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The geometric Iwasawa conjecture from a viewpoint of the arithmetic topology

By

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Abstract

For a local system on a complete hyperbolic threefold which is compact or of finite volume, the twisted Alexander polynomial and the Ruelle-Selberg L-functions are defined. Under a cohomological assumption, we have shown that their order and leading constants of the Taylor expansion at the origin are almost identical. These results may be considered as a solution of a geometric analogue of the Iwasawa conjecture in the algebraic number theory. We will interpret these results from a viewpoint of the arithmetic topology.

§ 1. Introduction

In the conference we have talked a geometric analog of the Iwasawa conjecture. Let X be a finite CW-complex of dimension three with a fixed base point x_0 such that there is a surjective homomorphism

$$\pi_1(X, x_0) \xrightarrow{\epsilon} \mathbb{Z}$$

and ρ a unitary representation of the fundamental group. The kernel of ϵ determines an infinite cyclic covering X_∞ of X and $H_\cdot(X_\infty, \mathbb{C})$, $H_\cdot(X_\infty, \rho)$, $H^\cdot(X_\infty, \mathbb{C})$ and $H^\cdot(X_\infty, \rho)$ become Λ-modules. Here Λ is the group ring $\mathbb{C}[\mathbb{Z}]$. Note that it is isomorphic to the Laurent polynomial ring $\mathbb{C}[t, t^{-1}]$ in a non-canonical way.

Suppose that all the dimensions of $H_\cdot(X_\infty, \mathbb{C})$ and $H_\cdot(X_\infty, \rho)$ are finite. Then due to Milnor [4] it is known that $H^1(X_\infty, \rho)$ becomes a finite dimensional complex vector space. *The twisted Alexander polynomial* $A^1_\rho(t)$ is defined to be the characteristic polynomial $\det[t - \tau^*]$ of the action of a generator $\tau \in \mathbb{Z}$ on $H^1(X_\infty, \rho)$, which generates

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the characteristic ideal $\mathrm{Char}_{\Lambda}(H^1(X_\infty,\,\rho))$. It will play the same role as the Iwasawa polynomial in the original Iwasawa theory.

In order to introduce a counterpart of the p-adic zeta function, we assume that X admits a hyperbolic structure of a finite volume. Then *the Ruelle L-function* $R_\rho(s)$ is defined to be

$$R_\rho(s)=\prod_{\gamma}P_\gamma(s)^{-1},\quad P_\gamma(s)=\det[1-\rho(\gamma)e^{-sl(\gamma)}],$$

where γ runs through the set of prime closed geodesics of X and $l(\gamma)$ denotes its length. It absolutely convergents if $\mathrm{Re}\,s$ is sufficiently large.

When X is compact, Fried has shown it is meromorphically continued on the whole plane. Moreover if $H^0(X,\,\rho)$ vanishes, he has also shown the order of $R_\rho(s)$ at the origin is $2\dim H^1(X,\rho)$ ([1]).

Suppose $H^0(X_\infty,\,\rho)$ vanishes. In [6], using Fried's result we have shown that if the action of τ^* on $H^1(X_\infty,\,\rho)$ is semisimple the identity

$$2\mathrm{ord}_{t=1}A^1_\rho(t)=\mathrm{ord}_{s=0}R_\rho(s), \tag{1}$$

holds, which may be considered as a geometric analog of the Iwasawa main conjecture. But since the compact case corresponds to a Galois representation which is unramified everywhere, it is desirable to generalize the result to a non-compact case. In fact when X has only one cusp and when ρ is a unitary character, we have shown the same identity holds if a certain condition of ρ at the cusp is satisfied [7].

