ induced by the quartic residue character χ_π. As mentioned before, it is of the form $\widetilde{\chi}_\pi((\nu)) = \chi_1(\nu)\,\overline{\nu}$ with a residue class character χ_1. For this purpose we first modify the character χ_π into χ_1 satisfying $\chi_1(i) = i$. After the preparation of supplementary simple characters χ_0 and χ_0', we shall treat the four cases separately in view of $\chi_\pi(i) = i^{(p-1)/4}$.

Let χ_0 be the character with conductor $(1+i)^3$ that gives the natural isomorphism $(\mathcal{O}/(1+i)^3)^\times \cong W$, namely,

$$\chi_0(\nu) \doteq \varepsilon \quad \text{for} \quad \nu \equiv \varepsilon \pmod{(1+i)^3}, \quad \varepsilon \in W = \{\pm 1, \pm i\}.$$

Let χ_0' be the character with conductor $(1+i)^2$ that gives the natural isomorphism $(\mathcal{O}/(1+i)^2)^\times = (\mathcal{O}/(2))^\times \cong \{\pm 1\}$, namely,

$$\chi_0'(\nu) \doteq \delta^2 \quad \text{for} \quad \nu \equiv \delta \pmod{(1+i)^2}, \quad \delta \in \{1, i\}.$$

Let π be a primary prime in $\mathcal{O}$; $\pi \equiv 1 \pmod{(1+i)^3}$. and let χ_π be the quartic residue character to the modulus (π).

Definition 2.18. The Hecke character $\widetilde{\chi}_\pi$ is fixed throughout as follows.

$$\text{(2.20)} \qquad \widetilde{\chi}_\pi((\nu)) \doteq \chi_1(\nu)\,\overline{\nu}, \quad \chi_1 \doteq \begin{cases} \chi_\pi \cdot \chi_0' & \text{for} \quad p = \pi\overline{\pi} \equiv 13 \pmod{16}, \\ \chi_\pi & \text{for} \quad p = \pi\overline{\pi} \equiv 5 \pmod{16}, \\ \chi_\pi \cdot \chi_0 & \text{for} \quad p = \pi\overline{\pi} \equiv 1 \pmod{16}, \\ \chi_\pi \cdot \overline{\chi}_0 & \text{for} \quad p = \pi\overline{\pi} \equiv 9 \pmod{16}. \end{cases}$$

For later use, we summarize these circumstances as a brief list :

(a) The case $p = \pi\overline{\pi} \equiv 13 \pmod{16}$. The conductor of $\widetilde{\chi}_\pi$ is $(\beta) = (2\pi)$.

$$(\mathcal{O}/(\beta))^\times \cong (\mathcal{O}/(\pi))^\times \times \{\pm 1\} \text{ by } \lambda \text{ to } (\kappa,\ \delta^2) :\ \lambda \equiv 2\kappa + \pi\delta \pmod{\beta}\ ; \quad \chi_1(\lambda) = \chi_\pi(2)\,\chi_\pi(\kappa)\,\delta^2. \tag{2.21}$$

(b) The case $p = \pi\overline{\pi} \equiv 5 \pmod{16}$. The conductor of $\widetilde{\chi}_\pi$ is $(\beta) = (\pi)$.

$$(\mathcal{O}/(\beta))^\times \cong (\mathcal{O}/(\pi))^\times \text{ by } \lambda \equiv \kappa \pmod{\beta}\ ;\quad \chi_1(\lambda) = \chi_\pi(\kappa). \tag{2.22}$$

(c) The case $p = \pi\overline{\pi} \equiv 1 \pmod 8$. The conductor of $\widetilde{\chi}_\pi$ is $(\beta) = ((1+i)^3\pi)$.

$$(\mathcal{O}/(\beta))^\times \cong (\mathcal{O}/(\pi))^\times \times W \text{ by } \lambda \text{ to } (\kappa,\ \varepsilon) :\ \lambda \equiv (1+i)^3\kappa + \pi\varepsilon \pmod{\beta}\ ;$$
$$\chi_1(\lambda) = \begin{cases} \overline{\chi}_\pi(1+i)\,\chi_\pi(\kappa)\,\varepsilon & (p \equiv 1 \pmod{16}), \\ \overline{\chi}_\pi(1+i)\,\chi_\pi(\kappa)\,\overline{\varepsilon} & (p \equiv 9 \pmod{16}). \end{cases} \tag{2.23}$$

2.4.2. Now we can evaluate the value of the associated L-series at $s = 1$, and in fact $L(1,\ \widetilde{\chi}_\pi)$ will be expressed by the corresponding elliptic Gauss sum.

Theorem 2.19. *Let $\widetilde{\chi}_\pi$ be the Hecke character for a primary prime π. Then*

$$\varpi^{-1} L(1,\ \widetilde{\chi}_\pi) = \begin{cases} -\dfrac{1+i}{2}\chi_\pi(2)\,\pi^{-1}\,\mathcal{G}_\pi(\chi_\pi,\ \varphi) & \text{if } p = \pi\overline{\pi} \equiv 13 \pmod{16}, \\ \pi^{-1}\,\mathcal{G}_\pi(\chi_\pi,\ \mathsf{Z}) & \text{if } p = \pi\overline{\pi} \equiv 5 \pmod{16}, \\ \dfrac{1}{2}\overline{\chi}_\pi(1+i)\,\pi^{-1}\,\mathcal{G}_\pi(\chi_\pi,\ \psi) & \text{if } p = \pi\overline{\pi} \equiv 1 \pmod{16}, \\ -\dfrac{1}{2}\overline{\chi}_\pi(1+i)\,\pi^{-1}\,\mathcal{G}_\pi(\chi_\pi,\ \psi) & \text{if } p = \pi\overline{\pi} \equiv 9 \pmod{16}. \end{cases} \tag{2.24}$$

Proof. We follow the formula (2.12) of Lemma 2.8.

(a) The case $p = \pi\overline{\pi} \equiv 13 \pmod{16}$. In view of (2.21), we have

$$\begin{aligned} \varpi^{-1} L(1,\ \widetilde{\chi}_\pi) &= \frac{1}{4\beta}\cdot \sum_{\lambda \,(\mathrm{mod}\, \beta)} \chi_1(\lambda)\,\mathsf{Z}(\lambda/\beta) \\ &= \frac{1}{8\pi}\chi_\pi(2)\cdot \sum_{\kappa\,(\mathrm{mod}\,\pi)} \chi_\pi(\kappa) \sum_{\delta=1,\,i} \delta^2\,\mathsf{Z}(\kappa/\pi + \delta/2) \\ &= -\frac{1+i}{2}\cdot\frac{\chi_\pi(2)}{\pi}\cdot\frac{1}{4}\sum_{\kappa\,(\mathrm{mod}\,\pi)} \chi_\pi(\kappa)\varphi(\kappa/\pi), \end{aligned}$$

since $\sum_{\delta=1,\,i} \delta^2\,\mathsf{Z}(u+\delta/2) = -(1+i)\,\varphi(u)$ by the definition (2.3).

(b) The case $p = \pi\overline{\pi} \equiv 5 \pmod{16}$. In view of (2.22), we obtain directly

$$\varpi^{-1} L(1,\ \widetilde{\chi}_\pi) = \frac{1}{\pi} \cdot \frac{1}{4} \sum_{\kappa \pmod{\pi}} \chi_\pi(\kappa)\, \mathsf{Z}(\kappa/\pi).$$

(c) The case $p = \pi\overline{\pi} \equiv 1 \pmod{16}$. In view of (2.23), we have

$$\begin{aligned}\varpi^{-1} L(1,\ \widetilde{\chi}_\pi) &= -\frac{1+i}{16} \cdot \frac{\overline{\chi}_\pi(1+i)}{\pi} \sum_{\kappa \pmod{\pi}} \chi_\pi(\kappa) \sum_{\varepsilon \in W} \varepsilon\, \mathsf{Z}(\kappa/\pi - \varepsilon(1+i)/4) \\ &= \frac{\overline{\chi}_\pi(1+i)}{\pi} \cdot \frac{1}{4} \sum_{\kappa \pmod{\pi}} \chi_\pi(\kappa)\, \psi(\kappa/\pi),\end{aligned}$$

since $\sum_{\varepsilon \in W} \varepsilon\, \mathsf{Z}(u - \varepsilon(1+i)/4) = -(1-i)\{\psi(u) + \psi(iu)\}$ by the definition (2.4).

