RIMS Kôkyûroku Bessatsu **4**
(2007), 153–164

# Ramification and tame characters of a finite flat representation of rank two

By

Shin HATTORI*

**Abstract**

Let $K$ be a complete discrete valuation field of mixed characteristic $(0,p)$ with perfect residue field $F$ and uniformizer $\pi$. In this paper, we propose an example of the main theorem of the paper [10]. Namely, we calculate the conductor $c(\mathcal{G})$ in the sense of Abbes and Saito for a finite flat group scheme $\mathcal{G}$ over $\mathcal{O}_K$ which is reducible, killed by $p$ and of rank $p^2$, and show that the $I_K$-module $\mathcal{G}(\bar{K})$ contains the fundamental character of level $c(\mathcal{G})$. For this purpose, we show that the Dieudonné functor of Breuil is compatible with the base extension $K(\pi^{1/p})/K$

## § 1. Introduction

Let $K$ be a complete discrete valuation field of mixed characteristic $(0,p)$ with perfect residue field $F$, $\pi = \pi_K$ be its uniformizer, $G_K$ be its absolute Galois group and $I_K$ be its inertia subgroup. For $j \in \mathbb{Q}_{>0}$, we define a tame character $\theta_j : I_K \to \bar{F}^\times$ to be $\theta_{l'}^{k'}$, where $k'/l'$ is the prime-to-$p$-denominator part of $j$ mod $\mathbb{Z}$ ([12]). In other words, we set $\theta_j(\sigma) = (\sigma(\pi^{1/l'})/\pi^{1/l'})^{k'}$ mod $\mathfrak{m}_{\bar{K}}$, where $\mathfrak{m}_{\bar{K}}$ is the maximal ideal of $\mathcal{O}_{\bar{K}}$. We refer any of $\mathbb{F}_p$-conjugates of $\theta_j$ as the fundamental character of level $j$.

Let $\mathcal{G}$ be a finite flat group scheme over $\mathcal{O}_K$. When $\mathcal{G}$ is killed by $p$ and monogenic, that is to say, when the affine algebra of $\mathcal{G}$ is generated over $\mathcal{O}_K$ by one element, it is well-known that the tame characters appearing in the $I_K$-module $\mathcal{G}(\bar{K})$ are determined by the slopes of the Newton polygon of a defining equation of $\mathcal{G}$, as follows.

**Proposition 1.1** ([12], Proposition 10). *Let $\mathcal{G}$ be as above and write the affine algebra of $\mathcal{G}$ as $\mathcal{O}_K[T]/(f(T))$ with $f(0) = 0$. Let $s_1, \ldots, s_r$ be the negatives of the slopes of the Newton polygon of $f(T)$. Then the semi-simplification of the $I_K$-module $\mathcal{G}(\bar{K}) \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p$ is the direct sum of fundamental characters of level $s_i$.*

On the other hand, for an elliptic modular form $f$ of level $N$ prime to $p$, we also have a description of the tame characters of the associated mod $p$ Galois representation

*Department of Mathematics, Hokkaido University. E-mail : `shin-h@math.sci.hokudai.ac.jp`

$\bar{\rho}_f$ ([9, Theorem 2.5, Theorem 2.6], [8, Section 4.3]). This is based on Raynaud's theory of prolongations of finite flat group schemes or the integral $p$-adic Hodge theory. However, for an analogous study of the associated mod $p$ Galois representation of a Hilbert modular form over a totally real number field, we encounter a local field of higher absolute ramification index. In this case, these two theories no longer work well and we need some other techniques to study the tame characters of a Galois representation.

In this paper, we show the following theorem, which suggests that the semi-simplification of a finite flat representation can be described by the ramification jumps of its finite flat model over $\mathcal{O}_K$.

**Theorem 1.2.** *Let $\mathcal{G}$ be a finite flat group scheme which is reducible, killed by $p$ and of rank $p^2$. Let $c(\mathcal{G})$ be its conductor in the sense of [2], [3]. Then the $I_K$-module $\mathcal{G}(\bar{K})$ contains the fundamental character of level $c(\mathcal{G})$.*

To prove the main theorem, firstly we show compatibility of the theory of Breuil ([5]) with the base extension from $K$ to $K_1 = K(\pi^{1/p})$ (Theorem 3.3). Using this theorem, we can write down a defining equation of $\mathcal{G}$ over $\mathcal{O}_{K_1}$ and calculate explicitly the tubular neighborhoods and conductor of $\mathcal{G}$ as in [10, Section 5].

In fact, we can show this more generally. In [10], we generalize Proposition 1.1 to the higher dimensional case (namely, the case where $\mathcal{G}$ is not monogenic) without any restriction on the absolute ramification index of $K$, on the residue field $F$ and on $\mathcal{G}$. There we show that we can, at least for the finite flat case, determine the semi-simplification of a Galois representation using the ramification theory of Abbes and Saito ([2], [3]). The main theorem of [10] is the following, whose proof is given there by totally different method from that of Theorem 1.2 in this paper.