In this report we will interpret (1) from a viewpoint of the arithmetic topology. Let γ be a prime closed geodesic of X such that $\epsilon(\gamma)\neq 0$. (Such a geodesic will be mentioned as *ϵ-inert.*) Then it is easy to to see that its inverse image in X_∞ becomes the infinite cyclic covering of γ. Let $X_\infty(\gamma)$ be its complement. Then we will show an identity of fractional ideals of Λ:

$$\mathrm{Char}_{\Lambda}(H^1(X_\infty(\gamma),\,\rho))\cdot(\det[1-\rho(\gamma)t^{-|\epsilon(\gamma)|}])^{-1}=\mathrm{Char}_{\Lambda}(H^1(X_\infty,\,\rho)).$$

This is a partial solution of the geometric Iwasawa conjecture. In fact substituting $t=\exp(\frac{l(\gamma)}{|\epsilon(\gamma)|}s)$ it shows us that $P_\gamma(s)^{-1}$ divides $\mathrm{Char}_{\Lambda}(H^1(X_\infty,\,\rho))$ in $\mathbb{C}[[s]]$, which also follows from (1). Moreover *a topological Euler system* which enjoys the same properties as the original one([5]) will be constructed and we will show a formal result which should be compared with [5]**Thorem 2.3.3.**

§ 2. A review of our results

Let X be a connected finite CW-complex with a fixed base point x_0 and Γ its fundamental group. Let ρ be its unitary representation of finite dimension and V_ρ the representation space. Suppose that there is a surjective homomorphism

$$\Gamma \xrightarrow{\epsilon} \mathbb{Z}. \tag{2}$$

By the Galois theory $\operatorname{Ker}\epsilon$ determines the infinite cyclic covering $X_\infty \xrightarrow{\pi} X$. In the following we will identify the group ring $\mathbb{C}[\mathbb{Z}]$ with the ring $\Lambda = \mathbb{C}[t, t^{-1}]$ of Laurent polynomials of complex coefficients. (Note that such an isomorphism is not canonical.) Thus (2) induces a ring homomorphism

$$\mathbb{C}[\Gamma] \xrightarrow{\epsilon} \Lambda.$$

Let $\tilde{X}$ be the universal covering of X. Then the chain complex $(C.(\tilde{X}), \partial)$ of complex coefficients is a complex of free $\mathbb{C}[\Gamma]$-module of finite rank and so is the cochain complex $(C^{\cdot}(\tilde{X}), d)$.

Following [2] let us consider a complex of finite dimensional vector spaces over $\mathbb{C}$:

$$C.(X, \rho) = C.(\tilde{X}) \otimes_{\mathbb{C}[\Gamma]} V_\rho.$$

and a complex of free Λ-modules of finite rank:

$$C.(X_\infty, \rho) = C.(\tilde{X}) \otimes_{\mathbb{C}[\operatorname{Ker}\epsilon]} V_\rho \simeq C.(\tilde{X}) \otimes_{\mathbb{C}[\Gamma]} (V_\rho \otimes_{\mathbb{C}} \Lambda).$$

Similary we set

$$C^{\cdot}(X, \rho) = C^{\cdot}(\tilde{X}) \otimes_{\mathbb{C}[\Gamma]} V_\rho$$

and

$$C^{\cdot}(X_\infty, \rho) = C^{\cdot}(\tilde{X}) \otimes_{\mathbb{C}[\operatorname{Ker}\epsilon]} V_\rho \simeq C^{\cdot}(\tilde{X}) \otimes_{\mathbb{C}[\Gamma]} (V_\rho \otimes_{\mathbb{C}} \Lambda),$$

which are the dual complex of $C.(X, \rho)$ over $\mathbb{C}$ and of $C.(X_\infty, \rho)$ over Λ, respectively. The homology or cohomology group of each complex will be denoted by

$$H.(X, \rho),\ H.(X_\infty, \rho),$$

and

$$H^{\cdot}(X, \rho),\ H^{\cdot}(X_\infty, \rho).$$

Note that both of $H_\cdot(X_\infty, \rho)$ and $H^\cdot(X_\infty, \rho)$ are finitely generated Λ-modules.