(c') The case $p = \pi\overline{\pi} \equiv 9 \pmod{16}$. In view of (2.23), we have

$$\begin{aligned}\varpi^{-1} L(1,\ \widetilde{\chi}_\pi) &= -\frac{1+i}{16} \cdot \frac{\overline{\chi}_\pi(1+i)}{\pi} \sum_{\kappa \pmod{\pi}} \chi_\pi(\kappa) \sum_{\varepsilon \in W} \overline{\varepsilon}\, \mathsf{Z}(\kappa/\pi - \varepsilon(1+i)/4) \\ &= -\frac{\overline{\chi}_\pi(1+i)}{\pi} \cdot \frac{1}{4} \sum_{\kappa \pmod{\pi}} \chi_\pi(\kappa)\, \psi(\kappa/\pi),\end{aligned}$$

since $\sum_{\varepsilon \in W} \overline{\varepsilon}\, \mathsf{Z}(u - \varepsilon(1+i)/4) = (1-i)\{\psi(u) - \psi(iu)\}$ also by the definition (2.4).

Thus we have completed the proof of Theorem 2.19.

§ 2.5. An explicit formula of the root number $C(\widetilde{\chi}_\pi)$

2.5.1. We require an important formula about the classical quartic Gauss sum. Let π be a primary prime in $\mathcal{O}$ and set $p = \pi\overline{\pi}$. The quartic residue character χ_π may be considered as a character on $(\mathbf{Z}/p\mathbf{Z})^\times$. Then the quartic Gauss sum is defined by

$$G_4(\pi) \doteq \sum_{r=1}^{p-1} \chi_\pi(r)\, e^{2\pi i r/p}. \tag{2.25}$$

Let $\widetilde{\pi}$ be the canonical quartic root of $-\pi$ as defined by (2.16). Then we have

Lemma 2.20 (**G_4-formula**).

$$G_4(\pi) = \chi_\pi(-2)\, \widetilde{\pi}^3\, \overline{\widetilde{\pi}} \tag{2.26}$$

Remark. As an immediate consequence we have a famous formula (cf. [1, Prop.9.10.1]): $G_4(\pi)^2 = -\chi_\pi(-1)\,\pi\ \ \overline{p}$, and also we obtain $G_4(\pi)^4 = \pi^3\,\overline{\pi}$. The ambiguity of the definition of $\widetilde{\pi}$ does not matter, for the right side of our G_4-formula depends only on $\widetilde{\pi}^2$.

Proof. This is only a slight modification of the celebrated formula of Matthews. He used the lattice $\theta\,\mathcal{O}$ instead of our $\varpi\,\mathcal{O}$, where $\theta = \sqrt{2}\,\varpi = 3.70814935\cdots$. Let $\wp_1(u)$ denote Weierstrass' $\wp$ with the period lattice $\theta\,\mathcal{O}$. Hence the relation $\wp(\varpi\,u) = 2\,\wp_1(\theta\,u)$ holds, so that $\wp_1'(u)^2 = 4\,\wp_1(u)^3 - \wp_1(u)$; further we have

$$\varphi(u) = \mathrm{sl}((1-i)\,\varpi\,u) = -2\,(1-i)\,\frac{\wp(\varpi\,u)}{\wp'(\varpi\,u)} = -\sqrt{2}\,(1-i)\,\frac{\wp_1(\theta\,u)}{\wp_1'(\theta\,u)} = \zeta_8^{-1}\,T(\theta\,u),$$

where $T(u) = -2\,\wp_1(u)\wp_1'(u)^{-1}$ after his notation. Matthews' formula states

Formula ([4, esp. p. 51])

$$G_4(\pi) = -\beta(\pi)\,\chi_\pi(2i)\prod_{r\in N} T(\theta\,r/\pi)\cdot p^{1/4}, \tag{2.27}$$

where $N = \{1,\,2,\,\cdots,\,(p-1)/2\}$, and the constant $\beta(\pi)$ is uniquely determined by the conditions $\beta(\pi) \equiv \prod_{r\in N} r \pmod{\pi}$ and $\beta(\pi)^2 = -1$.

Now we can derive the formula (2.26) from (2.27); indeed they are equivalent. We first note that since N is a half subset mod (π), there is a quarter subset S_0 such that $N = S_0 \cup iS_0$. Then $\beta(\pi) \equiv \prod_{r\in N} r \equiv \chi_\pi(i)\prod_{r\in S_0} r^2 \pmod{(\pi)}$; namely, $\beta(\pi) = \chi_\pi(i)\,\gamma(S_0)^2$. Also we can observe $\gamma(S_0)^4 = -\chi_\pi(-1)$. Hence we have

$$\begin{aligned} G_4(\pi) &= -\chi_\pi(-2)\,\gamma(S_0)^2\prod_{r\in N}\{\zeta_8\,\varphi(r/\pi)\}\cdot p^{1/4} \\ &= -\chi_\pi(-2)\,\gamma(S_0)^4\,\chi_\pi(-1)\cdot\gamma(S_0)^{-2}\prod_{\nu\in S_0}\varphi(\nu/\pi)^2\cdot p^{1/4} = \chi_\pi(-2)\,\widetilde{\pi}^3\,\overline{\widetilde{\pi}}. \end{aligned}$$

This finishes the proof of G_4-formula.

2.5.2. We are now ready to give the explicit value of the root number $C(\widetilde{\chi}_\pi)$.

Theorem 2.21. *Let $\widetilde{\chi}_\pi$ be the Hecke character for a primary prime π. Then*

$$C(\widetilde{\chi}_\pi) = \begin{cases} -i\,\widetilde{\pi}^{-1}\,\overline{\widetilde{\pi}} & \textit{if}\ \ p = \pi\,\overline{\pi} \equiv 13 \pmod{16}, \\ i\,\chi_\pi(2)\,\widetilde{\pi}^{-1}\,\overline{\widetilde{\pi}} & \textit{if}\ \ p = \pi\,\overline{\pi} \equiv 5 \pmod{16}, \\ -\overline{\chi}_\pi(1+i)\,\widetilde{\pi}^{-1}\,\overline{\widetilde{\pi}} & \textit{if}\ \ p = \pi\,\overline{\pi} \equiv 1 \pmod{16}, \\ -i\,\chi_\pi(1+i)\,\widetilde{\pi}^{-1}\,\overline{\widetilde{\pi}} & \textit{if}\ \ p = \pi\,\overline{\pi} \equiv 9 \pmod{16}. \end{cases} \tag{2.28}$$

Proof. We first evaluate some simple Gauss sums. The first three are easily verified by direct calculation.

$$g(\chi_0) \doteq \sum_{\varepsilon\in W} \varepsilon\, e^{2\pi i\,\mathrm{Re}(\varepsilon/(1+i)^3)} = -2-2i,\quad g(\overline{\chi}_0) \doteq \sum_{\varepsilon\in W} \overline{\varepsilon}\, e^{2\pi i\,\mathrm{Re}(\varepsilon/(1+i)^3)} = 2-2i$$

$$\textit{and}\quad g(\chi_0') \doteq \sum_{\delta=1,\,i} \delta^2\, e^{2\pi i\,\mathrm{Re}(\delta/2)} = -2.$$

The next sum is essentially nothing but the quartic Gauss sum :

(2.29)

$$g(\chi_\pi) \doteq \sum_{\kappa\,(\mathrm{mod}\,\pi)} \chi_\pi(\kappa)\, e^{2\pi i\,\mathrm{Re}(\kappa/\pi)} = \chi_\pi(-2)\,\tilde{\pi}^3\,\overline{\tilde{\pi}}\cdot\begin{cases} 1 & \textit{if}\ \ p\equiv 13,\,1\ (\mathrm{mod}\ 16)\\ -1 & \textit{if}\ \ p\equiv 5,\,9\ (\mathrm{mod}\ 16)\end{cases}$$

In fact, we first replace the sum over $\kappa\,(\mathrm{mod}\,\pi)$ by one over $r\,(\mathrm{mod}\,p)$, and then, by using $\mathrm{Re}(r\,\overline{\pi}/p) = ar/p$ where $\pi = a+b\,i$ $(a,\,b\in\mathbf{Z})$, we can calculate as follows :

$$g(\chi_\pi) = \sum_{r\,(\mathrm{mod}\,p)} \chi_\pi(r)\, e^{2\pi i\,\mathrm{Re}(r\,\overline{\pi}/p)} = \sum_{r\,(\mathrm{mod}\,p)} \chi_\pi(r)\, e^{2\pi i ar/p} = \overline{\chi}_\pi(a)\, G_4(\pi).$$

Furthermore, we know $\overline{\chi}_\pi(a) = 1$ or -1 for $p\equiv 13$, 1 or 5, 9 (mod 16), respectively (cf. [1, Chap. 9, Exerc. 34]), and finally, by applying G_4-formula we have (2.29).