**Theorem 1.3** ([10], Theorem 1.1).
*Let $\mathcal{G}$ be a finite flat group scheme over $\mathcal{O}_K$. Write $\{\mathcal{G}^j\}_{j\in\mathbb{Q}_{>0}}$ for the ramification filtration of $\mathcal{G}$ in the sense of [2] and [3]. Then the graded piece $\mathcal{G}^j(\bar{K})/\mathcal{G}^{j+}(\bar{K})$ is killed by $p$ and the $I_K$-module $\mathcal{G}^j(\bar{K})/\mathcal{G}^{j+}(\bar{K}) \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p$ is the direct sum of fundamental characters of level $j$.*

## § 2. Review of the ramification theory of Abbes and Saito

Let $K$ be a complete discrete valuation field with residue field $F$ which may be imperfect. Set $\pi = \pi_K$ to be a uniformizer of $K$. The separable closure of $K$ is denoted by $\bar{K}$ and the absolute Galois group of $K$ by $G_K$. In [2] and [3], Abbes and Saito defined the ramification theory of a finite flat $\mathcal{O}_K$-algebra of relative complete intersection. In this section, we gather the necessary definitions and briefly recall their theory.

Let $A$ be a finite flat $\mathcal{O}_K$-algebra and $\mathbb{A}$ be a complete Noetherian semi-local ring (with its topology defined by $\mathrm{rad}(\mathbb{A})$) which is of formally smooth over $\mathcal{O}_K$ and whose quotient ring $\mathbb{A}/\mathrm{rad}(\mathbb{A})$ is of finite type over $F$. A surjection of $\mathcal{O}_K$-algebras $\mathbb{A} \to A$ is called an embedding if $\mathbb{A}/\mathrm{rad}(\mathbb{A}) \to A/\mathrm{rad}(A)$ is an isomorphism. For an embedding $(\mathbb{A} \to A)$ and $j \in \mathbb{Q}_{>0}$, the $j$-th tubular neighborhood of $(\mathbb{A} \to A)$ is the $K$-affinoid variety $X^j(\mathbb{A} \to A)$ constructed as follows. Write $j = k/l$ with $k, l$ non-negative integers. Put $I = \mathrm{Ker}(\mathbb{A} \to A)$ and

$$\mathcal{A}_0^{k,l} = \mathbb{A}[I^l/\pi^k]^\wedge,$$

where $\wedge$ means the $\pi$-adic completion. Then $\mathcal{A}_0^{k,l}$ is a quotient ring of the Tate algebra $\mathcal{O}_K\langle T_1, \dots, T_r\rangle$ for some $r$. Its generic fiber $\mathcal{A}_K^j = \mathcal{A}_0^{k,l} \otimes_{\mathcal{O}_K} K$ is independent of the choice of a representation $j = k/l$ ([3, Lemma 1.4]) and set

$$X^j(\mathbb{A} \to A) = \mathrm{Sp}(\mathcal{A}_K^j).$$

We put $F(A) = \mathrm{Hom}_{\mathcal{O}_K\text{-alg.}}(A, \mathcal{O}_{\bar{K}})$ and

$$F^j(A) = \varprojlim \pi_0(X^j(\mathbb{A} \to A)_{\bar{K}}).$$

Here $\pi_0(X_{\bar{K}})$ denotes the set of geometric connected components of a $K$-affinoid variety $X$ and the projective limit is taken in the category of embeddings of $A$. Note that the projective family $\pi_0(X^j(\mathbb{A} \to A)_{\bar{K}})$ is constant ([3, Section 1.2]). These define contravariant functors $F$ and $F^j$ from the category of finite flat $\mathcal{O}_K$-algebras to the category of finite $G_K$-sets. Moreover, there are morphisms of functors $F \to F^j$ and $F^{j'} \to F^j$ for $j' \geq j > 0$.

Suppose that $A$ is of relative complete intersection over $\mathcal{O}_K$ and $A \otimes_{\mathcal{O}_K} K$ is etale over $K$. Then the natural map $F(A) \to F^j(A)$ is surjective. The family $\{F(A) \to F^j(A)\}_{j \in \mathbb{Q}_{>0}}$ is separated, exhaustive and its jumps are rational ([2, Proposition 6.4]). The conductor of $A$ is defined to be

$$c(A) = \inf\{j \in \mathbb{Q}_{>0} | F(A) \to F^j(A) \text{ is an isomorphism}\}.$$

If $B$ is the affine algebra of a finite flat group scheme $\mathcal{G}$ over $\mathcal{O}_K$ which is generically etale, then $B$ is of relative complete intersection (for example, [5, Proposition 2.2.2]) and the theory above can all be applied to $B$. By the functoriality, $F^j(B)$ is endowed with a $G_K$-module structure ([1, Lemme 2.1.1]) and the natural map $\mathcal{G}(\bar{K}) = F(B) \to F^j(B)$ is a $G_K$-homomorphism. Let $\mathcal{G}^j$ denote the schematic closure ([11]) in $\mathcal{G}$ of the kernel of this homomorphism. It is called the $j$-th ramification filtration of $\mathcal{G}$. We refer $c(B)$ as the conductor of $\mathcal{G}$, which is denoted also by $c(\mathcal{G})$. We put

$$\mathcal{G}^{j+}(\bar{K}) = \cup_{j' > j} \mathcal{G}^{j'}(\bar{K}).$$

We write the $j$-th tubular neighborhood of $B$ with respect to some embedding as $X_{\mathcal{G}}^{j}$ by abuse of notation.

