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Massera type theorems in hyperfunctions with reflexive Banach values

By

Yasunori Okada*

Abstract

For some classes of periodic linear ordinary differential equations and functional equations, it is known that the existence of a bounded solution in the future implies the existence of a periodic solution. They are called the Massera type phenomena. Being interested in such results, we introduced the notion of bounded hyperfunctions at infinity, and studied the Massera type phenomena for hyperfunction solutions to periodic linear functional equations.

In this article, we continue this study, and after recalling the terminologies, we will observe the Massera type phenomena in the settings of hyperfunctions with reflexive Banach values.

§ 1. Introduction

In 1950, Massera studied in [6] the existence of periodic solutions to periodic ordinary differential equations. In the linear case, he gave the result that for a 1-periodic linear ordinary differential equations of normal form with continuous coefficients, the existence of a bounded solution in the future implies that the existence of a 1-periodic solution. After Massera, many generalizations appeared in the case of periodic linear functional equations. The author studied in [7] such phenomena in the framework of hyperfunctions. We introduced a notion of bounded hyperfunctions at infinity and classes of operators, and gave the following result. (We recall relevant terminologies later.)

Theorem 1.1. *Let E be a sequentially complete Hausdorff locally convex space, K a closed interval in $\mathbb{R}$, and ω a positive number. Consider an ω-periodic operator P of type K on a strip domain $\mathbb{D}^1 + i]{-d}, d[$ for ${}^{E}\mathscr{O}_{L^\infty}$ with some $d > 0$ and an ω-periodic E-valued hyperfunction f. Assume that E satisfies the sequential Montel property* (M).

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Supported in part by JSPS Grant-in-Aid No. 22540173.
*Graduate School of Science, Chiba University, Chiba, 263-8522, Japan

Then the equation $Pu = f$ has an ω-periodic E-valued hyperfunction solution if and only if it has an ${}^E\mathscr{B}_{L^\infty}$-solution in a neighborhood of $+\infty$.

It is known that many useful function spaces appearing in the study of differential equations admit the sequential Montel property. But there are also many useful spaces which do not. For example, infinite dimensional Banach spaces never admit it. Therefore we are interested in the problem if we can observe similar phenomena for the case that E does not admit the sequential Montel property, for example, the case that E is a Banach space.

The purpose of this article is to give a partial answer. First, in the section 2, we briefly recall the notion of bounded hyperfunctions at infinity and that of operators of type K, which we introduced in [7]. In the section 3, we study some functional analytic properties of the spaces of holomorphic functions taking values in a reflexive locally convex space. After these preparations, we give our main result in the section 4, that is, a Massera type theorem in a reflexive Banach valued case. (See Theorem 4.4.)

§ 2. Bounded hyperfunctions at infinity and operators

We recall the notion of bounded hyperfunctions at infinity and that of operators of type K, introduced in [7]. The definition of bounded hyperfunctions is similar to the original cohomological definitions of hyperfunctions and Fourier hyperfunctions given in the one-dimensional case in Sato [8]. Refer also to Sato [9], Kawai [5], Sato-Kawai-Kashiwara [10], and Kaneko [4], for hyperfunctions, Fourier hyperfunctions, and related topics.

In this section, E denotes a sequentially complete Hausdorff locally convex space over $\mathbb{C}$. We denote by $\mathcal{N}(E)$ the family of continuous seminorms of E.

§ 2.1. Sheaf ${}^E\mathscr{B}_{L^\infty}$ of E-valued bounded hyperfunctions

In this subsection, we define the sheaf ${}^E\mathscr{B}_{L^\infty}$ of E-valued bounded hyperfunctions at infinity on a compactification $\mathbb{D}^1 := [-\infty, +\infty] = \mathbb{R} \sqcup \{\pm\infty\}$ of $\mathbb{R}$. In the scalar case (that is, the case $E = \mathbb{C}$), the space of the global sections of our sheaf can be identified with the space $\mathcal{B}_{L^\infty}$ of bounded hyperfunctions due to Chung-Kim-Lee [2].

