The geometric Iwasawa conjecture from a viewpoint of the arithmetic topology

By

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Abstract

For a local system on a complete hyperbolic threefold which is compact or of finite volume, the twisted Alexander polynomial and the Ruelle-Selberg L-functions are defined. Under a cohomological assumption, we have shown that their order and leading constants of the Taylor expansion at the origin are almost identical. These results may be considered as a solution of a geometric analogue of the Iwasawa conjecture in the algebraic number theory. We will interpret these results from a viewpoint of the arithmetic topology.

§1. Introduction

In the conference we have talked a geometric analog of the Iwasawa conjecture. Let $X$ be a finite CW-complex of dimension three with a fixed base point $x_0$ such that there is a surjective homomorphism

$$\pi_1(X, x_0) \to \mathbb{Z}$$

and $\rho$ a unitary representation of the fundamental group. The kernel of $\epsilon$ determines an infinite cyclic covering $X_\infty$ of $X$ and $H(X_\infty, \mathbb{C})$, $H(X_\infty, \rho)$, $H'(X_\infty, \mathbb{C})$ and $H'(X_\infty, \rho)$ become $\Lambda$-modules. Here $\Lambda$ is the group ring $\mathbb{C}[Z]$. Note that it is isomorphic to the Laurent polynomial ring $\mathbb{C}[t, t^{-1}]$ in a non-canonical way.

Suppose that all the dimensions of $H(X_\infty, \mathbb{C})$ and $H(X_\infty, \rho)$ are finite. Then due to Milnor [4] it is known that $H^1(X_\infty, \rho)$ becomes a finite dimensional complex vector space. The twisted Alexander polynomial $A^1_{\rho}(t)$ is defined to be the characteristic polynomial $\det[t-\tau^*]$ of the action of a generator $\tau \in \mathbb{Z}$ on $H^1(X_\infty, \rho)$, which generates

2000 Mathematics Subject Classification(s): 11F32, 11M36, 57M25, 57M27
Supported in part by the Grants-in-Aid for Scientific Research, JSPS
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the characteristic ideal \( \text{Char}_\Lambda(H^1(X_{\infty}, \rho)) \). It will play the same role as the Iwasawa polynomial in the original Iwasawa theory.

In order to introduce a counterpart of the \( p \)-adic zeta function, we assume that \( X \) admits a hyperbolic structure of a finite volume. Then the Ruelle L-function \( R_\rho(s) \) is defined to be

\[
R_\rho(s) = \prod_{\gamma} P_\gamma(s)^{-1}, \quad P_\gamma(s) = \det[1 - \rho(\gamma) e^{-s l(\gamma)}],
\]

where \( \gamma \) runs through the set of prime closed geodesics of \( X \) and \( l(\gamma) \) denotes its length. It absolutely convergent if \( \text{Re} \ s \) is sufficiently large.

When \( X \) is compact, Fried has shown it is meromorphically continued on the whole plane. Moreover if \( H^0(X, \rho) \) vanishes, he has also shown the order of \( R_\rho(s) \) at the origin is \( 2 \text{dim} H^1(X, \rho) \) ([1]).

Suppose \( H^0(X_{\infty}, \rho) \) vanishes. In [6], using Fried’s result we have shown that if the action of \( \tau^* \) on \( H^1(X_{\infty}, \rho) \) is semisimple the identity

\[
2\text{ord}_{t=1} A^1_\rho(t) = \text{ord}_{s=0} R_\rho(s),
\]

holds, which may be considered as a geometric analog of the Iwasawa main conjecture. But since the compact case corresponds to a Galois representation which is unramified everywhere, it is desirable to generalize the result to a non-compact case. In fact when \( X \) has only one cusp and when \( \rho \) is a unitary character, we have shown the same identity holds if a certain condition of \( \rho \) at the cusp is satisfied [7].