**Example 2.1.** For integers $0 \leq s_1, \ldots, s_r \leq e$, let $\mathcal{G} = \mathcal{G}(s_1, \ldots, s_r)$ denote the Raynaud $\mathbb{F}_{p^r}$-vector space scheme ([11]) over $\mathcal{O}_K$ defined by the $r$ equations

$$T_1^p = \pi^{s_1} T_2, T_2^p = \pi^{s_2} T_3, \ldots, T_r^p = \pi^{s_r} T_1.$$

We set

$$j_k = (ps_k + p^2 s_{k-1} + \ldots + p^k s_1 + p^{k+1} s_r + p^{k+2} s_{r-1} + \cdots + p^r s_{k+1})/(p^r - 1).$$

Then we have ([10, Theorem 5.5])

$$c(\mathcal{G}) = \sup_k j_k.$$

In this case, we see that the $I_K$-module $\mathcal{G}(\bar{K})$ is given by the fundamental character of level $c(\mathcal{G})$. For the proof, we refer to [10], where we take an appropriate syntomic cover of the affine algebra of $\mathcal{G}$ and compare its $j$-th tubular neighborhood with $X_{\mathcal{G}}^{j}$.

## § 3. Proof of Theorem 1.2

In this section, we assume that $K$ is as in Section 1 and write its residue field as $k$ in place of $F$, in accordance with [5].

Let $\mathcal{G}$ be a finite flat group scheme over $\mathcal{O}_K$ which is reducible, killed by $p$ and of $\mathbb{F}_p$-rank two. Namely, we have an exact sequence

$$0 \to \mathcal{G}(e-r) \to \mathcal{G} \to \mathcal{G}(e-s) \to 0$$

for some integers $0 \leq r, s \leq e$.

To state our result, let us recall the theory of filtered $\phi_1$-modules of Breuil ([5]). In the following, we take the divided power envelope of a $W$-algebra only with respect to the compatibility condition with the natural divided power structure on $pW$.

Let $e$ be the absolute ramification index of $K$, $W = W(k)$ and $\sigma$ be the Frobenius of $W$. We fix once and for all a uniformizer $\pi$ of $K$. Let $E(u) = u^e - pF(u)$ be the Eisenstein polynomial of $\pi$ over $W$ and set $S = S_\pi = (W[u]^{\mathrm{PD}})^{\wedge}$, where the divided power envelope of $W[u]$ is taken with respect to an ideal $(E(u))$ and $\wedge$ means the $\pi$-adic completion. The ring $S$ is endowed with a $\sigma$-semilinear map $\phi : u \mapsto u^p$, which we also call Frobenius, and the natural filtration induced by the divided power structure. We set $\phi_1 = p^{-1}\phi|_{\mathrm{Fil}^1 S}$ and $c = \phi_1(E(u)) \in S^{\times}$. We define $\phi$, $\phi_1$ and a filtration on $S_n = S/p^n$ similarly.

In [5], the following categories of filtered $\phi_1$-modules are defined. Set $'\mathcal{M}$ to be the category consisting of following data;

- an $S$-module $M$ and its $S$-submodule $\mathrm{Fil}^1 M$ containing $\mathrm{Fil}^1 S.M$,
- a $\phi$-semilinear map $\phi_1 : \mathrm{Fil}^1 M \to M$ satisfying
$$\phi_1(s_1 m) = \phi_1(s_1)\phi(m),$$
where $s_1 \in \mathrm{Fil}^1 S$, $m \in M$ and $\phi(m) = c^{-1}\phi_1(E(u)m)$.

Let $\mathcal{M}_1$ be the full subcategory of $'\mathcal{M}$ consisting of $M$ satisfying

- the $S_1$-module $M$ is free of finite rank,
- $\phi_1(\mathrm{Fil}^1 M)$ generates $M$ as an $S$-module.

and $\mathcal{M}$ be the minimal full subcategory of $'\mathcal{M}$ which contains $\mathcal{M}_1$ and stable under extension.

The category $\mathcal{M}$ is shown to be categorically anti-equivalent to the category $(p\text{-Gr}/\mathcal{O}_K)$ of finite flat group schemes over $\mathcal{O}_K$ which is killed by some $p$-power ([5]). Let us recall the definition of this equivalence. Let $\mathrm{Spf}(\mathcal{O}_K)_{\mathrm{syn}}$ be the category of formally syntomic $p$-adic formal schemes, endowed with the Grothendieck topology generated by the surjective families of formally syntomic morphisms. Write $(\mathrm{Ab}/\mathcal{O}_K)$ for the category of abelian sheaves on $\mathrm{Spf}(\mathcal{O}_K)_{\mathrm{syn}}$. The sheaves $\mathcal{O}_{n,\pi}$ and $\mathcal{J}_{n,\pi}$ are defined by the formula
$$\mathcal{O}_{n,\pi}(\mathfrak{X}) = \mathrm{H}^0_{\mathrm{crys}}((\mathfrak{X}_n/S_n)_{\mathrm{crys}}, \mathcal{O}_{\mathfrak{X}_n/S_n})$$
and
$$\mathcal{J}_{n,\pi}(\mathfrak{X}) = \mathrm{H}^0_{\mathrm{crys}}((\mathfrak{X}_n/S_n)_{\mathrm{crys}}, \mathcal{J}_{\mathfrak{X}_n/S_n}),$$
where $\mathfrak{X}_n = \mathfrak{X}/p^n$. We also set $\mathcal{O}_{\infty,\pi} = \varinjlim \mathcal{O}_{n,\pi}$ and $\mathcal{J}_{\infty,\pi} = \varinjlim \mathcal{J}_{n,\pi}$. We let the crystalline Frobenius map be denoted by $\phi : \mathcal{O}_{n,\pi} \to \mathcal{O}_{n,\pi}$. We can define the natural morphism $\phi_1 : \mathcal{J}_{n,\pi} \to \mathcal{O}_{n,\pi}$ which makes the following diagram commutative.
$$\begin{array}{ccc} \mathcal{J}_{n,\pi} & \xrightarrow{\phi_1} & \mathcal{O}_{n,\pi} \\ \uparrow & & \downarrow{\scriptstyle \times p} \\ \mathcal{J}_{n+1,\pi} & \xrightarrow{\phi} & \mathcal{O}_{n+1,\pi} \end{array}$$
Let $\mathcal{G} \in (p\text{-Gr}/\mathcal{O}_K)$ and $M \in \mathcal{M}$. Define
$$\mathrm{Mod}_K(\mathcal{G}) = \mathrm{Hom}_{(\mathrm{Ab}/\mathcal{O}_K)}(\mathcal{G}, \mathcal{O}_{\infty,\pi})$$
and
$$\mathrm{Gr}_K(M) = \mathrm{Hom}_{'\mathcal{M}}(M, \mathcal{O}_{\infty,\pi}).$$
Then the main theorem of [5] is the following.