Let Y be a connected subcomplex of X. Suppose that there is a connected subcomplex Y_∞ of X_∞ which is an infinite cyclic covering of Y and that the diagram:

$$\begin{array}{ccc} Y_\infty & \to & X_\infty \\ \downarrow & & \downarrow \\ Y & \to & X \end{array}$$

induces an isomorphism

$$\mathrm{Gal}(Y_\infty/Y) \simeq \mathrm{Gal}(X_\infty/X) \simeq \mathbb{Z}.$$

Then we have a complex of free Λ-modules of finite rank:

$$C_\cdot(Y_\infty, \rho) = C_\cdot(\tilde{Y}) \otimes_{\mathbb{C}[\pi_1(Y)]} (V_\rho \otimes_{\mathbb{C}} \Lambda).$$

Note that this is a subcomplex of $C_\cdot(X_\infty, \rho)$ whose quotient $C_\cdot(X_\infty, Y_\infty, \rho)$ is also a complex of free Λ-modules of finite rank. Taking the dual over Λ we have an exact sequence of bounded complexes of free Λ-modules:

$$0 \to C^\cdot(X_\infty, Y_\infty, \rho) \to C^\cdot(X_\infty, \rho) \to C^\cdot(Y_\infty, \rho) \to 0.$$

Thus we have an exact sequence of finitely generated Λ-modules:

(3) $\quad \to H^q(X_\infty, Y_\infty, \rho) \to H^q(X_\infty, \rho) \to H^q(Y_\infty, \rho) \to H^{q+1}(X_\infty, Y_\infty, \rho) \to$

Here is an example of $H^q(X_\infty, Y_\infty, \rho)$.

Let $X = D^2 \times S^1$, where D^2 is the two dimensional unit disk and S^1 is the unit circle. Let $Y = S^1 \times S^1$ be its boundary. Then the fundamental group of X is an infinite cyclic group and let

$$\pi_1(X) \simeq \mathbb{Z} \xrightarrow{\rho} U(n)$$

be a unitary representation. By the homotopy invariance of cohomology groups and by the Gysin isomorphism we have

$$H^q(X_\infty, Y_\infty, \rho) = \begin{cases} V_\rho & q = 2 \\ 0 & q \neq 2. \end{cases}$$

$\gamma \in \pi_1(X)$ acts on $H^2(X_\infty, Y_\infty, \rho) \simeq V_\rho$ by $\rho(\gamma)$, which makes it a Λ-module.

In the following, we will always assume that the dimension of X is three and that all $H_\cdot(X_\infty, \mathbb{C})$ and $H_\cdot(X_\infty, \rho)$ are finite dimensional vector spaces over $\mathbb{C}$. The arguments of §4 of [4] shows X_∞ is *a Riemann surface* in the cohomological sense.

Fact 2.1. *([4])*

1. *For $i \geq 3$, $H^i(X_\infty,\, \rho)$ vanishes.*
2. *For $0 \leq i \leq 2$, $H^i(X_\infty,\, \rho)$ is a finite dimensional vector space over $\mathbb{C}$ and there is a perfect pairng:*

$$H^i(X_\infty,\, \rho) \times H^{2-i}(X_\infty,\, \rho) \to \mathbb{C}.$$

The perfect pairing will be referred as *the Milnor duality.* It is easy to see that it is preserved by the action of $\mathrm{Gal}(X_\infty/X)$.

Thus each $H^i(X_\infty,\, \rho)$ is a torsion Λ-module and its characteristic ideal $\mathrm{Char}_\Lambda(H^i(X_\infty,\, \rho))$ is generated by

$$A^i_\rho(t) = \det[t - \tau^* \mid H^i(X_\infty,\, \rho)],$$

where τ^* is the action of t on $H^i(X_\infty,\, \rho)$.

Let $h^q(\rho)$ be the dimension of $H^q(X,\, \rho)$. Then in [6] we have shown the following results.