In this report we will interpret (1) from a viewpoint of the arithmetic topology. Let \( \gamma \) be a prime closed geodesic of \( X \) such that \( \epsilon(\gamma) \neq 0 \). (Such a geodesic will be mentioned as \( \epsilon \)-inert.) Then it is easy to see that its inverse image in \( X_{\infty} \) becomes the infinite cyclic covering of \( \gamma \). Let \( X_{\infty}(\gamma) \) be its complement. Then we will show an identity of fractional ideals of \( \Lambda \):

\[
\text{Char}_\Lambda(H^1(X_{\infty}(\gamma), \rho)) \cdot (\det[1 - \rho(\gamma) t^{-|\epsilon(\gamma)|}])^{-1} = \text{Char}_\Lambda(H^1(X_{\infty}, \rho)).
\]

This is a partial solution of the geometric Iwasawa conjecture. In fact substituting \( t = \exp(\frac{l(\gamma)}{|\epsilon(\gamma)|} s) \) it shows us that \( P_\gamma(s)^{-1} \) divides \( \text{Char}_\Lambda(H^1(X_{\infty}, \rho)) \) in \( \mathbb{C}[[s]] \), which also follows from (1). Moreover a topological Euler system which enjoys the same properties as the original one([5]) will be constructed and we will show a formal result which should be compared with [5]Thorem 2.3.3.
§ 2. A review of our results

Let $X$ be a connected finite CW-complex with a fixed base point $x_0$ and $\Gamma$ its fundamental group. Let $\rho$ be its unitary representation of finite dimension and $V_\rho$ the representation space. Suppose that there is a surjective homomorphism

\[ \Gamma \xrightarrow{\epsilon} \mathbb{Z}. \]

By the Galois theory $\text{Ker} \epsilon$ determines the infinite cyclic covering $X_\infty \xrightarrow{\epsilon} X$. In the following we will identify the group ring $\mathbb{C}[\mathbb{Z}]$ with the ring $\Lambda = \mathbb{C}[t, t^{-1}]$ of Laurent polynomials of complex coefficients. (Note that such an isomorphism is not canonical.) Thus (2) induces a ring homomorphism

\[ \mathbb{C}[\Gamma] \xrightarrow{\epsilon} \Lambda. \]

Let $\tilde{X}$ be the universal covering of $X$. Then the chain complex $(C.(\tilde{X}), \partial)$ of complex coefficients is a complex of free $\mathbb{C}[\Gamma]$-module of finite rank and so is the cochain complex $(C^\cdot(\tilde{X}), d)$.

Following [2] let us consider a complex of finite dimensional vector spaces over $\mathbb{C}$:

\[ C.(X, \rho) = C.(\tilde{X}) \otimes_{\mathbb{C}[\Gamma]} V_\rho, \]

and a complex of free $\Lambda$-modules of finite rank:

\[ C.(X_\infty, \rho) = C.(\tilde{X}) \otimes_{\mathbb{C}[Ker\epsilon]} V_\rho \simeq C.(\tilde{X}) \otimes_{\mathbb{C}[\Gamma]} (V_\rho \otimes_{\mathbb{C}} \Lambda). \]

Similarly we set

\[ C^\cdot(X, \rho) = C^\cdot(\tilde{X}) \otimes_{\mathbb{C}[\Gamma]} V_\rho \]

and

\[ C^\cdot(X_\infty, \rho) = C^\cdot(\tilde{X}) \otimes_{\mathbb{C}[Ker\epsilon]} V_\rho \simeq C^\cdot(\tilde{X}) \otimes_{\mathbb{C}[\Gamma]} (V_\rho \otimes_{\mathbb{C}} \Lambda), \]

which are the dual complex of $C.(X, \rho)$ over $\mathbb{C}$ and of $C.(X_\infty, \rho)$ over $\Lambda$, respectively. The homology or cohomology group of each complex will be denoted by

\[ H.(X, \rho), H.(X_\infty, \rho), \]

and

\[ H^\cdot(X, \rho), H^\cdot(X_\infty, \rho). \]
Note that both of $H.(X_{\infty}, \rho)$ and $H^r(X_{\infty}, \rho)$ are finitely generated $\Lambda$-modules.