We consider the following diagram

$$\begin{array}{ccc} \mathbb{C} = \mathbb{R} + i\mathbb{R} & \hookrightarrow & \mathbb{D}^1 + i\mathbb{R} \\ \cup & & \cup \\ \mathbb{R} = \,]-\infty, +\infty[& \hookrightarrow & \mathbb{D}^1 = [-\infty, +\infty] \end{array}$$

and identify $\mathbb{C}$ with an open subset $\mathbb{R} + i\mathbb{R}$ in $\mathbb{D}^1 + i\mathbb{R}$. Let ${}^E\mathscr{O}$ be the sheaf of E-valued holomorphic functions on $\mathbb{C}$. Refer to Bochnak-Siciak [1] for the properties of holomorphic functions taking values in a sequentially complete Hausdorff locally convex space.

Definition 2.1. We define the sheaf ${}^E\mathscr{O}_{L^\infty}$ of E-valued bounded holomorphic functions at infinity on $\mathbb{D}^1 + i\mathbb{R}$, as the sheaf associated with the presheaf given by the correspondence

$$\mathbb{D}^1 + i\mathbb{R} \overset{\text{open}}{\supset} U \mapsto \{f \in {}^E\mathscr{O}(U \cap \mathbb{C}); f \text{ is bounded.}\}.$$

For a compact set $L \Subset U$, a continuous seminorm $p \in \mathscr{N}(E)$, and a section $f \in {}^E\mathscr{O}(U \cap \mathbb{C})$, we use the notation

$$\|f\|_{L,p} := \sup_{w \in L \cap \mathbb{C}} p(f(w)). \tag{2.1}$$

Then, the space ${}^E\mathscr{O}_{L^\infty}(U)$ can be written as

$${}^E\mathscr{O}_{L^\infty}(U) := \{f \in {}^E\mathscr{O}(U \cap \mathbb{C}); \|f\|_{L,p} < +\infty, \forall L \Subset U, \forall p \in \mathscr{N}(E)\}, \tag{2.2}$$

and is endowed with a locally convex topology by the family of seminorms $\|\cdot\|_{L,p}$ with $L \Subset U$ and $p \in \mathscr{N}(E)$. We sometimes use $\mathscr{O}_{L^\infty}$ instead of ${}^{\mathbb{C}}\mathscr{O}_{L^\infty}$ for the scalar case ($E = \mathbb{C}$), and $\|f\|_L$ instead of $\|f\|_{L,|\cdot|}$ for $f \in \mathscr{O}_{L^\infty}(U)$.

Note that ${}^E\mathscr{O}_{L^\infty}|_{\mathbb{C}} = {}^E\mathscr{O}$, that is, ${}^E\mathscr{O}_{L^\infty}(U) = {}^E\mathscr{O}(U)$ for $U \subset \mathbb{C}$.

Definition 2.2 (Sheaf of E-valued bounded hyperfunctions at infinity)**.** We define the sheaf ${}^E\mathscr{B}_{L^\infty}$ of E-valued bounded hyperfunctions at infinity on $\mathbb{D}^1$ as the sheaf associated with the presheaf

$$\mathbb{D}^1 \overset{\text{open}}{\supset} \Omega \mapsto \varinjlim_{U} \frac{{}^E\mathscr{O}_{L^\infty}(U \setminus \Omega)}{{}^E\mathscr{O}_{L^\infty}(U)}, \tag{2.3}$$

where U runs through complex neighborhoods of Ω, that is, open sets in $\mathbb{D}^1 + i\mathbb{R}$ including Ω as a closed subset.

We also write ${}^E\mathscr{B}_{L^\infty}|_{\mathbb{R}}$ by ${}^E\mathscr{B}$, and sometimes abbreviate ${}^{\mathbb{C}}\mathscr{B}_{L^\infty}$ and ${}^{\mathbb{C}}\mathscr{B}$ as $\mathscr{B}_{L^\infty}$ and $\mathscr{B}$, respectively. These notations are compatible with the sheaf $\mathscr{B}$ of usual hyperfunctions due to Sato, and with the sheaf ${}^E\mathscr{B}$ of E-valued hyperfunctions introduced by Ion-Kawai [3] when E is a Fréchet space.