**Theorem 3.1** ([5]). *The functor* $\mathrm{Gr}_K$ *defines an anti-equivalence of categories* $\mathcal{M} \to (p\text{-Gr}/\mathcal{O}_K)$ *and its quasi-inverse is* $\mathrm{Mod}_K$.

Now let us return to our $\mathcal{G}$. Let $M = \mathrm{Mod}_K(\mathcal{G})$ be the filtered $\phi_1$-module of $\mathcal{G}$. Replacing $K$ with an unramified extension, we may assume that we have an exact sequence in $\mathcal{M}$

$$0 \to M(s) \to M \to M(r) \to 0,$$

where $M(s)$ is the filtered $\phi_1$-module defined by $M(s) = S_1 e$, $\mathrm{Fil}^1 M(s) = u^s S_1 e$ and $\phi_1(u^s e) = e$. By [7, Lemma 5.2.2], we may assume that $\tilde{M} = M/\mathrm{Fil}^p S.M$ is of the following type;

- $\tilde{M} = \tilde{S}_1 e_0 \oplus \tilde{S}_1 e_1$, where $\tilde{S}_1 = k[u]/(u^{ep})$
- $\mathrm{Fil}^1 \tilde{M} = \langle u^s e_0, u^r e_1 + f e_0 \rangle$, where $f \in u^{\sup(0, r+s-e)} \tilde{S}_1$
- $\phi_1(u^s e_0) = e_0$ and $\phi_1(u^r e_1 + f e_0) = e_1$.

Put $m = v_u(f)$. Then we have the following theorem.

**Theorem 3.2.** *If* $s, m \geq r$, *then* $c(\mathcal{G}) = p(e-r)/(p-1)$. *Otherwise,* $c(\mathcal{G})$ *is equal to*

$$\begin{cases} \sup(p(e-r)/(p-1), p(e-s)/(p-1)) \text{ if } m \geq (ps-r)/(p-1), \\ p(e-r)/(p-1) + (r-m) \text{ if } m < (ps-r)/(p-1). \end{cases}$$

*Moreover, the* $I_K$*-module* $\mathcal{G}(\bar{K})$ *contains the fundamental character of level* $c(\mathcal{G})$.

To prove this theorem, we first write down a defining equation of $\mathcal{G}$. This is possible after taking a base extension from $K$ to $K_1 = K(\pi_1)$, where $\pi_1 = \pi^{1/p}$ ([5, Proposition 3.1.2]) and using the theorem below.

**Theorem 3.3.** *Let* $S' = S_{\pi_1}$ *be the* $p$*-adic completion of the divided power envelope constructed as* $S$ *starting from* $E_1(v) = E(v^p) \in W[v]$ *and consider a map* $S \to S'$ *defined by* $u \mapsto v^p$. *Then we have a canonical isomorphism of filtered* $\phi_1$*-modules*

$$\mathrm{Mod}_{K_1}(\mathcal{G} \otimes_{\mathcal{O}_K} \mathcal{O}_{K_1}) \simeq \mathrm{Mod}_K(\mathcal{G}) \otimes_S S'.$$

Then we can calculate the tubular neighborhoods of $\mathcal{G} \times_{\mathcal{O}_K} \mathcal{O}_{K_1}$ and its conductor. As for the assertion on the tame character, we know from [6, Theorem 3.4.3] that it suffices to consider $\mathcal{G} \times_{\mathcal{O}_K} \mathcal{O}_{K_1}$. We can easily check from the shape of a defining equation of $\mathcal{G}$ over $\mathcal{O}_{K_1}$ that if $m > (ps-r)/(p-1)$ and $s < r$, then the $I_{K_1}$-module $\mathcal{G}(\bar{K})$ splits. Hence this assertion follows from the assertion on the conductor.

We omit the calculation here as it is just the same as in the proof of [10, Theorem 5.5]. In the rest of this paper, we prove Theorem 3.3.

**Lemma 3.4.** *The $S$-module $S'$ is free of finite rank.*

*Proof.* The $W[u]$-algebra $W[v]$ is free of finite rank. We have

$$(E(u))W[v] = (E_1(v)).$$

Therefore $W[v]^{\mathrm{PD}} = W[u]^{\mathrm{PD}} \otimes_{W[u]} W[v]$ from [4, Proposition 3.21] and $W[u]^{\mathrm{PD}} \to W[v]^{\mathrm{PD}}$ is also free of finite rank. Thus

$$(W[v]^{\mathrm{PD}})^{\wedge} = (W[u]^{\mathrm{PD}})^{\wedge} \otimes_{W[u]^{\mathrm{PD}}} W[v]^{\mathrm{PD}}.$$

This concludes the proof. □

Let the categories of filtered $\phi_1$-modules over $S'$ be denoted by $'\mathcal{M}'$ and $\mathcal{M}'$. From the lemma above, we can define a filtered $\phi_1$-module structure on $M' = M \otimes_S S'$ for any $M \in {'\mathcal{M}}$ by $\mathrm{Fil}^1 M' = (\mathrm{Fil}^1 M) \otimes_S S'$ and $\phi_{1,M'} = \phi_1 \otimes \phi$. If $M \in \mathcal{M}$, then we have $M' \in \mathcal{M}'$.