Let $Y$ be a connected subcomplex of $X$. Suppose that there is a connected subcomplex $Y_{\infty}$ of $X_{\infty}$ which is an infinite cyclic covering of $Y$ and that the diagram:

$$
\begin{array}{c}
Y_{\infty} \rightarrow X_{\infty} \\
\downarrow \quad \downarrow \\
Y \rightarrow X
\end{array}
$$

induces an isomorphism

$$
\text{Gal}(Y_{\infty}/Y) \simeq \text{Gal}(X_{\infty}/X) \simeq \mathbb{Z}.
$$

Then we have a complex of free $\Lambda$-modules of finite rank:

$$
C.(Y_{\infty}, \rho) = C.\left(\tilde{Y}\right) \otimes_{\mathbb{C}[\pi_{1}(Y)]} (V_\rho \otimes_{\mathbb{C}} \Lambda).
$$

Note that this is a subcomplex of $C.(X_{\infty}, \rho)$ whose quotient $C.(X_{\infty}, Y_{\infty}, \rho)$ is also a complex of free $\Lambda$-modules of finite rank. Taking the dual over $\Lambda$ we have an exact sequence of bounded complexes of free $\Lambda$-modules:

$$
0 \rightarrow C^r(X_{\infty}, Y_{\infty}, \rho) \rightarrow C^r(X_{\infty}, \rho) \rightarrow C^r(Y_{\infty}, \rho) \rightarrow 0.
$$

Thus we have an exact sequence of finitely generated $\Lambda$-modules:

$$
(3) \quad \rightarrow H^q(X_{\infty}, Y_{\infty}, \rho) \rightarrow H^q(X_{\infty}, \rho) \rightarrow H^q(Y_{\infty}, \rho) \rightarrow H^{q+1}(X_{\infty}, Y_{\infty}, \rho) \rightarrow
$$

Here is an example of $H^q(X_{\infty}, Y_{\infty}, \rho)$.

Let $X = D^2 \times S^1$, where $D^2$ is the two dimensional unit disk and $S^1$ is the unit circle. Let $Y = S^1 \times S^1$ be its boundary. Then the fundamental group of $X$ is an infinite cyclic group and let

$$
\pi_{1}(X) \simeq \mathbb{Z} \xrightarrow{\rho} U(n)
$$

be a unitary representation. By the homotopy invariance of cohomology groups and by the Gysin isomorphism we have

$$
H^q(X_{\infty}, Y_{\infty}, \rho) = \begin{cases} V_\rho & q = 2 \\
0 & q \neq 2. \end{cases}
$$

$\gamma \in \pi_{1}(X)$ acts on $H^2(X_{\infty}, Y_{\infty}, \rho) \simeq V_\rho$ by $\rho(\gamma)$, which makes it a $\Lambda$-module.

In the following, we will always assume that the dimension of $X$ is three and that all $H.(X_{\infty}, \mathbb{C})$ and $H.(X_{\infty}, \rho)$ are finite dimensional vector spaces over $\mathbb{C}$. The arguments of §4 of [4] shows $X_{\infty}$ is a Riemann surface in the cohomological sense.
Fact 2.1. ([4])

1. For $i \geq 3$, $H^i(X_{\infty}, \rho)$ vanishes.

2. For $0 \leq i \leq 2$, $H^i(X_{\infty}, \rho)$ is a finite dimensional vector space over $\mathbb{C}$ and there is a perfect pairing:

$$H^i(X_{\infty}, \rho) \times H^{2-i}(X_{\infty}, \rho) \rightarrow \mathbb{C}.$$ 

The perfect pairing will be referred as the Milnor duality. It is easy to see that it is preserved by the action of $\text{Gal}(X_{\infty}/X)$.

Thus each $H^i(X_{\infty}, \rho)$ is a torsion $\Lambda$-module and its characteristic ideal $\text{Char}_\Lambda(H^i(X_{\infty}, \rho))$ is generated by

$$A^i_\rho(t) = \det[t - \tau^* | H^i(X_{\infty}, \rho)],$$

where $\tau^*$ is the action of $t$ on $H^i(X_{\infty}, \rho)$.

Let $h^q(\rho)$ be the dimension of $H^q(X, \rho)$. Then in [6] we have shown the following results.