A section of ${}^E\mathscr{B}_{L^\infty}$ on a compact set admits a boundary value representation. In the sequel, we use the conventions $B_d :=]-d, d[$ and $\dot{B}_d := B_d \setminus \{0\}$ for $d > 0$. We cite the following proposition, which is a part of [7, Proposition 2.3].

Proposition 2.3. *For a compact set $K \subset \mathbb{D}^1$, we have*

$${}^E\mathscr{B}_{L^\infty}(K) = \varinjlim_{\Omega, d > 0} \frac{{}^E\mathscr{O}_{L^\infty}(\Omega + i\dot{B}_d)}{{}^E\mathscr{O}_{L^\infty}(\Omega + iB_d)}, \tag{2.4}$$

where Ω runs through open neighborhoods of K in $\mathbb{D}^1$.

§ 2.2. Operators of type K

We recall the notion of operators of type K in [7, §3].

Definition 2.4. Let $U \subset \mathbb{D}^1 + i\mathbb{R}$ be an open set and K a closed interval $[a, b] \subset \mathbb{R}$. We admit the case $a = b$, i.e., $K = \{a\}$. A family $P = \{P_V : {}^E\mathscr{O}_{L^\infty}(V + K) \to {}^E\mathscr{O}_{L^\infty}(V)\}_{V \subset U}$ of linear maps is said to be an operator of type K for ${}^E\mathscr{O}_{L^\infty}$ on U, if each P_V is continuous and each diagram below commutes for any pair $V_1 \supset V_2$ in U.

$$\begin{array}{ccc}
{}^E\mathscr{O}_{L^\infty}(V_1 + K) & \xrightarrow{P_{V_1}} & {}^E\mathscr{O}_{L^\infty}(V_1) \\
\downarrow & & \downarrow \\
{}^E\mathscr{O}_{L^\infty}(V_2 + K) & \xrightarrow{P_{V_2}} & {}^E\mathscr{O}_{L^\infty}(V_2)
\end{array}$$

Here the vertical arrows are the restriction maps.

Note that the meaning of the vector sum $V + K$ can be naturally defined also in case $V \not\subset \mathbb{C}$.

An operator P of type K automatically induces a family $\{P_\Omega : {}^E\mathscr{B}_{L^\infty}(\Omega + K) \to {}^E\mathscr{B}_{L^\infty}(\Omega)\}_{\Omega \subset U \cap \mathbb{D}^1}$ of linear maps, where the following diagram commutes

$$\begin{array}{ccc}
{}^E\mathscr{B}_{L^\infty}(\Omega_1 + K) & \xrightarrow{P_{\Omega_1}} & {}^E\mathscr{B}_{L^\infty}(\Omega_1) \\
\downarrow & & \downarrow \\
{}^E\mathscr{B}_{L^\infty}(\Omega_2 + K) & \xrightarrow{P_{\Omega_2}} & {}^E\mathscr{B}_{L^\infty}(\Omega_2)
\end{array}$$

for any $\Omega_1 \supset \Omega_2$ in $U \cap \mathbb{D}^1$.

Consider the case $U \subset \mathbb{C}$. Then the sets V and $V + K$ are included in $\mathbb{C}$, and the entries of the family P are linear maps $P_V : {}^E\mathscr{O}(V + K) \to {}^E\mathscr{O}(V)$. Therefore, in this case, we say that P is an operator of type K for ${}^E\mathscr{O}$ on U.

Let P be again an operator of type K for ${}^E\mathscr{O}_{L^\infty}$ on U, and we define the notion of ${}^E\mathscr{O}_{L^\infty}$-solutions on an open set $V \subset U$ to an equation given by P. For $f \in {}^E\mathscr{O}_{L^\infty}(V)$, we say that u is an ${}^E\mathscr{O}_{L^\infty}$-solution to the equation $Pu = f$ on V, or simply an ${}^E\mathscr{O}_{L^\infty}(V)$-solution to $Pu = f$, if u belongs to ${}^E\mathscr{O}_{L^\infty}(V + K)$ and satisfies $P_V u = f$. Note that the domain of definition of u is not V but $V + K$. Similarly, for $f \in {}^E\mathscr{B}_{L^\infty}(\Omega)$, an ${}^E\mathscr{B}_{L^\infty}$-solution to $Pu = f$ on Ω is a section $u \in {}^E\mathscr{B}_{L^\infty}(\Omega + K)$ satisfying $P_\Omega u = f$. Moreover, when f is a germ of ${}^E\mathscr{B}_{L^\infty}$ at $+\infty$ (that is, $f \in ({}^E\mathscr{B}_{L^\infty})_{+\infty}$), it makes sense to consider an $({}^E\mathscr{B}_{L^\infty})_{+\infty}$-solution to an equation $Pu = f$.