Theorem 2.1. *Suppose that $H^0(X_\infty,\, \rho)$ vanishes. Then we have*

$$h^1(\rho) \leq \mathrm{ord}_{t=1} A^1_\rho(t),$$

and the identity holds if the action of τ^* on $H^1(X_\infty,\, \rho)$ is semisimple. Moreover suppose that all $h^q(\rho)$ vanish. Then we have

$$|A^1_\rho(1)| = \delta |\tau^*_{\mathbb{C}}(X, \rho)|^{-1},$$

where δ is an explicit positive constant. Here $\tau^*_{\mathbb{C}}(X, \rho)$ is the Frantz-Milnor-Reidemeister torsion, which is a geometric invariant of a representation.([3])

When X is a mapping torus we can say more.

Theorem 2.2. *([6]) Let f be an automorphism of a connected finite CW-complex of dimension two S and X its mapping torus. Let ρ be a unitary representation of the fundamental group of X which satisfies $H^0(S,\, \rho) = 0$. Suppose that the surjective homomorphism*

$$\Gamma \xrightarrow{\epsilon} \mathbb{Z}$$

is induced by the structure map

$$X \to S^1,$$

and that the action of f^* on $H^1(S, \rho)$ is semisimple. Then the order of $A^1_\rho(t)$ at $t=1$ is $h^1(\rho)$ and we have

$$\lim_{t\to 1} |(t-1)^{-h^1(\rho)} A^1_\rho(t)| = |\tau^*_{\mathbb{C}}(X,\rho)|^{-1}.$$

Note that in **Theorem 2.2** X_∞ is $S \times \mathbb{R}$.

In order to introduce an analytic object -*the Ruelle L-function*- we need a geometric structure on X. Let X be a connected hyperbolic threefold of finite volume. Thus its fundamental group may be considered as a torsion-free cofinite discrete subgroup Γ_g of $PSL_2(\mathbb{C})$ and let ρ be its unitary representation. By the one to one correspondence between the set of loxodromic conjugacy classes of Γ_g and one of closed geodesics of X, *the Ruelle L-function* is defined to be a product of the inverse of the characteristic polynomials of $\rho(\gamma)$ over prime closed geodesics:

$$R_\rho(s) = \prod_\gamma P_\gamma(s)^{-1},\ P_\gamma(s) = \det[1-\rho(\gamma)e^{-sl(\gamma)}].$$

Here s is a complex number and $l(\gamma)$ is the length of γ. It absolutely convergents for s whose real part is sufficiently large.

Let X be a compact hyperbolic threefold satisfying $H^0(X, \rho) = 0$. Due to Fried([1]), it is known that $R_\rho(s)$ is meromorphically continued in the whole plane and that its order at $s = 0$ is $2h^1(\rho)$. Moreover he has shown its absolute value of the leading constant is equal to $|\tau^*_{\mathbb{C}}(X,\rho)|^{-2}$.

In the following we always assume that X admits an infinite cyclic covering X_∞. (i.e. the first Betti number of X is positive.) Thus combining Fried's results and **Theorem 2.1** and **Theorem 2.2** we obtain the following theorem.

Theorem 2.3. *Let X be a compact hyperbolic threefold and ρ a unitary representaion of the fundamental group.*

1. *Suppose that $H^0(X_\infty, \rho)$ vanishes. Then*

$$2h^1(\rho) = \mathrm{ord}_{s=0} R_\rho(s) \le 2\mathrm{ord}_{t=1} A^1_\rho(t),$$

and the identity holds if the action of τ^* on $H^1(X_\infty, \rho)$ is semisimple. Moreover if all $h^q(\rho)$ vanish, we have

$$|R_\rho(0)| = \delta_\rho |A^1_\rho(1)|^2,$$

where δ_ρ is an explicit constant.