**Theorem 2.1.** Suppose that $H^0(X_{\infty}, \rho)$ vanishes. Then we have

$$h^1(\rho) \leq \text{ord}_{t=1} A^1_\rho(t),$$

and the identity holds if the action of $\tau^*$ on $H^1(X_{\infty}, \rho)$ is semisimple. Moreover suppose that all $h^q(\rho)$ vanish. Then we have

$$|A^1_\rho(1)| = \delta |\tau^*_C(X, \rho)|^{-1},$$

where $\delta$ is an explicit positive constant. Here $\tau^*_C(X, \rho)$ is the Frantz-Milnor-Reidemeister torsion, which is a geometric invariant of a representation. ([3])

When $X$ is a mapping torus we can say more.

**Theorem 2.2.** ([6]) Let $f$ be an automorphism of a connected finite CW-complex of dimension two $S$ and $X$ its mapping torus. Let $\rho$ be a unitary representation of the fundamental group of $X$ which satisfies $H^0(S, \rho) = 0$. Suppose that the surjective homomorphism

$$\Gamma \rightarrow \mathbb{Z}$$

is induced by the structure map

$$X \rightarrow S^1,$$
and that the action of $f^*$ on $H^1(S, \rho)$ is semisimple. Then the order of $A^{1}_{\rho}(t)$ at $t = 1$ is $h^{1}(\rho)$ and we have
\[ \lim_{t \to 1} |(t-1)^{-h^{1}(\rho)}A^{1}_{\rho}(t)| = |\tau^{*}_{C}(X, \rho)|^{-1}. \]

Note that in **Theorem 2.2** $X_{\infty}$ is $S \times \mathbb{R}$.

In order to introduce an analytic object -the Ruelle L-function- we need a geometric structure on $X$. Let $X$ be a connected hyperbolic threefold of finite volume. Thus its fundamental group may be considered as a torsion-free cofinite discrete subgroup $\Gamma_{g}$ of $\text{PSL}_{2}(\mathbb{C})$ and let $\rho$ be its unitary representation. By the one to one correspondence between the set of loxodromic conjugacy classes of $\Gamma_{g}$ and one of closed geodesics of $X$, the Ruelle L-function is defined to be a product of the inverse of the characteristic polynomials of $\rho(\gamma)$ over prime closed geodesics:
\[ R_{\rho}(s) = \prod_{\gamma} P_{\gamma}(s)^{-1}, \quad P_{\gamma}(s) = \det[1 - \rho(\gamma)e^{-sl(\gamma)}]. \]

Here $s$ is a complex number and $l(\gamma)$ is the length of $\gamma$. It absolutely convergents for $s$ whose real part is sufficiently large.

Let $X$ be a compact hyperbolic threefold satisfying $H^0(X, \rho) = 0$. Due to Fried([1]), it is known that $R_{\rho}(s)$ is meromorphically continued in the whole plane and that its order at $s = 0$ is $2h^{1}(\rho)$. Moreover he has shown its absolute value of the leading constant is equal to $|\tau^{*}_{C}(X, \rho)|^{-2}$.

In the following we always assume that $X$ admits an infinite cyclic covering $X_{\infty}$. (i.e. the first Betti number of $X$ is positive.) Thus combining Fried’s results and **Theorem 2.1** and **Theorem 2.2** we obtain the following theorem.

**Theorem 2.3.** Let $X$ be a compact hyperbolic threefold and $\rho$ a unitary representation of the fundamental group.

1. Suppose that $H^0(X_{\infty}, \rho)$ vanishes. Then
\[ 2h^{1}(\rho) = \text{ord}_{s=0} R_{\rho}(s) \leq 2\text{ord}_{t=1} A^{1}_{\rho}(t), \]

and the identity holds if the action of $\tau^{*}$ on $H^1(X_{\infty}, \rho)$ is semisimple. Moreover if all $h^q(\rho)$ vanish, we have
\[ |R_{\rho}(0)| = \delta_{\rho} |A^{1}_{\rho}(1)|^2, \]

where $\delta_{\rho}$ is an explicit constant.
2. Suppose that $X$ is homeomorphic to a mapping torus of an automorphism $f$ of a compact surface $S$ and that the surjective homomorphism $\epsilon$ is induced by the structure map:

$$X \to S^1.$$ 

If $H^0(S, \rho)$ vanishes, we have

$$2h^1(\rho) = \text{ord}_{s=0} R_\rho(s) \leq 2\text{ord}_{t=1} A^1_\rho(t),$$

and the identity holds if the action of $f^*$ on $H^1(S, \rho)$ is semisimple. Moreover if this condition is satisfied, we have

$$\lim_{s \to 0} |s^{-2h^1(\rho)} R_\rho(s)| = \lim_{t \to 1} |(t-1)^{-h^1(\rho)} A^1_\rho(t)|^2.$$ 