For a presheaf $\mathcal{F}$ on $\mathrm{Spf}(\mathcal{O}_K)_{\mathrm{syn}}$, we let $\mathcal{F}|_{\mathcal{O}_{K_1}}$ denote the restriction of $\mathcal{F}$ to $\mathrm{Spf}(\mathcal{O}_{K_1})_{\mathrm{syn}}$. If $\mathcal{F}$ is a sheaf on $\mathrm{Spf}(\mathcal{O}_K)_{\mathrm{syn}}$, then $\mathcal{F}|_{\mathcal{O}_{K_1}}$ is also a sheaf on $\mathrm{Spf}(\mathcal{O}_{K_1})_{\mathrm{syn}}$. By [5, Corollaire 2.3.3], we have the following exact sequences in $(\mathrm{Ab}/\mathcal{O}_{K_1})$.

(1)
$$0 \to \mathcal{O}_{r,\pi}|_{\mathcal{O}_{K_1}} \overset{\times p^s}{\to} \mathcal{O}_{r+s,\pi}|_{\mathcal{O}_{K_1}} \to \mathcal{O}_{s,\pi}|_{\mathcal{O}_{K_1}} \to 0$$

(2)
$$0 \to \mathcal{J}_{r,\pi}|_{\mathcal{O}_{K_1}} \overset{\times p^s}{\to} \mathcal{J}_{r+s,\pi}|_{\mathcal{O}_{K_1}} \to \mathcal{J}_{s,\pi}|_{\mathcal{O}_{K_1}} \to 0$$

Consider an $\mathcal{O}_{K_1}$-algebra

$$\mathfrak{A}' = \mathcal{O}_{K_1}\langle X_1', \dots, X_r'\rangle/(f_1, \dots, f_s),$$

where $\mathcal{O}_{K_1}\langle X_1', \dots, X_r'\rangle$ is the $\pi$-adic completion $\mathcal{O}_{K_1}[X_1', \dots, X_r']^{\wedge}$ and $f_1, \dots, f_s$ is a transversally regular sequence in that ring. Then $\mathrm{Spf}(\mathfrak{A}') \in \mathrm{Spf}(\mathcal{O}_{K_1})_{\mathrm{syn}}$. Put

$$\mathfrak{A}_i' = \mathcal{O}_{K_1}\langle X_0'^{p^{-i}}, \dots, X_r'^{p^{-i}}\rangle/(X_0' - \pi_1, f_1, \dots, f_s)$$

and $\mathfrak{A}_\infty' = \varinjlim \mathfrak{A}_i'$. Note that the formal scheme $\mathrm{Spf}(\mathfrak{A}_i')$ is a covering of $\mathrm{Spf}(\mathfrak{A}')$ in $\mathrm{Spf}(\mathcal{O}_{K_1})_{\mathrm{syn}}$. The $W$-algebra $\mathfrak{A}_i'$ is isomorphic to

$$\begin{aligned}
&\mathcal{O}_K[T]/(T^p - \pi)\langle X_0'^{p^{-i}}, \dots, X_r'^{p^{-i}}\rangle/(X_0' - T, f_1, \dots, f_s) \\
&\quad = W[u,T]/(E(u), T^p - u)\langle X_0'^{p^{-i}}, \dots, X_r'^{p^{-i}}\rangle/(X_0' - T, f_1, \dots, f_s) \\
&\quad = W\langle X_0'^{p^{-i}}, \dots, X_r'^{p^{-i}}\rangle/(E(X_0'^{p}), f_1, \dots, f_s).
\end{aligned}$$

We also set

$$A_\infty' = \mathfrak{A}_\infty'/p = k[X_0'^{p^{-\infty}}, \dots, X_r'^{p^{-\infty}}]/(X_0'^{ep}, \bar{f}_1, \dots, \bar{f}_s),$$

where $\bar{f}_i$ is the image of $f_i$. Put $\mathcal{O}_{r,\pi}(\mathfrak{A}_\infty') = \varinjlim_i \mathcal{O}_{r,\pi}(\mathfrak{A}_i')$ and $\mathcal{J}_{r,\pi}(\mathfrak{A}_\infty') = \varinjlim_i \mathcal{J}_{r,\pi}(\mathfrak{A}_i')$.

**Lemma 3.5.** *There exists a canonical isomorphism*

$$\mathcal{O}_{n,\pi}(\mathfrak{A}'_\infty) = \mathrm{H}^0_{\mathrm{crys}}(\mathfrak{A}'_\infty/p^n/S_n) \to (W_n(A'_\infty) \otimes_{W_n,\sigma^n} W_n[u])^{\mathrm{PD}}.$$

Here the divided power envelope is taken with respect to the kernel of a surjection

$$\begin{aligned} W_n(A'_\infty) \otimes_{W_n,\sigma^n} W_n[u] &\to \mathfrak{A}'_\infty/p^n \\ (x_0, \dots, x_{n-1}) \otimes 1 &\mapsto \sum_{k=0}^{n-1} p^k \hat{x}_k^{p^{n-k}} \\ 1 \otimes u &\mapsto X_0'^p \end{aligned}$$

where $\hat{x}_k$ denotes a lifting of $x_k$ in $\mathfrak{A}'_\infty/p^n$.