Note also that we sometimes omit the subscripts V and Ω in P_V and P_Ω if it causes no confusion. A simple example is the differentiation ∂_w.

§ 2.3. Periodicity for bounded hyperfunctions and operators

We take a positive constant ω and give a notion of ω-periodicity for our hyperfunctions and operators. Roughly speaking, we denote by T_ω the ω-translation operator $f \mapsto f(\cdot + \omega)$, and by $T_\omega - 1$ the ω-difference operator $f \mapsto f(\cdot + \omega) - f(\cdot)$. Then we define the ω-periodicity for bounded holomorphic functions and bounded hyperfunctions by the equation $(T_\omega - 1)f = 0$, and the ω-periodicity for operators of type K by the commutativity with T_ω. Let us see this process a little bit more precisely.

As we have seen in (2.2), a section $f \in {}^E\mathscr{O}_{L^\infty}(V + \omega)$ is actually a section $f \in {}^E\mathscr{O}((V + \omega) \cap \mathbb{C})$ satisfying $\|f\|_{L,p} < +\infty$ for any $L \Subset V + \omega$ and $p \in \mathcal{N}(E)$. We define $T_\omega f \in {}^E\mathscr{O}(V \cap \mathbb{C})$ by $(T_\omega f)(w) := f(w + \omega)$ for $w \in V \cap \mathbb{C}$. Then, it immediately follows that $\|T_\omega f\|_{L,p} = \|f\|_{L+\omega,p} < +\infty$ for $L \Subset V$ and $p \in \mathcal{N}(E)$, which implies the continuity of $T_\omega : {}^E\mathscr{O}_{L^\infty}(V + \omega) \to {}^E\mathscr{O}_{L^\infty}(V)$. Since these maps for open sets $V \subset \mathbb{D}^1 + i\mathbb{R}$ commute with restrictions, they form an operator of type $\{\omega\}$ for ${}^E\mathscr{O}_{L^\infty}$ on $\mathbb{D}^1 + i\mathbb{R}$. Similarly $T_\omega - 1$ becomes an operator of type $[0, \omega]$, which can be seen from the estimate $\|(T_\omega - 1)f\|_{L,p} \le \|f\|_{L+\omega,p} + \|f\|_{L,p} \le 2\|f\|_{L+[0,\omega],p}$.

A section $f \in {}^E\mathscr{O}_{L^\infty}(V + [0, \omega])$, (resp. $f \in {}^E\mathscr{B}_{L^\infty}(\Omega + [0, \omega])$) is called ω-periodic if it satisfies $(T_\omega - 1)f = 0$ in ${}^E\mathscr{O}_{L^\infty}(V)$, (resp. in ${}^E\mathscr{B}_{L^\infty}(\Omega)$), and an operator $P = \{P_V\}_{V \subset U}$ of type $K \subset \mathbb{R}$ on a strip domain U is called ω-periodic if the diagram

$$\begin{array}{ccc} {}^E\mathscr{O}_{L^\infty}(V + \omega + K) & \xrightarrow{P_{V+\omega}} & {}^E\mathscr{O}_{L^\infty}(V + \omega) \\ \Big\downarrow{\scriptstyle T_\omega} & & \Big\downarrow{\scriptstyle T_\omega} \\ {}^E\mathscr{O}_{L^\infty}(V + K) & \xrightarrow{P_V} & {}^E\mathscr{O}_{L^\infty}(V) \end{array}$$

commutes for any $V \subset U$. Note that ω-periodic operator induces the commutative diagram

$$\begin{array}{ccc} {}^E\mathscr{B}_{L^\infty}(\Omega + \omega + K) & \xrightarrow{P_{\Omega+\omega}} & {}^E\mathscr{B}_{L^\infty}(\Omega + \omega) \\ \Big\downarrow{\scriptstyle T_\omega} & & \Big\downarrow{\scriptstyle T_\omega} \\ {}^E\mathscr{B}_{L^\infty}(\Omega + K) & \xrightarrow{P_\Omega} & {}^E\mathscr{B}_{L^\infty}(\Omega) \end{array}$$

for any $\Omega \subset U \cap \mathbb{D}^1$, and preserves the ω-periodicity of its operands.