Next we will consider a non-compact case. Let $X$ be a hyperbolic threefold of finite volume with one cusp and $\rho$ a unitary character of the fundamental group. The fundamental group at the cusp will be denoted by $\Gamma_\infty$. Here is a generalization of Fried's results.

**Theorem 2.4.** ([7] [8]) $R_\rho(s)$ is meromorphically continued on the whole plane and satisfies an analog of the Riemann hypothesis. Moreover it satisfies the following properties at the origin.

1. Suppose $\rho|_{\Gamma_\infty}$ is trivial. Then we have

$$\text{ord}_{s=0} R_\rho(s) = 2(h^1(\rho) - 2h^0(\rho) - 1).$$

2. Suppose $\rho|_{\Gamma_\infty}$ is nontrivial, then

$$\text{ord}_{s=0} R_\rho(s) = 2h^1(\rho).$$

Moreover if $h^1(\rho)$ vanishes we have

$$|R_\rho(0)| = |\tau^*_{\infty}(X, \rho)|^{-2}.$$ 

We remark that the "error term" $-2$ in the RHS of the first identity is caused by a pathology of the Hodge theory. Note that in the second case the assumption automatically implies vanishing of $h^0(\rho)$. Thus we have

**Theorem 2.5.** Let $X$ be a hyperbolic threefold of finite volume with one cusp and $\rho$ a unitary character of the fundamental group such that $h^0(\rho)$ vanishes.
1. Suppose $\rho|_{\Gamma_\infty}$ is trivial. Then we have

$$\text{ord}_{s=0} R_\rho(s) + 2 \leq 2\text{ord}_{t=1} A^1_\rho(t),$$

and the identity holds if the action of $\tau^*$ on $H^1(X_\infty, \rho)$ is semisimple.

2. Suppose $\rho|_{\Gamma_\infty}$ is nontrivial. Then we have

$$\text{ord}_{s=0} R_\rho(s) \leq 2\text{ord}_{t=1} A^1_\rho(t),$$

and the identity holds if the action of $\tau^*$ on $H^1(X_\infty, \rho)$ is semisimple. Moreover if all $h^q(\rho)$ vanish we have

$$|R_\rho(0)| = \delta_\rho |A^1_\rho(1)|^2.$$

In either case if we make a change of variables:

$$t = s + 1,$$

under a suitable assumption, our theorem implies two ideals in $\mathbb{C}[[s]]$ generated by $R_\rho(s)$ and $A^1_\rho(s)^2$ coincide. Thus our theorem may be considered as a solution of a geometric analog of the Iwasawa main conjecture.

In particular we may say for each prime closed geodesic $\gamma$, $P_\gamma(s)^{-1}$ divides $A^1_\rho$. In the next section we will explain this phenomenon from a viewpoint of the arithmetic topology.

§ 3. An explanation from the arithmetic topology

Let $X$ be a hyperbolic threefold of finite volume and $\rho$ a unitary representation of the fundamental group. We assume that $H^0(X_\infty, \rho)$ vanishes.