*Proof.* We repeat exactly the same argument as in [5, Lemme 2.3.2]. Indeed, this surjection induces a PD-thickening

$$(W_n(A'_\infty) \otimes_{W_n,\sigma^n} W_n[u])^{\mathrm{PD}} \to \mathfrak{A}'_\infty/p^n$$

of $\mathfrak{A}'_\infty/p^n$ over $S_n$ and thus we have the natural projection

$$\mathrm{H}^0_{\mathrm{crys}}(\mathfrak{A}'_\infty/p^n/S_n) \to (W_n(A'_\infty) \otimes_{W_n,\sigma^n} W_n[u])^{\mathrm{PD}}.$$

Its inverse map is defined as follows. For any affine PD-thickening $U \to T$ of $\mathfrak{A}'_\infty/p^n$ over $S_n$, we have a map

$$\begin{aligned} (W_n(A'_\infty) \otimes_{W_n,\sigma^n} W_n[u])^{\mathrm{PD}} &\to \Gamma(U, \mathcal{O}_U) \\ (x_0, \dots, x_{n-1}) \otimes 1 &\mapsto \sum_{k=0}^{n-1} p^k \hat{t}_k^{p^{n-k}} \\ 1 \otimes u &\mapsto u, \end{aligned}$$

where $\hat{t}_k$ is a lifting of $x_k$ in $\Gamma(T, \mathcal{O}_T)$. This is a well-defined ring homomorphism, patches in the non-affine case and induces the inverse map of the natural projection. □

Let us define a morphism $\Psi_M : \mathrm{Gr}_K(M)|_{\mathcal{O}_{K_1}} \to \mathrm{Gr}_{K_1}(M')$ of $(\mathrm{Ab}/\mathcal{O}_{K_1})$ as follows. For any $\mathfrak{X}' \in \mathrm{Spf}(\mathcal{O}_{K_1})_{\mathrm{syn}}$, we want to set

$$\Psi_{M,\mathfrak{X}'} : \mathrm{Hom}_{'\mathcal{M}}(M, \mathcal{O}_{n,\pi}(\mathfrak{X}')) \to \mathrm{Hom}_{'\mathcal{M}'}(M \otimes_S S', \mathcal{O}_{n,\pi_1}(\mathfrak{X}'))$$

by $f \mapsto (m \otimes s' \mapsto s'.\mathrm{pr}^*_{\mathfrak{X}'}(f(m)))$, where

$$\mathrm{pr}^*_{\mathfrak{X}'} : \mathcal{O}_{n,\pi}(\mathfrak{X}') = \mathrm{H}^0_{\mathrm{crys}}(\mathfrak{X}'_n/S_n) \to \mathrm{H}^0_{\mathrm{crys}}(\mathfrak{X}'_n/S'_n) = \mathcal{O}_{n,\pi_1}(\mathfrak{X}')$$

is the natural pull-back. The map $\mathrm{pr}^*_{\mathfrak{X}'}$ respects the filtration. To show the compatibility with $\phi_1$, note that we have a diagram

$$\begin{array}{ccccccc}
\mathcal{J}_{n+1,\pi}|_{\mathcal{O}_{K_1}} & \longrightarrow & \mathcal{J}_{n,\pi}|_{\mathcal{O}_{K_1}} & \xrightarrow{\phi_1} & \mathcal{O}_{n,\pi}|_{\mathcal{O}_{K_1}} & \xrightarrow{\times p} & \mathcal{O}_{n+1,\pi}|_{\mathcal{O}_{K_1}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{J}_{n+1,\pi_1} & \longrightarrow & \mathcal{J}_{n,\pi_1} & \xrightarrow{\phi_1} & \mathcal{O}_{n,\pi_1} & \xrightarrow{\times p} & \mathcal{O}_{n+1,\pi_1},
\end{array}$$

where the vertical arrows are the pull-backs and the left and right squares are commutative. The composites of the horizontal maps are $\phi$. From the exact sequences (1) and (2), we see that the middle square is also commutative. In other words, the map $\mathrm{pr}^*_{\mathfrak{X}'}$ is compatible with $\phi_1$. Therefore, we get a morphism of $(\mathrm{Ab}/\mathcal{O}_{K_1})$

$$\Psi_M : \mathrm{Gr}_K(M)|_{\mathcal{O}_{K_1}} \to \mathrm{Gr}_{K_1}(M').$$

**Theorem 3.6.** *The canonical map $\Psi_M$ is an isomorphism.*

*Proof.*

As the functor $\mathrm{Gr}_K$ is exact, by devissage we may assume that $pM = 0$. The sheaves of both sides come from finite flat group schemes $\mathrm{Gr}_K(M) \times_{\mathcal{O}_K} \mathcal{O}_{K_1}$ and $\mathrm{Gr}_{K_1}(M')$. Thus the bijectivity can be checked after taking the functor $\mathrm{Mod}_{K_1}$. In other words, it suffices to show that

$$\Phi_M : M' = M \otimes_S S' \to \mathrm{Hom}_{(\mathrm{Ab}/\mathcal{O}_{K_1})}(\mathrm{Hom}_{'\mathcal{M}}(M, \mathcal{O}_{1,\pi}|_{\mathcal{O}_{K_1}}), \mathcal{O}_{1,\pi_1}),$$

defined by $m \otimes s' \mapsto (f \mapsto s'.\mathrm{pr}^*(f(m)))$ is an isomorphism of $'\mathcal{M}'$. Here $\mathrm{pr}^*$ denotes the pull-back map $\mathcal{O}_{1,\pi}|_{\mathcal{O}_{K_1}} \to \mathcal{O}_{1,\pi_1}$.