2. *Suppose that X is homeomorphic to a mapping torus of an automorphism f of a compact surface S and that the surjective homomorphism ϵ is induced by the structure map:*

$$X \to S^1.$$

If $H^0(S, \rho)$ vanishes, we have

$$2h^1(\rho) = \mathrm{ord}_{s=0}R_\rho(s) \leq 2\mathrm{ord}_{t=1}A^1_\rho(t),$$

and the identiy holds if the action of f^* on $H^1(S, \rho)$ is semisimple. Moreover if this condition is satisfied, we have

$$\lim_{s\to 0}|s^{-2h^1(\rho)}R_\rho(s)| = \lim_{t\to 1}|(t-1)^{-h^1(\rho)}A^1_\rho(t)|^2.$$

Next we will consider a non-compact case. Let X be a hyperbolic threefold of finite volume with one cusp and ρ a unitary character of the fundamental group. The fundamental group at the cusp will be denoted by Γ_∞. Here is a generalization of Fried's results.

Theorem 2.4. *([7] [8]) $R_\rho(s)$ is meromorphically continued on the whole plane and satisfies an analog of the Riemann hypothesis. Moreover it satisfies the following properties at the origin.*

1. *Suppose $\rho|_{\Gamma_\infty}$ is trivial. Then we have*

$$\mathrm{ord}_{s=0}R_\rho(s) = 2(h^1(\rho) - 2h^0(\rho) - 1).$$

2. *Suppose $\rho|_{\Gamma_\infty}$ is nontrivial, then*

$$\mathrm{ord}_{s=0}R_\rho(s) = 2h^1(\rho).$$

Moreover if $h^1(\rho)$ vanishes we have

$$|R_\rho(0)| = |\tau^*_{\mathbb{C}}(X,\rho)|^{-2}.$$

We remark that the "error term" -2 in the RHS of the first identity is caused by a pathology of the Hodge theory. Note that in the second case the assumption automatically implies vanishing of $h^0(\rho)$. Thus we have

Theorem 2.5. *Let X be a hyperbolic threefold of finite volume with one cusp and ρ a unitary character of the fundamental group such that $h^0(\rho)$ vanishes.*

1. *Suppose $\rho|_{\Gamma_\infty}$ is trivial. Then we have*

$$\mathrm{ord}_{s=0}R_\rho(s) + 2 \leq 2\mathrm{ord}_{t=1}A^1_\rho(t),$$

and the identiy holds if the action of τ^* on $H^1(X_\infty, \rho)$ is semisimple.

2. *Suppose $\rho|_{\Gamma_\infty}$ is nontrivial. Then we have*

$$\mathrm{ord}_{s=0}R_\rho(s) \leq 2\mathrm{ord}_{t=1}A^1_\rho(t),$$

and the identiy holds if the action of τ^* on $H^1(X_\infty, \rho)$ is semisimple. Moreover if all $h^q(\rho)$ vanish we have

$$|R_\rho(0)| = \delta_\rho |A^1_\rho(1)|^2.$$

In either case if we make a change of variables:

$$t = s + 1,$$

under a suitable assumption, our theorem implies two ideals in $\mathbb{C}[[s]]$ generated by $R_\rho(s)$ and $A^1_\rho(s)^2$ coincide. Thus our theorem may be considered as a solution of *a geometric analog of the Iwasawa main conjecture.*

In particular we may say for each prime closed geodesic γ, $P_\gamma(s)^{-1}$ *divides* A^1_ρ. In the next section we will explain this phenomenon from a viewpoint of the arithmetic topology.

§ 3. An explanation from the arithmetic topology

Let X be a hyperbolic threefold of finite volume and ρ a unitary representation of the fundamental group. We assume that $H^0(X_\infty, \rho)$ vanishes.

Note that ϵ induces a map from a set of prime closed geodesics Σ_{prim} to $\mathbb{Z}$. Thus it is decomposed into two subsets:

$$\Sigma^\iota_{prim} = \{\gamma \in \Sigma_{prim} \mid \epsilon(\gamma) \neq 0\}$$

and its complement Σ^σ_{prim}. An element of Σ^σ_{prim} (resp. Σ^ι_{prim}) will be referred as ϵ-*split* (resp. ϵ-*inert*). For $\gamma \in \Sigma^\iota_{prim}$ its ϵ-*inertia degree* $m_\epsilon(\gamma)$ is defined to be the absolute value of $\epsilon(\gamma)$.