We return to the proof of Theorem 2.21. Using the formula (2.11) of Lemma 2.7 :

$$C(\tilde{\chi}_\pi) = -i\,\beta^{-1} \sum_{\lambda\,(\mathrm{mod}\,\beta)} \chi_1(\lambda)\, e^{2\pi i\,\mathrm{Re}(\lambda/\beta)},$$

we treat each of the four cases according to the definition of χ_1, especially in view of (2.21), (2.22) and (2.23). Thus we can easily obtain

(a) The case $p = \pi\,\overline{\pi} \equiv 13$ (mod 16).

$$C(\tilde{\chi}_\pi) = -i\,2^{-1}\,\pi^{-1}\,\chi_\pi(2)\,g(\chi_0')\,g(\chi_\pi) = -i\,\tilde{\pi}^{-1}\,\overline{\tilde{\pi}}.$$

(b) The case $p = \pi\,\overline{\pi} \equiv 5$ (mod 16).

$$C(\tilde{\chi}_\pi) = -i\,\pi^{-1}\,g(\chi_\pi) = i\,\chi_\pi(2)\cdot\tilde{\pi}^{-1}\,\overline{\tilde{\pi}}.$$

(c) The case $p = \pi\,\overline{\pi} \equiv 1$ (mod 16).

$$C(\tilde{\chi}_\pi) = -i\,(1+i)^{-3}\,\pi^{-1}\,\overline{\chi}_\pi(1+i)\,g(\chi_0)\,g(\chi_\pi) = -\chi_\pi(1+i)\,\tilde{\pi}^{-1}\,\overline{\tilde{\pi}}.$$

(c') The case $p = \pi\,\overline{\pi} \equiv 9$ (mod 16).

$$C(\tilde{\chi}_\pi) = -i\,(1+i)^{-3}\,\pi^{-1}\,\overline{\chi}_\pi(1+i)\,g(\overline{\chi}_0)\,g(\chi_\pi) = -i\,\chi_\pi(1+i)\,\tilde{\pi}^{-1}\,\overline{\tilde{\pi}}.$$

These complete the proof of Theorem 2.21.

§ 2.6. Rationality of the elliptic Gauss sum coefficient

2.6.1. Now we can mention about the rationality of the coefficients of elliptic Gauss sums. More precisely, the coefficient itself is not always rational but it is shown that the essential factor of this is certainly a rational integer, which seems also the most important part in respect of arithmetical nature. The following theorem, together with two corollaries, is the main result of §2.

Theorem 2.22. *Let α_π be the coefficient of the elliptic Gauss sum. Then*

$$\alpha_\pi = \begin{cases} \overline{\alpha}_\pi & \text{if } \ p = \pi\overline{\pi} \equiv 13 \pmod{16}, \\ i\,\chi_\pi(2)\,\overline{\alpha}_\pi & \text{if } \ p = \pi\overline{\pi} \equiv 5 \pmod{16}, \\ -\overline{\chi}_\pi(1+i)\,\overline{\alpha}_\pi & \text{if } \ p = \pi\overline{\pi} \equiv 1 \pmod{16}, \\ -i\,\overline{\chi}_\pi(1+i)\,\overline{\alpha}_\pi & \text{if } \ p = \pi\overline{\pi} \equiv 9 \pmod{16}. \end{cases} \tag{2.30}$$

Proof. By Theorem 2.16, Theorem 2.19 and Theorem 2.21 we have already known both the explicit values of $L(1,\, \widetilde{\chi}_\pi)$ and $C(\widetilde{\chi}_\pi)$. To prove Theorem 2.22, we have only to substitute them for the both sides of the central value equation : $L(1,\, \widetilde{\chi}_\pi) = C(\widetilde{\chi}_\pi)\,\overline{L(1,\, \widetilde{\chi}_\pi)}$ (cf. Lemma 2.7). For example, suppose that $p = \pi\overline{\pi} \equiv 13 \pmod{16}$. In this case we have

$$\varpi^{-1}\, L(1,\, \widetilde{\chi}_\pi) = \frac{1+i}{2}\,\chi_\pi(2)\,\widetilde{\pi}^{-1}\,\alpha_\pi \quad \text{and} \quad C(\widetilde{\chi}_\pi) = -i\,\widetilde{\pi}^{-1}\,\overline{\widetilde{\pi}},$$

and hence the central value equation implies $\chi_\pi(2)\,\alpha_\pi = -\overline{\chi}_\pi(2)\,\overline{\alpha}_\pi$. This immediately proves $\alpha_\pi = \overline{\alpha}_\pi$ since $\chi_\pi(2)^2 = \chi_\pi(i)^2 = -1$. In other cases, the argument is quite similar, so we omit the details. Thus the proof is finished.

Before stating the corollaries of Theorem 2.22, we give a proof of the integrality of $\mathcal{G}_\pi(\chi_\pi,\, \mathsf{Z})$, which has been postponed until now. (cf. §2.3.1)

***Proof of* Claim (Z).** Assume that $p = \pi\overline{\pi} \equiv 5 \pmod{16}$ and $\mathcal{G}_\pi(\chi_\pi,\, \mathsf{Z}) = \alpha_\pi\,\widetilde{\pi}^3$. We know $\alpha_\pi \in F$. Further, Theorem 2.22 shows $\overline{\chi}_\pi(1+i)\,\alpha_\pi = \chi_\pi(1+i)\,\overline{\alpha}_\pi$, and hence $\alpha_\pi = \chi_\pi(1+i)\,a_\pi$ for some $a_\pi \in \mathbf{Q}$. On the other hand, by Lemma 2.12 and the subsequent discussions, we know $((1+i) - i\,\overline{\chi}_\pi(1+i))\,\alpha_\pi \in \mathcal{O}$. Namely, a_π, $-(1+2i)\,a_\pi$, $-a_\pi$ or $(1-2i)\,a_\pi$ is algebraic integer in $\mathcal{O}$, when $\chi_\pi(1+i) = 1,\, -1,\, i$ or $-i$, accordingly. This means that $a_\pi \in \mathbf{Z}$ holds already. The proof is completed.

Corollary 2.23. *Suppose that* $p = \pi\overline{\pi} \equiv 5 \pmod 8$. *There exists a rational integer* a_π *such that* $a_\pi \equiv 1 \pmod 2$, *and the coefficient* α_π *of the elliptic Gauss sum is expressed by* a_π *as follows. In particular,* $|\alpha_\pi|^2 = a_\pi^2$.

$$\alpha_\pi = \begin{cases} a_\pi & \textit{if} \quad p = \pi\overline{\pi} \equiv 13 \pmod{16}, \\ a_\pi\, \chi_\pi(1+i) & \textit{if} \quad p = \pi\overline{\pi} \equiv 5 \pmod{16}. \end{cases} \tag{2.31}$$

Proof. We can derive the integrality of the coefficient α_π from Theorem 2.16, and the rationality from Theorem 2.22. The congruence property $a_\pi \equiv 1 \pmod 2$ follows from $\alpha_\pi \equiv 1 \pmod{(1+i)}$ in Theorem 2.16.

Corollary 2.24. *Suppose that* $p = \pi\overline{\pi} \equiv 1 \pmod 8$. *There exists a rational integer* a_π, *and the coefficient* α_π *of the elliptic Gauss sum is expressed by* a_π *as follows. In particular,* $|\alpha_\pi|^2 = 2\,a_\pi^2$ *or* $|\alpha_\pi|^2 = a_\pi^2$ *according as* $\chi_\pi(2) = 1$ *or* $\chi_\pi(2) = -1$.

$$\alpha_\pi = \begin{cases} a_\pi \cdot i\sqrt{2} & \textit{if} \quad \chi_\pi(1+i) = 1, \\ a_\pi \cdot \sqrt{2} & \textit{if} \quad \chi_\pi(1+i) = -1, \\ a_\pi \cdot \zeta_8 & \textit{if} \quad \chi_\pi(1+i) = i, \\ a_\pi \cdot i\,\zeta_8 & \textit{if} \quad \chi_\pi(1+i) = -i, \end{cases} \quad \textit{and} \quad p \equiv 1 \pmod{16}, \tag{2.32}$$

$$\alpha_\pi = \begin{cases} a_\pi \cdot i\,\zeta_8 & \textit{if} \quad \chi_\pi(1+i) = 1, \\ a_\pi \cdot \zeta_8 & \textit{if} \quad \chi_\pi(1+i) = -1, \\ a_\pi \cdot i\sqrt{2} & \textit{if} \quad \chi_\pi(1+i) = i, \\ a_\pi \cdot \sqrt{2} & \textit{if} \quad \chi_\pi(1+i) = -i, \end{cases} \quad \textit{and} \quad p \equiv 9 \pmod{16}. \tag{2.33}$$

Proof. In this case we have $\alpha_\pi \in \zeta_8\, \mathcal{O}$, which combined with the rationality relation (2.30) will immediately give an explicit form of the coefficient. For example, consider the case of $p \equiv 1 \pmod{16}$ and $\chi_\pi(1+i) = 1$. Put $\alpha_\pi = (c+di)\,\zeta_8$ with $c, d \in \mathbf{Z}$. By (2.30) we see $\alpha_\pi = -\overline{\alpha}_\pi$, and hence $c + di = i(c - di)$, which means $c = d$. Namely, we have $\alpha_\pi = c\,(1+i)\,\zeta_8 = a_\pi \cdot i\sqrt{2}$ by putting $a_\pi = c$. We omit the details for other seven cases.