Note that $\epsilon$ induces a map from a set of prime closed geodesics $\Sigma_{\text{prim}}$ to $\mathbb{Z}$. Thus it is decomposed into two subsets:

$$\Sigma^\epsilon_{\text{prim}} = \{ \gamma \in \Sigma_{\text{prim}} | \epsilon(\gamma) \neq 0 \}$$
and its complement $\Sigma^\iota_{\text{prim}}$. An element of $\Sigma^\epsilon_{\text{prim}}$ (resp. $\Sigma^\iota_{\text{prim}}$) will be referred as $\epsilon$-split (resp. $\epsilon$-inert). For $\gamma \in \Sigma^\epsilon_{\text{prim}}$ its $\epsilon$-inertia degree $m_\epsilon(\gamma)$ is defined to be the absolute value of $\epsilon(\gamma)$. 
Let $\gamma \in \Sigma_{\text{prim}}^{\iota}$ be of $\epsilon$-inertia degree 1. We may regard it as a smooth imbedded $S^1$ and let $C_\infty$ be a connected component of $X_\infty \times_X S^1$. Thus we have a diagram:

\[
\begin{array}{c}
C_\infty \rightarrow X_\infty \\
p \downarrow \quad \downarrow \pi \\
S^1 \overset{\gamma}{\rightarrow} X.
\end{array}
\]

(4)

We claim that $C_\infty$ is the universal covering of $S^1$ and that the diagram induces an isomorphism:

\[
\text{Gal}(C_\infty/S^1) \simeq \text{Gal}(X_\infty/X) \simeq \mathbb{Z}.
\]

In fact (4) implies the diagram:

\[
\begin{array}{c}
\pi_1(C_\infty) \rightarrow \pi_1(X_\infty) \\
p_* \downarrow \quad \downarrow \\
\mathbb{Z} = \pi_1(S^1) \overset{\gamma_*}{\rightarrow} \pi_1(X) \\
\downarrow \epsilon \\
\mathbb{Z},
\end{array}
\]

which satisfies

\[
\gamma_*(1) = \gamma.
\]

If $C_\infty$ were not $\mathbb{R}$, it should be a circle. In particular the image of $p_*$ becomes a nontrivial subgroup of $\pi_1(S^1)$. But the image of $\epsilon \cdot \gamma_*$ is a subgroup of $\mathbb{Z}$ which is torsion free, the above diagram shows that $\epsilon(\gamma)$ should be zero. This contradicts to the choice of $\gamma$. Moreover since $\epsilon$-inertia degree of $\gamma$ is one, $\gamma_*$ gives a splitting of $\epsilon$ and we have

\[
\text{Gal}(C_\infty/S^1) \overset{\gamma_*}{\simeq} \text{Gal}(X_\infty/X) \simeq \mathbb{Z}.
\]

Let $N(\gamma)$ be a small tubular neighborhood of $\gamma$ and $N_\infty(\gamma)$ its lift to $X_\infty$ along $C_\infty$:

\[
N_\infty(\gamma) = \pi^{-1}(N(\gamma)).
\]

We set

\[
X_\infty(\gamma) = X_\infty \setminus N_\infty(\gamma).
\]

By the excision we have

\[
H^q(X_\infty, X_\infty(\gamma), \rho) \simeq H^q(N_\infty, \partial N_\infty(\gamma), \rho),
\]

and the computation of the previous section implies

\[
H^q(N_\infty, \partial N_\infty(\gamma), \rho) = \begin{cases} \\
\Lambda/|\det[t - \rho(\gamma)]| & q = 2 \\
0 & q \neq 2.
\end{cases}
\]
Thus the exact sequence (3) and our assumption show the vanishing of $H^0(X_{\infty}(\gamma), \rho)$ and an exact sequence of $\Lambda$-modules:

$$0 \to H^1(X_{\infty}, \rho) \xrightarrow{\text{Res}} H^1(X_{\infty}(\gamma), \rho) \to \Lambda/(\det[t - \rho(\gamma)]) \to 0.$$ 

In particular we know the dimension of $H^1(X_{\infty}(\gamma), \rho)$ is finite and we have an identity of fractional ideals of $\Lambda$:

$$\text{Char}_\Lambda(H^1(X_{\infty}, \rho)) = \text{Char}_\Lambda(H^1(X_{\infty}(\gamma), \rho)) \cdot (\det[1 - \rho(\gamma)t^{-1}])^{-1}.$$ 