Let $\gamma \in \Sigma^{\iota}_{prim}$ be of ϵ-inertia degree 1. We may regard it as a smooth imbedded S^1 and let C_∞ be a connected component of $X_\infty \times_X S^1$. Thus we have a diagram:

$$\begin{array}{ccc} C_\infty & \to & X_\infty \\ p\downarrow & & \downarrow\pi \\ S^1 & \xrightarrow{\gamma} & X. \end{array} \tag{4}$$

We claim that C_∞ is the universal covering of S^1 and that the diagram induces an isomorphism:

$$\mathrm{Gal}(C_\infty/S^1) \simeq \mathrm{Gal}(X_\infty/X) \simeq \mathbb{Z}.$$

In fact (4) implies the diagram:

$$\begin{array}{ccc} \pi_1(C_\infty) & \to & \pi_1(X_\infty) \\ p_*\downarrow & & \downarrow \\ \mathbb{Z} = \pi_1(S^1) & \xrightarrow{\gamma_*} & \pi_1(X) \\ & & \downarrow\epsilon \\ & & \mathbb{Z}, \end{array}$$

which satisfies

$$\gamma_*(1) = \gamma.$$

If C_∞ were not $\mathbb{R}$, it should be a circle. In particular the image of p_* becomes a nontrivial subgroup of $\pi_1(S^1)$. But the image of $\epsilon \cdot \gamma_*$ is a subgroup of $\mathbb{Z}$ which is torsion free, the above diagram shows that $\epsilon(\gamma)$ should be zero. This contradicts to the choice of γ. Moreover since ϵ-inertia degree of γ is one, γ_* gives a splitting of ϵ and we have

$$\mathrm{Gal}(C_\infty/S^1) \overset{\gamma_*}{\simeq} \mathrm{Gal}(X_\infty/X) \simeq \mathbb{Z}.$$

Let $N(\gamma)$ be a small tubular neighborhood of γ and $N_\infty(\gamma)$ its lift to X_∞ along C_∞:

$$N_\infty(\gamma) = \pi^{-1}(N(\gamma)).$$

We set

$$X_\infty(\gamma) = X_\infty \setminus N_\infty(\gamma).$$

By the exicision we have

$$H^q(X_\infty, X_\infty(\gamma), \rho) \simeq H^q(N_\infty, \partial N_\infty(\gamma), \rho),$$

and the computation of the previous section implies

$$H^q(N_\infty, \partial N_\infty(\gamma), \rho) = \begin{cases} \Lambda/(\det[t - \rho(\gamma)]) & q = 2 \\ 0 & q \neq 2. \end{cases}$$

Thus the exact sequence (3) and our assumption show the vanishing of $H^0(X_\infty(\gamma),\, \rho)$ and an exact sequence of Λ-modules:

$$0 \to H^1(X_\infty,\, \rho) \overset{\mathrm{Res}}{\to} H^1(X_\infty(\gamma),\, \rho) \to \Lambda/(\det[t-\rho(\gamma)]) \to 0.$$

In particular we know the dimension of $H^1(X_\infty(\gamma),\, \rho)$ is finite and we have an identity of fractional ideals of Λ:

$$\mathrm{Char}_\Lambda(H^1(X_\infty,\, \rho)) = \mathrm{Char}_\Lambda(H^1(X_\infty(\gamma),\, \rho)) \cdot (\det[1-\rho(\gamma)t^{-1}])^{-1}.$$