2.6.2. The substance of Theorem 2.22 and the corollaries can be stated by the language of Hecke L-values in various ways. The following is one of them. It shows that there is a close relation between the value $L(1, \widetilde{\chi}_\pi)$ and the quartic Gauss sum $G_4(\pi)$, especially between their arguments. Roughly speaking, the argument of $L(1, \widetilde{\chi}_\pi)$ is parallel to one of $\widetilde{\pi}^{-1}$, and the argument of $G_4(\pi)$ is parallel to one of $\widetilde{\pi}^2$. Hence by eliminating the factors of $\widetilde{\pi}$ from their formulas, the result can be obtained.

Theorem 2.25. *Let a_π be a rational integer given in Corollary 2.23 or 2.24.*

$$\varpi^{-2} L(1, \widetilde{\chi}_\pi)^2 = \begin{cases} 2^{-1}\, i\, \chi_\pi(2)\, p^{1/4}\, a_\pi^2 \cdot G_4(\pi)^{-1} & \text{if } \ p \equiv 13 \pmod{16}, \\ i\, p^{1/4}\, a_\pi^2 \cdot G_4(\pi)^{-1} & \text{if } \ p \equiv 5 \pmod{16}, \\ -2^{-1}\, \overline{\chi}_\pi(1+i)\, p^{1/4}\, a_\pi^2 \cdot G_4(\pi)^{-1} & \text{if } \ p \equiv 1 \pmod{16},\ \chi_\pi(2) = 1, \\ -2^{-2} \overline{\chi}_\pi(1+i)\, p^{1/4}\, a_\pi^2 \cdot G_4(\pi)^{-1} & \text{if } \ p \equiv 1 \pmod{16},\ \chi_\pi(2) = -1, \\ 2^{-1}\, i\, \overline{\chi}_\pi(1+i)\, p^{1/4}\, a_\pi^2 \cdot G_4(\pi)^{-1} & \text{if } \ p \equiv 9 \pmod{16},\ \chi_\pi(2) = 1, \\ 2^{-2} i\, \overline{\chi}_\pi(1+i)\, p^{1/4}\, a_\pi^2 \cdot G_4(\pi)^{-1} & \text{if } \ p \equiv 9 \pmod{16},\ \chi_\pi(2) = -1. \end{cases}$$

Proof. Consider the case $p = \pi\overline{\pi} \equiv 13 \pmod{16}$. By Theorems 2.16 and 2.19, we have $\varpi^{-1} L(1, \widetilde{\chi}_\pi) = 2^{-1}(1+i)\,\chi_\pi(2)\,\alpha_\pi\,\widetilde{\pi}^{-1}$. Theorem 2.22 shows $\alpha_\pi^2 = a_\pi^2$. These combined with G_4-formula : $G_4(\pi) = -\chi_\pi(2)\,\widetilde{\pi}^2\,p^{1/4}$ implies the result. We omit the similar discussions for other cases. Obviously the formula for $|L(1, \widetilde{\chi}_\pi)|^2$ is very simple.

Corollary 2.26. $L(1, \widetilde{\chi}_\pi) \neq 0$ *in case* $p = \pi\overline{\pi} \equiv 5 \pmod 8$.

Proof. Because $|\alpha_\pi|^2 = a_\pi^2 \equiv 1 \pmod 2$.

Remark. The case $p = \pi\overline{\pi} \equiv 1 \pmod 8$. According to some observation it is plausible that a_π is even and $\frac{1}{2}a_\pi \equiv 1 \pmod 2$ and hence $L(1, \widetilde{\chi}_\pi)$ never vanishes if $p \equiv 1 \pmod{16}$ and $\chi_\pi(1+i) \neq 1$, or if $p \equiv 9 \pmod{16}$ and $\chi_\pi(1+i) \neq -i$. On the other hand, we can observe that $L(1, \widetilde{\chi}_\pi)$ happens to vanish very often in the contrary cases. For examples it seems $L(1, \widetilde{\chi}_\pi) = 0$ holds for each prime as follows :

$$p = 113,\ 257,\ 593,\ 1201,\ 1217,\ 2129,\ 2593, \ldots, \quad (p \equiv 1 \pmod{16},\quad \chi_\pi(1+i) = 1),$$
$$p = 89,\ 601,\ 1097,\ 1193,\ 1433,\ 1481,\ 1721, \ldots, \quad (p \equiv 9 \pmod{16},\quad \chi_\pi(1+i) = -i).$$

while any reason for these is not known at all. (cf. Table 2-3, 2-4)

Appendix. Formulas of special elliptic functions ($\mathbf{Z}[i]$-case)

Addition and multiplication formulas of the functions $\varphi(u)$ and $\psi(u)$ are minimally selected. For definitions of these functions cf. §2.1.1. These formulas and more of this lemniscatic case are well known, one may refer to the book [2, esp. Chap. 8].

1. Addition Formula

(i) $$\varphi(u+v) = \frac{\varphi(u)\,\psi(u) + \varphi(v)\,\psi(v)}{\varphi(u)\,\varphi(v) + \psi(u)\,\psi(v)}, \quad \psi(u+v) = \frac{\varphi(u)\,\psi(u) - \varphi(v)\,\psi(v)}{\varphi(u)\,\psi(v) - \varphi(v)\,\psi(u)}$$

2. Multiplication Formula

$$\text{(ii)}\ \varphi(2u)=\frac{2\varphi(u)\,\psi(u)\,(1+\varphi(u)^2)}{1+\varphi(u)^4}=\frac{2\varphi(u)\,\psi(u)}{1-\varphi(u)^2\psi(u)^2},\quad \psi(2u)=\frac{\psi(u)^2-\varphi(u)^2}{1+\varphi(u)^2\,\psi(u)^2}$$

$$\text{(iii)}\ \varphi((1+i)u)=\frac{(1+i)\varphi(u)\,\psi(u)}{1-\varphi(u)^2},\quad \psi((1+i)u)=\frac{1-i\,\varphi(u)^2}{1+i\,\varphi(u)^2}=-i\cdot\frac{1+i\,\psi(u)^2}{1-i\,\psi(u)^2}$$

Tables of the elliptic Gauss sum coefficients

For convenience' and interest's sake, we append some tables of the coefficients of elliptic Gauss sums on the following pages. The computation was made by UBASIC.

Table 1 (The cubic charcter case).

In the table, the coefficient α_π is expressed as $\alpha_\pi = a_\pi \cdot \chi_\pi(3)$, $a_\pi \cdot \overline{\chi}_\pi(3)$, or $a_\pi \cdot \chi_\pi(3) \cdot \sqrt{-3}$, for the case $p \equiv 7 \pmod 9$, $p \equiv 4 \pmod 9$, or $p \equiv 1 \pmod 9$, respectively. (cf. Theorem 1.19) We can observe the size of a_π being remarkably small.

Table 2 (The quartic character case).

In Tables 2.1, 2.2, the coefficient is given as $\alpha_\pi = a_\pi$ or $\alpha_\pi = a_\pi \cdot \chi_\pi(1+i)$, for the case $p \equiv 13 \pmod{16}$ or $p \equiv 5 \pmod{16}$, respectively. (cf. Corollary 2.23)

It is very notable that the magnitude of a_π seems to be remarkably small. In fact, thanks to Mr. Naruo Kanou's computation by PARI/GP, we know

$$-49 \le a_\pi \le 49 \quad \text{for} \quad 13 \le p \le 3999949, \quad p \equiv 13 \pmod{16},$$
$$-43 \le a_\pi \le 47 \quad \text{for} \quad 37 \le p \le 3999893, \quad p \equiv 5 \pmod{16},$$

and further, $a_\pi = \pm 1$ for 32.3% p's in the former case, and 46.9% p's in the latter case.