More generally let $\gamma$ be an element of $\Sigma_{\text{prim}}^\iota$. Then the subgroup

$$m_\epsilon(\gamma)\mathbb{Z} \subseteq \mathbb{Z} = \text{Gal}(X_{\infty}/X)$$

determines a cyclic covering $X_{m_\epsilon(\gamma)}$ of $X$ with degree $m_\epsilon(\gamma)$. Note that $X_{\infty}$ is its infinite cyclic covering satisfying

$$\text{Gal}(X_{\infty}/X_{m_\epsilon(\gamma)}) = m_\epsilon(\gamma)\mathbb{Z},$$

and that $\gamma$ lifts to a smooth embedded $S^1$ in $X_{m_\epsilon(\gamma)}$ which is mapped to $\pm m_\epsilon(\gamma)$ by

$$\pi_1(X_{m_\epsilon(\gamma)}) \to \text{Gal}(X_{\infty}/X_{m_\epsilon(\gamma)}) = m_\epsilon(\gamma)\mathbb{Z}.$$ 

Now the previous argument shows the vanishing of $H^0(X_{\infty}(\gamma), \rho)$ and an exact sequence:

$$0 \to H^1(X_{\infty}, \rho) \xrightarrow{\text{Res}} H^1(X_{\infty}(\gamma), \rho) \to \Lambda/(\det[t^{m_\epsilon(\gamma)} - \rho(\gamma)]) \to 0$$

and

$$\text{Char}_\Lambda(H^1(X_{\infty}, \rho)) = \text{Char}_\Lambda(H^1(X_{\infty}(\gamma), \rho)) \cdot (\det[1 - \rho(\gamma)t^{-m_\epsilon(\gamma)}])^{-1}.$$ 

Thus we have proved the following theorem.

**Theorem 3.1.** Suppose $H^0(X_{\infty}, \rho)$ vanishes. Then for $\gamma \in \Sigma_{\text{prim}}^\iota$, $H^0(X_{\infty}(\gamma), \rho)$ also vanishes and we have an exact sequence of $\Lambda$-modules:

$$(5) \quad 0 \to H^1(X_{\infty}, \rho) \xrightarrow{\text{Res}} H^1(X_{\infty}(\gamma), \rho) \to \Lambda/(\det[t^{m_\epsilon(\gamma)} - \rho(\gamma)]) \to 0.$$ 

In particular the dimension of $H^1(X_{\infty}(\gamma), \rho)$ is finite and we have an identity of fractional ideals of $\Lambda$:

$$\text{Char}_\Lambda(H^1(X_{\infty}, \rho)) = \text{Char}_\Lambda(H^1(X_{\infty}(\gamma), \rho)) \cdot (\det[1 - \rho(\gamma)t^{-m_\epsilon(\gamma)}])^{-1}.$$
Note that the Euler factor $P_\gamma(s)$ of the Ruelle L-function is given by

$$P_\gamma(s) = \det[1 - \rho(\gamma)t^{-m_e(\gamma)}]|_{t=\exp[l(\gamma)/m_e(\gamma)]}.$$

Since we have

$$\exp\left(\frac{l(\gamma)}{m_e(\gamma)}s\right) - 1 = \frac{l(\gamma)s}{m_e(\gamma)} + O(s^2),$$

localizing at $s = t - 1$, the fact that $(\det[1 - \rho(\gamma)t^{-m_e(\gamma)}])^{-1}$ divides $\text{Char}_\Lambda(H^1(X_\infty, \rho))$ implies the divisibility of $A_\rho^1$ by $P_\gamma(s)^{-1}$ in $\mathbb{C}[[s]]$ for $\gamma \in \Sigma_{\text{prim}}^i$.

We can formulate this fact in terms of an analog of the Euler system ([5]).