More generally let γ be an element of Σ^ι_{prim}. Then the subgroup

$$m_\epsilon(\gamma)\mathbb{Z} \subseteq \mathbb{Z} = \mathrm{Gal}(X_\infty/X)$$

determines a cyclic covering $X_{m_\epsilon(\gamma)}$ of X with degree $m_\epsilon(\gamma)$. Note that X_∞ is its infinite cyclic covering satisfying

$$\mathrm{Gal}(X_\infty/X_{m_\epsilon(\gamma)}) = m_\epsilon(\gamma)\mathbb{Z},$$

and that γ lifts to a smooth embedded S^1 in $X_{m_\epsilon(\gamma)}$ which is mapped to $\pm m_\epsilon(\gamma)$ by

$$\pi_1(X_{m_\epsilon(\gamma)}) \to \mathrm{Gal}(X_\infty/X_{m_\epsilon(\gamma)}) = m_\epsilon(\gamma)\mathbb{Z}.$$

Now the previous argument shows the vanishing of $H^0(X_\infty(\gamma),\, \rho)$ and an exact sequence:

$$0 \to H^1(X_\infty,\, \rho) \overset{\mathrm{Res}}{\to} H^1(X_\infty(\gamma),\, \rho) \to \Lambda/(\det[t^{m_\epsilon(\gamma)}-\rho(\gamma)]) \to 0$$

and

$$\mathrm{Char}_\Lambda(H^1(X_\infty,\, \rho)) = \mathrm{Char}_\Lambda(H^1(X_\infty(\gamma),\, \rho)) \cdot (\det[1-\rho(\gamma)t^{-m_\epsilon(\gamma)}])^{-1}.$$

Thus we have proved the following theorem.

Theorem 3.1. *Suppose $H^0(X_\infty,\, \rho)$ vanishes. Then for $\gamma \in \Sigma^\iota_{prim}$, $H^0(X_\infty(\gamma),\, \rho)$ also vanishes and we have an exact sequence of Λ-modules:*

$$0 \to H^1(X_\infty,\, \rho) \overset{\mathrm{Res}}{\to} H^1(X_\infty(\gamma),\, \rho) \to \Lambda/(\det[t^{m_\epsilon(\gamma)}-\rho(\gamma)]) \to 0. \tag{5}$$

In particular the dimension of $H^1(X_\infty(\gamma),\, \rho)$ is finite and we have an identity of fractional ideals of Λ:

$$\mathrm{Char}_\Lambda(H^1(X_\infty,\, \rho)) = \mathrm{Char}_\Lambda(H^1(X_\infty(\gamma),\, \rho)) \cdot (\det[1-\rho(\gamma)t^{-m_\epsilon(\gamma)}])^{-1}.$$

Note that the Euler factor $P_\gamma(s)$ of the Ruelle L-function is given by

$$P_\gamma(s) = \det[1 - \rho(\gamma)t^{-m_\epsilon(\gamma)}]|_{t=\exp[\frac{l(\gamma)s}{m_\epsilon(\gamma)}]}.$$

Since we have

$$\exp(\frac{l(\gamma)}{m_\epsilon(\gamma)}s) - 1 = \frac{l(\gamma)s}{m_\epsilon(\gamma)} + O(s^2),$$

localizing at $s = t-1$, the fact that $(\det[1-\rho(\gamma)t^{-m_\epsilon(\gamma)}])^{-1}$ divides $\mathrm{Char}_\Lambda(H^1(X_\infty, \rho))$ implies the divisibility of A^1_ρ by $P_\gamma(s)^{-1}$ in $\mathbb{C}[[s]]$ for $\gamma \in \Sigma^\iota_{prim}$.

We can formulate this fact in terms of an analog of *the Euler system*([5]).