For the case of $p \equiv 1 \pmod 8$, we need to add a few remarks. Firstly, in this case, as was mentioned in §2.3.2, there is an ambiguity caused by the choice of Π or Π' in defining the quartic root $\widetilde{\pi}$ of $-\pi$. In our computation, we take the quarter subset S_0 such as $S_0 \cup i\,S_0 = \{1, 2, \ldots, (p-1)/2\}$. And we set $\gamma(S_0) = \zeta_8$ or $\overline{\zeta}_8$ when $\gamma(S_0)^2 \equiv i$ or $-i \pmod{\pi}$, accordingly. This means that we have chosen an appropriate Π for temporary convenience. Anyway in the case $p \equiv 1 \pmod 8$, only the quantity α_π^2 has an invariant meaning with respect to each π. Secondly, in the case $p \equiv 1 \pmod 8$, it is very likely that a_π is always divisible by 2. This suggests that it might be better to choose the constant r to be $r = 2$ in the definition (2.13) of the elliptic Gauss sum, though unfortunately we could not prove this fact yet. So the half of the coefficient $\frac{1}{2}\alpha_\pi$ is given in the table for simplicity. In Tables 2.3, 2.4, we can see and check that each value of the coefficient is in exact conformity with the statement of Corollary 2.24.

Table 1. The elliptic Gauss sum coefficients

$\mathcal{G}_\pi(\chi_\pi,\varphi)=\alpha_\pi\,\tilde{\pi}^2$			$\mathcal{G}_\pi(\chi_\pi,\varphi^{-1})=\alpha_\pi\,\tilde{\pi}^2$			$\mathcal{G}_\pi(\chi_\pi,\psi)=\alpha_\pi\,\tilde{\pi}^2$		
p	π	α_π	p	π	α_π	p	π	α_π
7	$1+3\rho$	$1\cdot\rho$	13	$4+3\rho$	$-1\cdot\bar\rho$	19	$-2+3\rho$	$-1\cdot\rho\cdot\sqrt{-3}$
43	$7+6\rho$	$1\cdot\bar\rho$	31	$1+6\rho$	$-1\cdot\rho$	37	$7+3\rho$	$-1\cdot\rho\cdot\sqrt{-3}$
61	$4+9\rho$	$1\cdot1$	67	$7+9\rho$	$2\cdot1$	73	$1+9\rho$	0
79	$10+3\rho$	$1\cdot\rho$	103	$-2+9\rho$	$2\cdot1$	109	$7+12\rho$	$-1\cdot\rho\cdot\sqrt{-3}$
97	$-8+3\rho$	$-2\cdot\rho$	139	$13+3\rho$	$2\cdot\bar\rho$	127	$13+6\rho$	$1\cdot\bar\rho\cdot\sqrt{-3}$
151	$-5+9\rho$	$-2\cdot1$	157	$13+12\rho$	$-1\cdot\bar\rho$	163	$-11+3\rho$	$-1\cdot\rho\cdot\sqrt{-3}$
223	$-11+6\rho$	$1\cdot\bar\rho$	193	$16+9\rho$	$2\cdot1$	181	$4+15\rho$	$1\cdot\bar\rho\cdot\sqrt{-3}$
241	$16+15\rho$	$1\cdot\bar\rho$	211	$1+15\rho$	$-1\cdot\rho$	199	$13+15\rho$	$1\cdot\bar\rho\cdot\sqrt{-3}$
277	$19+12\rho$	$1\cdot\rho$	229	$-5+12\rho$	$-1\cdot\bar\rho$	271	$19+9\rho$	0
313	$19+3\rho$	$4\cdot\rho$	283	$19+6\rho$	$2\cdot\rho$	307	$1+18\rho$	0
331	$10+21\rho$	$-5\cdot\rho$	337	$13+21\rho$	$-1\cdot\bar\rho$	379	$22+15\rho$	$-2\cdot\bar\rho\cdot\sqrt{-3}$
349	$-17+3\rho$	$4\cdot\rho$	373	$4+21\rho$	$-4\cdot\bar\rho$	397	$-11+12\rho$	$-1\cdot\rho\cdot\sqrt{-3}$
367	$22+9\rho$	$1\cdot1$	409	$-8+15\rho$	$-4\cdot\rho$	433	$13+24\rho$	$1\cdot\bar\rho\cdot\sqrt{-3}$
421	$1+21\rho$	$4\cdot\rho$	463	$22+21\rho$	$2\cdot\bar\rho$	487	$-2+21\rho$	$-1\cdot\rho\cdot\sqrt{-3}$
439	$-5+18\rho$	$4\cdot1$	499	$25+18\rho$	$2\cdot1$	523	$-17+9\rho$	0
457	$7+24\rho$	$1\cdot\bar\rho$	571	$-5+21\rho$	$-1\cdot\bar\rho$	541	$25+21\rho$	$2\cdot\rho\cdot\sqrt{-3}$
547	$13+27\rho$	$-2\cdot1$	607	$-23+3\rho$	$5\cdot\bar\rho$	577	$19+27\rho$	0
601	$25+24\rho$	$-2\cdot\bar\rho$	643	$-11+18\rho$	$2\cdot1$	613	$28+9\rho$	$-3\cdot1\cdot\sqrt{-3}$
619	$22+27\rho$	$4\cdot1$	661	$-20+9\rho$	$2\cdot1$	631	$-14+15\rho$	$1\cdot\bar\rho\cdot\sqrt{-3}$
673	$-8+21\rho$	$4\cdot\rho$	733	$31+12\rho$	$-1\cdot\bar\rho$	739	$7+30\rho$	$-1\cdot\rho\cdot\sqrt{-3}$
691	$19+30\rho$	$1\cdot\rho$	751	$31+21\rho$	$5\cdot\bar\rho$	757	$28+27\rho$	$3\cdot1\cdot\sqrt{-3}$
709	$28+3\rho$	$-2\cdot\rho$	769	$-17+15\rho$	$-1\cdot\rho$	811	$31+6\rho$	$1\cdot\bar\rho\cdot\sqrt{-3}$
727	$31+18\rho$	$1\cdot1$	787	$-2+27\rho$	$-4\cdot1$	829	$13+33\rho$	$1\cdot\bar\rho\cdot\sqrt{-3}$
853	$31+27\rho$	$4\cdot1$	823	$19+33\rho$	$-4\cdot\rho$	883	$34+21\rho$	$-1\cdot\rho\cdot\sqrt{-3}$
907	$7+33\rho$	$-5\cdot\bar\rho$	859	$10+33\rho$	$2\cdot\rho$	919	$-17+18\rho$	0
997	$13+36\rho$	$1\cdot1$	877	$31+3\rho$	$-1\cdot\bar\rho$	937	$-29+3\rho$	$2\cdot\rho\cdot\sqrt{-3}$
1033	$37+21\rho$	$-2\cdot\rho$	967	$34+27\rho$	$2\cdot1$	991	$-26+9\rho$	$3\cdot1\cdot\sqrt{-3}$
1051	$-29+6\rho$	$-2\cdot\bar\rho$	1021	$25+36\rho$	$2\cdot1$	1009	$-8+27\rho$	$-3\cdot1\cdot\sqrt{-3}$
1069	$37+12\rho$	$1\cdot\rho$	1039	$37+15\rho$	$-1\cdot\rho$	1063	$34+3\rho$	$-1\cdot\rho\cdot\sqrt{-3}$
1087	$-17+21\rho$	$7\cdot\rho$	1093	$7+36\rho$	$2\cdot1$	1117	$37+9\rho$	$3\cdot1\cdot\sqrt{-3}$
1123	$34+33\rho$	$1\cdot\bar\rho$	1129	$-32+3\rho$	$-1\cdot\bar\rho$	1153	$16+39\rho$	$-4\cdot\rho\cdot\sqrt{-3}$
1213	$28+39\rho$	$1\cdot\rho$	1201	$40+21\rho$	$-4\cdot\bar\rho$	1171	$25+39\rho$	$-1\cdot\rho\cdot\sqrt{-3}$
1231	$10+39\rho$	$4\cdot\rho$	1237	$37+33\rho$	$-1\cdot\rho$	1279	$-5+33\rho$	$-2\cdot\bar\rho\cdot\sqrt{-3}$
1249	$40+27\rho$	$-5\cdot1$	1291	$-26+15\rho$	$5\cdot\rho$	1297	$7+39\rho$	$-1\cdot\rho\cdot\sqrt{-3}$
1303	$-14+27\rho$	$-2\cdot1$	1327	$19+42\rho$	$-1\cdot\rho$	1423	$31+42\rho$	$1\cdot\bar\rho\cdot\sqrt{-3}$
1321	$40+9\rho$	$-2\cdot1$	1381	$4+39\rho$	$2\cdot\bar\rho$	1459	$43+30\rho$	$-1\cdot\rho\cdot\sqrt{-3}$
1429	$43+15\rho$	$-2\cdot\bar\rho$	1399	$43+18\rho$	$2\cdot1$	1531	$19+45\rho$	0
1447	$37+39\rho$	$7\cdot\rho$	1453	$-23+21\rho$	$-1\cdot\bar\rho$	1549	$28+45\rho$	0
1483	$1+39\rho$	$-2\cdot\rho$	1471	$-35+6\rho$	$-1\cdot\rho$	1567	$-38+3\rho$	$2\cdot\rho\cdot\sqrt{-3}$
1609	$13+45\rho$	$7\cdot1$	1489	$40+3\rho$	$-7\cdot\bar\rho$	1621	$-35+9\rho$	$-3\cdot1\cdot\sqrt{-3}$
1627	$43+6\rho$	$1\cdot\bar\rho$	1543	$43+9\rho$	$2\cdot1$	1657	$-23+24\rho$	$1\cdot\bar\rho\cdot\sqrt{-3}$
1663	$-26+21\rho$	$-2\cdot\rho$	1579	$37+42\rho$	$-1\cdot\rho$	1693	$43+39\rho$	$2\cdot\rho\cdot\sqrt{-3}$
1699	$-17+30\rho$	$1\cdot\rho$	1597	$43+36\rho$	$2\cdot1$	1747	$-14+33\rho$	$1\cdot\bar\rho\cdot\sqrt{-3}$