We have $\mathrm{rank}_{S'_1}(M \otimes_S S') = \mathrm{rank}_{S_1}(M)$ and

$$\begin{aligned}
&\mathrm{rank}_{S'_1}(\mathrm{Hom}_{(\mathrm{Ab}/\mathcal{O}_{K_1})}(\mathrm{Hom}_{'\mathcal{M}}(M, \mathcal{O}_{1,\pi}|_{\mathcal{O}_{K_1}}), \mathcal{O}_{1,\pi_1})) \\
&\quad = \mathrm{rank}_{S'_1}(\mathrm{Mod}_{K_1}(\mathrm{Gr}_K(M) \times_{\mathcal{O}_K} \mathcal{O}_{K_1})) = \mathrm{rank}_{S_1}(M).
\end{aligned}$$

By [5, Lemme 3.3.2], it suffices to show $\mathrm{Ker}(\Phi_M) \subseteq \mathrm{Fil}^p S'_1 M'$.

Take an adapted basis $\{e_1, \ldots, e_d\}$ of $M$. Let $m = \sum_{i=1}^d s'_i e_i$ be an element of $\mathrm{Ker}(\Phi_M)$. Consider the affine algebra $R_M$ of $\mathrm{Gr}_K(M)$ and the element $f \in \mathrm{Hom}_S^{'\mathcal{M}}(M, \mathcal{O}_{1,\pi}(R_M)) \simeq \mathrm{Gr}_K(M)(R_M)$ corresponding to $\mathrm{id}_{R_M}$. Then, from the proof of [5, Proposition 3.1.5], we have

$$f(e_i) \equiv \bar{X}_{i,0} + u\bar{X}_{i,1} + \cdots + u^{p-1}\bar{X}_{i,p-1} \mod \mathcal{J}_{1,\pi}^{[p]}(R_M),$$

where $X_{i,0}, \ldots, X_{i,p-1}$ are the generators of $R_M$ as in [5, p.507] and $\bar{X}_{i,k}$ is the image of $X_{i,k}$ in $R_M/p$. Here we regard $\bar{X}_{i,k}$ as an element of $\mathcal{O}_{1,\pi}(R_M)$ by the natural map $(R_M/p) \otimes_{k,\sigma} k[u] \to \mathcal{O}_{1,\pi}(R_M)$. Let us write $f_1$ for the image of $f$ by the natural map

$$\mathrm{Hom}_{'\mathcal{M}}(M, \mathcal{O}_{1,\pi}(R_M)) \to \mathrm{Hom}_{'\mathcal{M}}(M, \mathcal{O}_{1,\pi}(R'_M)),$$

where $R'_M = R_M \otimes_{\mathcal{O}_K} \mathcal{O}_{K_1}$. As $m \in \mathrm{Ker}(\Phi_M)$, we have

$$\sum s'_i \mathrm{pr}^*_{R'_M}(f_1(e_i)) = 0.$$

Let $\bar{X}'_{i,k}$ be the image of $\bar{X}_{i,k}$ by the natural map $(R'_M/p) \otimes_{k,\sigma} k[v] \to \mathcal{O}_{1,\pi_1}(R'_M)$. Now we claim that $\mathrm{pr}^*_{R'_M}(\bar{X}_{i,k}) = \bar{X}'_{i,k}$. It is sufficient to show this equality on an appropriate syntomic cover of $R'_M$. Thus we may consider $\mathrm{pr}^*_{R'_{M,\infty}} : \mathcal{O}_{1,\pi}(R'_{M,\infty}) \to \mathcal{O}_{1,\pi_1}(R'_{M,\infty})$, where $R'_{M,\infty}$ is the ring constructed from $\mathfrak{A}' = R'_M$ as in the proof of Lemma 3.5. Then the composite

$$((R'_{M,\infty}/p) \otimes_{k,\sigma} k[u])^{\mathrm{PD}} \xrightarrow{\mathrm{pr}^*} \mathrm{H}^0_{\mathrm{crys}}((R'_{M,\infty}/p)/S'_1) \xrightarrow{\mathrm{can.}} ((R'_{M,\infty}/p) \otimes_{k,\sigma} k[v])^{\mathrm{PD}}$$

maps $1 \otimes u$ to $1 \otimes v^p$ and $r \otimes 1$ to $\hat{r}^p \otimes 1$, where $\hat{r}$ is a lifting of $r$ by the canonical surjection $((R'_{M,\infty}/p) \otimes_{k,\phi} k[v])^{\mathrm{PD}} \to R'_{M,\infty}/p$. We may take $\hat{r}$ to be $r^{1/p} \otimes 1$. Thus the claim follows.