First of all we remark that using the homology exact sequence:

$$\to H_q(X_\infty(\gamma), \rho) \to H_q(X_\infty, \rho) \to H_q(X_\infty, X_\infty(\gamma), \rho) \to H_{q-1}(X_\infty(\gamma), \rho) \to$$

and by the isomorphism

$$H_q(X_\infty, X_\infty(\gamma), \rho) \simeq H_q(D^2, S^1, \rho)$$

derived from the excision and the homotopy invariance of the homology group one may check that the dimension of $H_.(X_\infty(\gamma), \mathbb{C})$ and $H_.(X_\infty(\gamma), \rho)$ are finite. Taking the dual of (5) over $\mathbb{C}$, the Milnor duality shows an exact sequence of Λ-modules:

$$0 \to \Lambda/(\det[t^{m_\epsilon(\gamma)} - \rho(\gamma)]) \to H^1(X_\infty(\gamma), \rho) \overset{\mathrm{Cor}}{\to} H^1(X_\infty, \rho) \to 0.$$

Let us fix a nonzero element c_∞ of $H^1(X_\infty, \rho)$ and choose its any lift $c'(\gamma)_\infty$ to $H^1(X_\infty(\gamma), \rho)$. Then

$$c(\gamma)_\infty = F_\gamma(t)c'(\gamma)_\infty, \quad F_\gamma(t) = \det[t^{m_\epsilon(\gamma)} - \rho(\gamma)]$$

is independent of a choice of the lift and satisfies

$$\mathrm{Cor}(c(\gamma)_\infty) = F_\gamma(t)c_\infty.$$

More generally, for elements $\{\gamma_1, \cdots, \gamma_N\}$ of Σ^ι_{prim}, we set

$$X_\infty(\gamma_1 \cdots \gamma_N) = X_\infty \setminus N_\infty(\gamma_1) \cup \cdots \cup N_\infty(\gamma_N).$$

Using **Theorem 3.1**, an induction argument shows that we have an exact sequence of Λ-modules:

$$0 \to \Lambda/(F_{\gamma_N}(t)) \to H^1(X_\infty(\gamma_1 \cdots \gamma_N), \rho) \overset{\mathrm{Cor}}{\to} H^1(X_\infty(\gamma_1 \cdots \gamma_{N-1}), \rho) \to 0.$$

Therefore we can successively choose an element $c_\infty(\gamma_1 \cdots \gamma_N)$ of $H^1(X_\infty(\gamma_1 \cdots \gamma_N),\, \rho)$ so that

$$\mathrm{Cor}(c_\infty(\gamma_1 \cdots \gamma_N)) = F_{\gamma_N}(t) c_\infty(\gamma_1 \cdots \gamma_{N-1}).$$

Thus $\{c_\infty(\gamma_1 \cdots \gamma_N)\}$ has the same property as the Euler system [5] §2.1. If we apply the co-restriction map "Cor" N-times to $c_\infty(\gamma_1 \cdots \gamma_N)$, we obtain an element $d_\infty(\gamma_1 \cdots \gamma_N)$ of $H^1(X_\infty,\, \rho)$ which satisfies

$$d_\infty(\gamma_1 \cdots \gamma_N) = \prod_{i=1}^{N} F_{\gamma_i}(t) \cdot c_\infty.$$

Now our solution of the geometric Iwasawa conjecture is formally described in the following way.

If two elements c and c' of $H^1(X_\infty,\, \rho)$ have a relation:

$$c' = f \cdot c, \quad f \in \Lambda,$$

f^{-1} will be denoted by $ind_\Lambda(c, c')$. (Note that in fact since $H^1(X_\infty,\, \rho)$ is a torsion Λ-module it is *formally* defined.) In particular our *topological Euler system* gives the ϵ-inert part of the Euler product:

$$ind_\Lambda(c_\infty,\, d_\infty(\prod_{\gamma \in \Sigma^\iota_{prim}} \gamma)) = \prod_{\gamma \in \Sigma^\iota_{prim}} F_{\gamma_i}^{-1} \in \mathbb{C}[[s]].$$

The following statement is a formal reformulation of **Theorem 3.1**, which should be compared with [5] **Theorem 2.3.3.**

Theorem 3.2. *(formal) In* $\mathbb{C}[[s]]$, $ind_\Lambda(c_\infty,\, d_\infty(\prod_{\gamma \in \Sigma^\iota_{prim}} \gamma))$ *divides* $\mathrm{Char}_\Lambda(H^1(X_\infty,\, \rho))$.

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