Table 2.1. The elliptic Gauss sum coefficients

$$p = \pi\overline{\pi} \equiv 13 \pmod{16}, \quad \mathcal{G}_\pi(\chi_\pi, \varphi) = \alpha_\pi \widetilde{\pi}^3$$

p	π	α_π	p	π	α_π	p	π	α_π
13	$3+2i$	1	2221	$-45+14i$	-5	4973	$67+22i$	-1
29	$-5+2i$	1	2237	$11+46i$	-1	5021	$11+70i$	-1
61	$-5+6i$	-1	2269	$-37+30i$	1	5101	$51+50i$	-1
109	$3+10i$	1	2333	$43+22i$	-1	5197	$-29+66i$	-1
157	$11+6i$	1	2381	$35+34i$	1	5261	$19+70i$	3
173	$-13+2i$	-1	2477	$19+46i$	5	5309	$-53+50i$	-7
269	$-13+10i$	3	2557	$-21+46i$	5	5437	$-69+26i$	1
317	$11+14i$	-1	2621	$11+50i$	3	5501	$-5+74i$	1
349	$-5+18i$	1	2749	$43+30i$	-1	5581	$35+66i$	1
397	$19+6i$	-1	2797	$51+14i$	1	5693	$43+62i$	5
461	$19+10i$	-1	2861	$19+50i$	-1	5741	$-29+70i$	-1
509	$-5+22i$	-1	2909	$-53+10i$	-5	5821	$75+14i$	-5
541	$-21+10i$	1	2957	$-29+46i$	1	5869	$-45+62i$	-1
557	$19+14i$	-1	3037	$11+54i$	-1	5981	$59+50i$	1
653	$-13+22i$	-1	3181	$-45+34i$	-5	6029	$-77+10i$	1
701	$-5+26i$	1	3229	$27+50i$	7	6173	$-53+58i$	-1
733	$27+2i$	1	3373	$3+58i$	1	6221	$-61+50i$	1
797	$11+26i$	-1	3389	$-5+58i$	-5	6269	$-37+70i$	1
829	$27+10i$	-5	3469	$-45+38i$	5	6301	$75+26i$	-5
877	$-29+6i$	-1	3517	$59+6i$	-3	6317	$-29+74i$	-1
941	$-29+10i$	3	3533	$-13+58i$	-1	6397	$59+54i$	5
1021	$11+30i$	-3	3581	$59+10i$	1	6637	$-61+54i$	1
1069	$-13+30i$	-1	3613	$43+42i$	5	6653	$-53+62i$	-1
1117	$-21+26i$	-5	3677	$59+14i$	3	6701	$35+74i$	-5
1181	$-5+34i$	1	3709	$-53+30i$	-3	6733	$3+82i$	-5
1213	$27+22i$	-3	3821	$-61+10i$	1	6781	$75+34i$	-5
1229	$35+2i$	3	3853	$3+62i$	1	6829	$-77+30i$	-1
1277	$11+34i$	-1	3917	$-61+14i$	-1	7069	$75+38i$	3
1373	$-37+2i$	-3	4013	$-13+62i$	-3	7213	$83+18i$	5
1453	$3+38i$	-3	4093	$27+58i$	-3	7229	$-85+2i$	-1
1549	$35+18i$	1	4157	$59+26i$	-1	7309	$35+78i$	1
1597	$-21+34i$	-5	4253	$-53+38i$	1	7517	$11+86i$	-1
1613	$-13+38i$	-1	4349	$43+50i$	-1	7549	$-85+18i$	-1
1693	$-37+18i$	-1	4397	$-61+26i$	3	7741	$75+46i$	-5
1709	$35+22i$	1	4493	$67+2i$	-3	7757	$19+86i$	-5
1741	$-29+30i$	3	4621	$-61+30i$	-1	7789	$83+30i$	-1
1789	$-5+42i$	1	4637	$59+34i$	-1	7853	$67+58i$	-1
1901	$35+26i$	-1	4733	$-37+58i$	3	7901	$-85+26i$	-1
1933	$-13+42i$	1	4813	$67+18i$	1	7933	$43+78i$	3
1949	$43+10i$	1	4861	$-69+10i$	3	7949	$35+82i$	-1
1997	$-29+34i$	1	4877	$-61+34i$	5	8093	$-37+82i$	3
2029	$-45+2i$	1	4909	$3+70i$	1	8221	$11+90i$	-1
2141	$-5+46i$	-3	4957	$-69+14i$	-1	8237	$-29+86i$	1

Table 2.2. The elliptic Gauss sum coefficients

$$p = \pi\overline{\pi} \equiv 5 \pmod{16}, \quad \mathcal{G}_\pi(\chi_\pi, \mathsf{Z}) = \alpha_\pi\, \widetilde{\pi}^3$$