Now we have

$$\sum_{i=1}^{d} s'_i(\bar{X}'_{i,0} + v^p \bar{X}'_{i,1} + \cdots + v^{p(p-1)} \bar{X}'_{i,p-1}) = 0$$

in $\mathcal{O}_{1,\pi_1}(R'_M)/\mathcal{J}^{[p]}_{1,\pi_1}(R'_M)$. This equation also holds in

$$\mathcal{O}_{1,\pi_1}(R'_{M,\infty})/\mathcal{J}^{[p]}_{1,\pi_1}(R'_{M,\infty})$$

and its subring

$$(R'_{M,\infty}/p)[v]/(v^p - X'_0) = (R'_{M,\infty}/p)[v]/(v^p - \pi_1)$$

(see [5, Lemme 2.3.2]). As $R'_{M,\infty}$ is the direct limit of syntomic covers of $R'_M$, $R'_M/p$ is a subring of $R'_{M,\infty}/p$. Thus the equation

$$\sum_{i=1}^{d} s'_i(\bar{X}'_{i,0} + v^p \bar{X}'_{i,1} + \cdots + v^{p(p-1)} \bar{X}'_{i,p-1}) = 0$$

holds in $(R'_M/p)[v]/(v^p - \pi_1)$. Let us write $\bar{s}'_i$ for $s'_i$ mod $v \in k$. Taking mod $v$, we have

$$\sum_{i=1}^{d} \bar{s}'_i \bar{X}'_{i,0} = 0$$

in $(R'_M/p)[v]/(v, v^p - \pi_1) = R'_M/\pi_1 = R_M/\pi$. From the proof of [5, Proposition 3.1.1], we know that $X_{1,0}, \ldots, X_{d,0}$ are linearly independent over $k$ in $R_M/\pi$. Thus $\bar{s}'_i = 0$ and $s'_i \in vS'_1 + \mathrm{Fil}^p S'_1$ for all $i$. Take $s'^{(1)}_i \in S'_1$ satisfying $s'_i - vs'^{(1)}_i \in \mathrm{Fil}^p S'_1$. Then we have

$$v \sum_{i=1}^{d} s'^{(1)}_i(\bar{X}'_{i,0} + v^p \bar{X}'_{i,1} + \cdots + v^{p(p-1)} \bar{X}'_{i,p-1}) = 0$$

in $(R'_M/p)[v]/(v^p - \pi_1)$. However,

$$R'_M/p \simeq (\mathcal{O}_{K_1}/p)^{\oplus N} \simeq (k[T]/(T^{ep}))^{\oplus N}$$

for some $N$ and

$$(k[T]/(T^{ep}))[v]/(v^p - T) \simeq k[v]/(v^{ep^2}).$$

Thus $(R'_M/p)[v]/(v^p - \pi_1)$ is finite flat over $k[v]/(v^{ep^2})$, and we have

$$\sum_{i=1}^{d} s'^{(1)}_i(\bar{X}'_{i,0} + v^p \bar{X}'_{i,1} + \cdots + v^{p(p-1)}\bar{X}'_{i,p-1}) \in v^{ep^2-1}(R'_M/p)[v]/(v^p - \pi_1).$$

Taking mod $v$ and repeating this procedure show $s'_i \in v^{ep^2}S'_1 + \mathrm{Fil}^p S'_1 = \mathrm{Fil}^p S'_1$. In other words, $m \in \mathrm{Fil}^p S'_1 M'$. This concludes the proof. □

**Remark 3.7.** In general, let $L$ be a totally ramified extension over $K$ of degree $e'$ whose uniformizer is denoted by $\pi_L$. When we define $S_L = S_{\pi_L}$ as above, there exists a morphism $S \to S_L$ respecting the filtration and $\phi_1$ if and only if $\pi_L^{e'} = \pi\zeta_{p-1}^i$ for some $i$.

**Acknowledgments.** The author would like to thank Takeshi Saito for many valuable discussions. He also would like to thank an anonymous referee for a number of comments improving the paper.

## References

[1] A. Abbes and A. Mokrane: *Sous-groupes canoniques et cycles évanescents p-adiques pour les variétés abéliennes*, Publ. Math. IHES **99** (2004), 117-162.

[2] A. Abbes and T. Saito: *Ramification of local fields with imperfect residue fields I*, Amer. J. Math. **124** (2002), 879-920.

[3] A. Abbes and T. Saito: *Ramification of local fields with imperfect residue fields II*, Documenta Math. Extra volume: Kazuya Kato's Fiftieth Birthday (2003), 5-72.

[4] P. Berthelot and A. Ogus: *Notes on crystalline cohomology*, Princeton Univ. Press and Univ. of Tokyo Press, (1978).

[5] C. Breuil: *Groupes p-divisibles, groupes finis et modules filtrés*, Ann. of Math. (2) **152** (2000), 489-549.

[6] C. Breuil: *Integral p-adic Hodge theory*, Advanced Studies in Pure Math. **36** (2002), pp.51-80.

[7] C. Breuil, B. Conrad, F. Diamond and R. Taylor: *On the modularity of elliptic curves over* $\mathbb{Q}$*: wild* 3*-adic exercises*, J. Amer. Math. Soc., **14** (2001), 843-939.

[8] C. Breuil and A. Mézard: *Multiplicités modulaires et représentations de* $\mathrm{GL}_2(\mathbb{Z}_p)$ *et de* $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ *en* $l = p$, Duke Math. J. **115** (2002), 205-310.

[9] B. Edixhoven: *The weight in Serre's conjectures on modular forms*, Invent. Math. **109** (1992), 563-594.

[10] S. Hattori: *Tame characters and ramification of finite flat group schemes*, to appear in J. of Number Theory.

[11] M. Raynaud: *Schémas en groupes de type* $(p, ..., p)$, Bull. Soc. Math. France **102** (1974), 241-280.

[12] J.-P. Serre: *Propriétés galoisiennes des points d'ordre fini des courbes elliptiques*, Invent. Math. **15** (1972), 259-331.