First of all we remark that using the homology exact sequence:

$$\rightarrow H_q(X_\infty(\gamma), \rho) \rightarrow H_q(X_\infty, \rho) \rightarrow H_q(X_\infty, X_\infty(\gamma), \rho) \rightarrow H_{q-1}(X_\infty(\gamma), \rho) \rightarrow$$

and by the isomorphism

$$H_q(X_\infty, X_\infty(\gamma), \rho) \simeq H_q(D^2, S^1, \rho)$$

derived from the excision and the homotopy invariance of the homology group one may check that the dimension of $H_q(X_\infty(\gamma), \mathbb{C})$ and $H_q(X_\infty(\gamma), \rho)$ are finite. Taking the dual of (5) over $\mathbb{C}$, the Milnor duality shows an exact sequence of $\Lambda$-modules:

$$0 \rightarrow \Lambda/(\det[t^{m_e(\gamma)} - \rho(\gamma)]) \rightarrow H^1(X_\infty(\gamma), \rho) \xrightarrow{\text{Cor}} H^1(X_\infty, \rho) \rightarrow 0.$$

Let us fix a nonzero element $c_\infty$ of $H^1(X_\infty, \rho)$ and choose its any lift $c'(\gamma)_\infty$ to $H^1(X_\infty(\gamma), \rho)$. Then

$$c(\gamma)_\infty = F_\gamma(t)c'(\gamma)_\infty, \quad F_\gamma(t) = \det[t^{m_e(\gamma)} - \rho(\gamma)]$$

is independent of a choice of the lift and satisfies

$$\text{Cor}(c(\gamma)_\infty) = F_\gamma(t)c_\infty.$$

More generally, for elements $\{\gamma_1, \cdots, \gamma_N\}$ of $\Sigma_{\text{prim}}^i$, we set

$$X_\infty(\gamma_1 \cdots \gamma_N) = X_\infty \setminus N_\infty(\gamma_1) \cup \cdots \cup N_\infty(\gamma_N).$$

Using Theorem 3.1, an induction argument shows that we have an exact sequence of $\Lambda$-modules:

$$0 \rightarrow \Lambda/(F_\gamma(t)) \rightarrow H^1(X_\infty(\gamma_1 \cdots \gamma_N), \rho) \xrightarrow{\text{Cor}} H^1(X_\infty(\gamma_1 \cdots \gamma_{N-1}), \rho) \rightarrow 0.$$
Therefore we can successively choose an element $c_{\infty}(\gamma_1 \cdots \gamma_N)$ of $H^1(X_{\infty}(\gamma_1 \cdots \gamma_N), \rho)$ so that

$$\text{Cor}(c_{\infty}(\gamma_1 \cdots \gamma_N)) = F_{\gamma_N}(t) c_{\infty}(\gamma_1 \cdots \gamma_{N-1}).$$

Thus $\{c_{\infty}(\gamma_1 \cdots \gamma_N)\}$ has the same property as the Euler system [5] §2.1. If we apply the co-restriction map “Cor” $N$-times to $c_{\infty}(\gamma_1 \cdots \gamma_N)$, we obtain an element $d_{\infty}(\gamma_1 \cdots \gamma_N)$ of $H^1(X_{\infty}, \rho)$ which satisfies

$$d_{\infty}(\gamma_1 \cdots \gamma_N) = \prod_{i=1}^{N} F_{\gamma_i}(t) \cdot c_{\infty}.$$

Now our solution of the geometric Iwasawa conjecture is formally described in the following way.

If two elements $c$ and $c'$ of $H^1(X_{\infty}, \rho)$ have a relation:

$$c' = f \cdot c, \quad f \in \Lambda,$$

$f^{-1}$ will be denoted by $\text{ind}_{\Lambda}(c, c')$. (Note that in fact since $H^1(X_{\infty}, \rho)$ is a torsion $\Lambda$-module it is formally defined.) In particular our topological Euler system gives the $\epsilon$-inert part of the Euler product:

$$\text{ind}_{\Lambda}(c_{\infty}, d_{\infty}(\prod_{\gamma \in \Sigma_{\text{prin}}} \gamma)) = \prod_{\gamma \in \Sigma_{\text{prin}}} F_{\gamma_i}^{-1} \in \mathbb{C}[[s]].$$

The following statement is a formal reformulation of Theorem 3.1, which should be compared with [5] Theorem 2.3.3.

**Theorem 3.2.** (formal) In $\mathbb{C}[[s]]$, $\text{ind}_{\Lambda}(c_{\infty}, d_{\infty}(\prod_{\gamma \in \Sigma_{\text{prin}}} \gamma))$ divides $\text{Char}_{\Lambda}(H^1(X_{\infty}, \rho))$.

**References**