p	π	α_π	p	π	α_π	p	π	α_π
37	$-1+6i$	$1\cdot i$	2357	$-41+26i$	$-1\cdot(-1)$	5477	$-1+74i$	$-5\cdot 1$
53	$7+2i$	$1\cdot 1$	2389	$-25+42i$	$1\cdot(-1)$	5557	$-9+74i$	$1\cdot(-1)$
101	$-1+10i$	$1\cdot 1$	2437	$-49+6i$	$-1\cdot i$	5573	$47+58i$	$1\cdot 1$
149	$7+10i$	$1\cdot(-1)$	2549	$7+50i$	$-5\cdot 1$	5653	$-73+18i$	$3\cdot 1$
181	$-9+10i$	$-1\cdot(-1)$	2677	$39+34i$	$-3\cdot 1$	5669	$-65+38i$	$1\cdot i$
197	$-1+14i$	$-1\cdot(-i)$	2693	$47+22i$	$-1\cdot i$	5701	$15+74i$	$1\cdot 1$
229	$15+2i$	$-1\cdot(-1)$	2741	$-25+46i$	$-3\cdot i$	5717	$71+26i$	$-1\cdot(-1)$
277	$-9+14i$	$-1\cdot i$	2789	$-17+50i$	$-1\cdot(-1)$	5749	$-57+50i$	$-1\cdot 1$
293	$-17+2i$	$1\cdot(-1)$	2837	$-41+34i$	$5\cdot 1$	5813	$-73+22i$	$1\cdot(-i)$
373	$7+18i$	$-1\cdot 1$	2917	$-1+54i$	$-3\cdot i$	5861	$31+70i$	$5\cdot i$
389	$-17+10i$	$1\cdot 1$	3061	$55+6i$	$-1\cdot(-i)$	6037	$-41+66i$	$-1\cdot 1$
421	$15+14i$	$1\cdot(-i)$	3109	$47+30i$	$1\cdot(-i)$	6053	$47+62i$	$1\cdot(-i)$
613	$-17+18i$	$-1\cdot(-1)$	3221	$55+14i$	$-1\cdot i$	6101	$-25+74i$	$-1\cdot(-1)$
661	$-25+6i$	$-1\cdot(-i)$	3253	$-57+2i$	$-3\cdot 1$	6133	$7+78i$	$-1\cdot i$
677	$-1+26i$	$3\cdot 1$	3301	$-49+30i$	$-1\cdot(-i)$	6197	$71+34i$	$-1\cdot 1$
709	$15+22i$	$1\cdot i$	3413	$7+58i$	$-1\cdot(-1)$	6229	$-73+30i$	$5\cdot i$
757	$-9+26i$	$1\cdot(-1)$	3461	$31+50i$	$-1\cdot(-1)$	6277	$79+6i$	$-5\cdot i$
773	$-17+22i$	$1\cdot i$	3541	$-25+54i$	$1\cdot(-i)$	6373	$-17+78i$	$-1\cdot(-i)$
821	$-25+14i$	$1\cdot i$	3557	$-49+34i$	$1\cdot(-1)$	6389	$55+58i$	$1\cdot(-1)$
853	$23+18i$	$1\cdot 1$	3637	$39+46i$	$5\cdot i$	6421	$39+70i$	$-1\cdot(-i)$
997	$31+6i$	$1\cdot i$	3701	$55+26i$	$-1\cdot(-1)$	6469	$63+50i$	$-1\cdot(-1)$
1013	$23+22i$	$1\cdot(-i)$	3733	$-57+22i$	$1\cdot(-i)$	6581	$-41+70i$	$-1\cdot(-i)$
1061	$31+10i$	$1\cdot 1$	3797	$-41+46i$	$5\cdot i$	6661	$-81+10i$	$-3\cdot 1$
1093	$-33+2i$	$-1\cdot(-1)$	3877	$31+54i$	$1\cdot i$	6709	$-25+78i$	$-1\cdot i$
1109	$-25+22i$	$1\cdot(-i)$	3989	$-25+58i$	$1\cdot(-1)$	6869	$55+62i$	$-3\cdot i$
1237	$-9+34i$	$1\cdot 1$	4021	$39+50i$	$3\cdot 1$	6917	$79+26i$	$3\cdot 1$
1301	$-25+26i$	$1\cdot(-1)$	4133	$-17+62i$	$1\cdot(-i)$	6949	$15+82i$	$1\cdot(-1)$
1381	$15+34i$	$-1\cdot(-1)$	4229	$-65+2i$	$-1\cdot(-1)$	6997	$39+74i$	$-1\cdot(-1)$
1429	$23+30i$	$-1\cdot i$	4261	$-65+6i$	$3\cdot i$	7013	$-17+82i$	$-1\cdot(-1)$
1493	$7+38i$	$-1\cdot(-i)$	4357	$-1+66i$	$-1\cdot(-1)$	7109	$47+70i$	$1\cdot i$
1621	$39+10i$	$-1\cdot(-1)$	4373	$23+62i$	$-1\cdot i$	7237	$-81+26i$	$-7\cdot 1$
1637	$31+26i$	$-1\cdot 1$	4421	$-65+14i$	$-1\cdot(-i)$	7253	$23+82i$	$-5\cdot 1$
1669	$15+38i$	$-3\cdot i$	4517	$-49+46i$	$1\cdot(-i)$	7333	$63+58i$	$3\cdot 1$
1733	$-17+38i$	$-3\cdot i$	4549	$-65+18i$	$-1\cdot(-1)$	7349	$-25+82i$	$-1\cdot 1$
1861	$31+30i$	$-1\cdot(-i)$	4597	$-41+54i$	$1\cdot(-i)$	7477	$-9+86i$	$-1\cdot(-i)$
1877	$-41+14i$	$3\cdot i$	4789	$55+42i$	$1\cdot(-1)$	7541	$71+50i$	$1\cdot 1$
1973	$23+38i$	$-1\cdot(-i)$	4933	$-33+62i$	$-1\cdot(-i)$	7573	$87+2i$	$5\cdot 1$
2053	$-17+42i$	$-1\cdot 1$	5077	$71+6i$	$-1\cdot(-i)$	7589	$-65+58i$	$1\cdot 1$
2069	$-25+38i$	$1\cdot(-i)$	5189	$-17+70i$	$-1\cdot i$	7621	$15+86i$	$3\cdot i$
2213	$47+2i$	$1\cdot(-1)$	5237	$71+14i$	$-1\cdot i$	7669	$87+10i$	$3\cdot(-1)$
2293	$23+42i$	$-1\cdot(-1)$	5333	$-73+2i$	$1\cdot 1$	7717	$-81+34i$	$1\cdot(-1)$
2309	$47+10i$	$5\cdot 1$	5381	$-65+34i$	$-1\cdot(-1)$	7829	$-73+50i$	$5\cdot 1$
2341	$15+46i$	$-1\cdot(-i)$	5413	$63+38i$	$1\cdot i$	7877	$-49+74i$	$-1\cdot 1$

Table 2.3. The elliptic Gauss sum coefficients

$$p = \pi\overline{\pi} \equiv 1 \pmod{16}, \quad \mathcal{G}_\pi(\chi_\pi, \psi) = \alpha_\pi \widetilde{\pi}^3$$

p	π	$\frac{1}{2}\alpha_\pi$	$\chi_\pi(1+i)$	p	π	$\frac{1}{2}\alpha_\pi$	$\chi_\pi(1+i)$
17	$1+4i$	$1\cdot i\zeta_8$	$-i$	2657	$49+16i$	0	1
97	$9+4i$	$-1\cdot\zeta_8$	i	2689	$33+40i$	$-3\cdot\sqrt{2}$	-1
113	$-7+8i$	0	1	2753	$-7+52i$	$-1\cdot\zeta_8$	i
193	$-7+12i$	$-1\cdot i\zeta_8$	$-i$	2801	$49+20i$	$-1\cdot i\zeta_8$	$-i$
241	$-15+4i$	$1\cdot i\zeta_8$	$-i$	2833	$-23+48i$	$1\cdot\sqrt{2}$	-1
257	$1+16i$	0	1	2897	$-31+44i$	$-1\cdot\zeta_8$	i
337	$9+16i$	$1\cdot\sqrt{2}$	-1	3041	$-55+4i$	$-1\cdot\zeta_8$	i
353	$17+8i$	$1\cdot\sqrt{2}$	-1	3089	$-55+8i$	$2\cdot i\sqrt{2}$	1
401	$1+20i$	$1\cdot i\zeta_8$	$-i$	3121	$-39+40i$	0	1
433	$17+12i$	$-1\cdot\zeta_8$	i	3137	$1+56i$	$1\cdot\sqrt{2}$	-1
449	$-7+20i$	$1\cdot\zeta_8$	i	3169	$-55+12i$	$-1\cdot i\zeta_8$	$-i$
577	$1+24i$	$1\cdot\sqrt{2}$	-1	3217	$9+56i$	0	1
593	$-23+8i$	0	1	3313	$57+8i$	$-2\cdot i\sqrt{2}$	1
641	$25+4i$	$1\cdot\zeta_8$	i	3329	$25+52i$	$-1\cdot\zeta_8$	i
673	$-23+12i$	$1\cdot i\zeta_8$	$-i$	3361	$-15+56i$	$1\cdot\sqrt{2}$	-1
769	$25+12i$	$1\cdot i\zeta_8$	$-i$	3457	$-39+44i$	$-5\cdot i\zeta_8$	$-i$
881	$25+16i$	$1\cdot\sqrt{2}$	-1	3617	$41+44i$	$-5\cdot i\zeta_8$	$-i$
929	$-23+20i$	$1\cdot\zeta_8$	i	3697	$49+36i$	$1\cdot i\zeta_8$	$-i$
977	$-31+4i$	$1\cdot i\zeta_8$	$-i$	3761	$25+56i$	0	1
1009	$-15+28i$	$-1\cdot\zeta_8$	i	3793	$33+52i$	$-1\cdot i\zeta_8$	$-i$
1153	$33+8i$	$1\cdot\sqrt{2}$	-1	3889	$17+60i$	$1\cdot\zeta_8$	i
1201	$25+24i$	0	1	4001	$49+40i$	$-1\cdot\sqrt{2}$	-1
1217	$-31+16i$	0	1	4049	$-55+32i$	$1\cdot\sqrt{2}$	-1
1249	$-15+32i$	$2\cdot i\sqrt{2}$	1	4129	$-23+60i$	$-1\cdot i\zeta_8$	$-i$
1297	$1+36i$	$3\cdot i\zeta_8$	$-i$	4177	$9+64i$	$-1\cdot\sqrt{2}$	-1
1361	$-31+20i$	$1\cdot i\zeta_8$	$-i$	4241	$65+4i$	$1\cdot i\zeta_8$	$-i$
1409	$25+28i$	$-1\cdot i\zeta_8$	$-i$	4273	$57+32i$	$1\cdot\sqrt{2}$	-1
1489	$33+20i$	$5\cdot i\zeta_8$	$-i$	4289	$65+8i$	$1\cdot\sqrt{2}$	-1
1553	$-23+32i$	$1\cdot\sqrt{2}$	-1	4337	$49+44i$	$1\cdot\zeta_8$	i
1601	$1+40i$	$1\cdot\sqrt{2}$	-1	4481	$65+16i$	0	1
1697	$41+4i$	$-3\cdot\zeta_8$	i	4513	$-47+48i$	0	1
1777	$-39+16i$	$-1\cdot\sqrt{2}$	-1	4561	$-31+60i$	$-1\cdot\zeta_8$	i
1873	$33+28i$	$-1\cdot\zeta_8$	i	4657	$-39+56i$	$-2\cdot i\sqrt{2}$	1
1889	$17+40i$	$1\cdot\sqrt{2}$	-1	4673	$-7+68i$	$-1\cdot\zeta_8$	i
2017	$9+44i$	$-1\cdot i\zeta_8$	$-i$	4721	$25+64i$	$-1\cdot\sqrt{2}$	-1
2081	$41+20i$	$-1\cdot\zeta_8$	i	4801	$65+24i$	$1\cdot\sqrt{2}$	-1
2113	$33+32i$	$-2\cdot i\sqrt{2}$	1	4817	$41+56i$	0	1
2129	$-23+40i$	0	1	4993	$-63+32i$	0	1
2161	$-15+44i$	$-1\cdot\zeta_8$	i	5009	$65+28i$	$1\cdot\zeta_8$	i
2273	$-47+8i$	$1\cdot\sqrt{2}$	-1	5153	$-23+68i$	$1\cdot\zeta_8$	i
2417	$49+4i$	$1\cdot i\zeta_8$	$-i$	5233	$-7+72i$	0	1
2593	$17+48i$	0	1	5281	$41+60i$	$1\cdot i\zeta_8$	$-i$
2609	$-47+20i$	$5\cdot i\zeta_8$	$-i$	5297	$-71+16i$	$1\cdot\sqrt{2}$	-1

Table 2.4. The elliptic Gauss sum coefficients

$$p = \pi\overline{\pi} \equiv 9 \pmod{16}, \quad \mathcal{G}_\pi(\chi_\pi, \psi) = \alpha_\pi \widetilde{\pi}^3$$

p	π	$\frac{1}{2}\alpha_\pi$	$\chi_\pi(1+i)$	p	π	$\frac{1}{2}\alpha_\pi$	$\chi_\pi(1+i)$
41	$5+4i$	$-1\cdot i\zeta_8$	1	2521	$-35+36i$	$1\cdot\zeta_8$	-1
73	$-3+8i$	$1\cdot i\sqrt{2}$	i	2617	$-51+4i$	$-1\cdot\zeta_8$	-1
89	$5+8i$	0	$-i$	2633	$-43+28i$	$1\cdot\zeta_8$	-1
137	$-11+4i$	$-1\cdot i\zeta_8$	1	2713	$-3+52i$	$-1\cdot\zeta_8$	-1
233	$13+8i$	$1\cdot i\sqrt{2}$	i	2729	$5+52i$	$-1\cdot i\zeta_8$	1
281	$5+16i$	$1\cdot i\sqrt{2}$	i	2777	$29+44i$	$-1\cdot i\zeta_8$	1
313	$13+12i$	$1\cdot i\zeta_8$	1	2857	$-51+16i$	0	$-i$
409	$-3+20i$	$-1\cdot\zeta_8$	-1	2953	$53+12i$	$-1\cdot\zeta_8$	-1
457	$21+4i$	$1\cdot i\zeta_8$	1	2969	$37+40i$	0	$-i$
521	$-11+20i$	$-1\cdot i\zeta_8$	1	3001	$-51+20i$	$1\cdot\zeta_8$	-1
569	$13+20i$	$-3\cdot\zeta_8$	-1	3049	$45+32i$	0	$-i$
601	$5+24i$	0	$-i$	3209	$53+20i$	$-3\cdot i\zeta_8$	1
617	$-19+16i$	$2\cdot\sqrt{2}$	$-i$	3257	$-11+56i$	0	$-i$
761	$-19+20i$	$-1\cdot\zeta_8$	-1	3433	$-27+52i$	$-1\cdot i\zeta_8$	1
809	$5+28i$	$-1\cdot\zeta_8$	-1	3449	$-43+40i$	0	$-i$
857	$29+4i$	$1\cdot\zeta_8$	-1	3529	$-35+48i$	0	$-i$
937	$-19+24i$	$1\cdot i\sqrt{2}$	i	3593	$53+28i$	$1\cdot\zeta_8$	-1
953	$13+28i$	$-1\cdot i\zeta_8$	1	3673	$37+48i$	$-1\cdot i\sqrt{2}$	i
1033	$-3+32i$	$2\cdot\sqrt{2}$	$-i$	3769	$13+60i$	$1\cdot i\zeta_8$	1
1049	$5+32i$	$1\cdot i\sqrt{2}$	i	3833	$53+32i$	$-1\cdot i\sqrt{2}$	i
1097	$29+16i$	0	$-i$	3881	$-59+20i$	$-1\cdot i\zeta_8$	1
1129	$-27+20i$	$3\cdot i\zeta_8$	1	3929	$-35+52i$	$1\cdot\zeta_8$	-1
1193	$13+32i$	0	$-i$	4057	$-59+24i$	$-2\cdot\sqrt{2}$	$-i$
1289	$-35+8i$	$1\cdot i\sqrt{2}$	i	4073	$37+52i$	$-3\cdot i\zeta_8$	1
1321	$5+36i$	$-1\cdot i\zeta_8$	1	4153	$-43+48i$	$-1\cdot i\sqrt{2}$	i
1433	$37+8i$	0	$-i$	4201	$-51+40i$	$3\cdot i\sqrt{2}$	i
1481	$-35+16i$	0	$-i$	4217	$-11+64i$	$-1\cdot i\sqrt{2}$	i
1609	$-3+40i$	$1\cdot i\sqrt{2}$	i	4297	$61+24i$	$1\cdot i\sqrt{2}$	i
1657	$-19+36i$	$-3\cdot\zeta_8$	-1	4409	$53+40i$	$-2\cdot\sqrt{2}$	$-i$
1721	$-11+40i$	0	$-i$	4441	$29+60i$	$1\cdot i\zeta_8$	1
1753	$-27+32i$	$-1\cdot i\sqrt{2}$	i	4457	$-19+64i$	0	$-i$
1801	$-35+24i$	$1\cdot i\sqrt{2}$	i	4649	$5+68i$	$1\cdot i\zeta_8$	1
1913	$-43+8i$	$2\cdot\sqrt{2}$	$-i$	4729	$45+52i$	$-1\cdot\zeta_8$	-1
1993	$-43+12i$	$-1\cdot\zeta_8$	-1	4793	$13+68i$	$-1\cdot\zeta_8$	-1
2089	$45+8i$	$-1\cdot i\sqrt{2}$	i	4889	$-67+20i$	$-5\cdot\zeta_8$	-1
2137	$29+36i$	$1\cdot\zeta_8$	-1	4937	$29+64i$	0	$-i$
2153	$37+28i$	$-1\cdot\zeta_8$	-1	4969	$37+60i$	$-1\cdot\zeta_8$	-1
2281	$45+16i$	$2\cdot\sqrt{2}$	$-i$	5081	$-59+40i$	0	$-i$
2297	$-19+44i$	$-1\cdot i\zeta_8$	1	5113	$53+48i$	$-1\cdot i\sqrt{2}$	i
2377	$21+44i$	$-5\cdot\zeta_8$	-1	5209	$5+72i$	$-2\cdot\sqrt{2}$	$-i$
2393	$37+32i$	$-1\cdot i\sqrt{2}$	i	5273	$-67+28i$	$1\cdot i\zeta_8$	1
2441	$29+40i$	$1\cdot i\sqrt{2}$	i	5417	$-59+44i$	$1\cdot\zeta_8$	-1
2473	$13+48i$	0	$-i$	5449	$-43+60i$	$-1\cdot\zeta_8$	-1