Now we cite a result concerning ω-periodicity.

Proposition 2.5 ([7, Proposition 3.8]). *Let $\Omega \subset \mathbb{R}$ be an open interval and K the closed interval $[0, \omega]$. The restriction maps ${}^E\mathscr{B}_{L^\infty}(\mathbb{D}^1) \to {}^E\mathscr{B}(\mathbb{R})$ and ${}^E\mathscr{B}(\mathbb{R}) \to {}^E\mathscr{B}(\Omega + K)$ induce the following isomorphisms respectively.*

$$\{f \in {}^E\mathscr{B}_{L^\infty}(\mathbb{D}^1); (T_\omega - 1)f = 0\} \to \{f \in {}^E\mathscr{B}(\mathbb{R}); (T_\omega - 1)f = 0\}, \tag{2.5}$$

$$\{f \in {}^E\mathscr{B}(\mathbb{R}); (T_\omega - 1)f = 0\} \to \{f \in {}^E\mathscr{B}(\Omega + K); (T_\omega - 1)f = 0\}. \tag{2.6}$$

Moreover, any ω-periodic hyperfunction $g \in {}^{E}\mathscr{B}(\mathbb{R})$ has an ω-periodic defining function $f \in {}^{E}\mathscr{O}_{L^\infty}(\mathbb{D}^1 + i\dot{B}_d)$ with some $d > 0$.

Consider an equation $Pu = f$ on $\mathbb{R}$, where P is an ω-periodic operator of type $K \subset \mathbb{R}$ for ${}^{E}\mathscr{O}_{L^\infty}$ on $\mathbb{D}^1 + iB_d$, and f is an ω-periodic E-valued hyperfunction on $\mathbb{R}$. We can take a unique ω-periodic extension $\tilde{f} \in {}^{E}\mathscr{B}_{L^\infty}(\mathbb{D}^1)$ of f using the isomorphism (2.5), and associate an equation $P\tilde{u} = \tilde{f}$ on $\mathbb{D}^1$ to the original equation $Pu = f$. Under this situation, we give the following corollary of Proposition 2.5, which is explained at the end of section 3 in [7].

Corollary 2.6. *The restriction ${}^{E}\mathscr{B}_{L^\infty}(\mathbb{D}^1) \to {}^{E}\mathscr{B}(\mathbb{R})$ induces the isomorphism between the spaces of the ω-periodic solutions.*

$$\{\tilde{u} \in {}^{E}\mathscr{B}_{L^\infty}(\mathbb{D}^1); (T_\omega - 1)\tilde{u} = 0,\ P\tilde{u} = \tilde{f}\} \xrightarrow{\sim} \{u \in {}^{E}\mathscr{B}(\mathbb{R}); (T_\omega - 1)u = 0,\ Pu = f\}.$$

§ 3. Duality results on ${}^{E}\mathscr{O}$

Throughout this section, E denotes a reflexive Hausdorff locally convex space over $\mathbb{C}$. We denote by E' its strong dual space. By the very definition of the reflexivity, the standard embedding $\imath : E \to E''$ given by $\imath(x)(y) = y(x)$ for $x \in E$ and $y \in E'$ becomes a topological isomorphism. Since the reflexivity implies the sequential completeness, we can consider ${}^{E}\mathscr{O}$ as we did in the previous section, as well as ${}^{E'}\mathscr{O}$ by the same reason. We study some functional analytic properties on ${}^{E}\mathscr{O}$.

Note that, unlike in other sections, we do not consider $\mathbb{D}^1 + i\mathbb{R}$, (nor ${}^{E}\mathscr{O}_{L^\infty}$, ${}^{E}\mathscr{B}_{L^\infty}$) in this section. Instead, we take the Riemann sphere $\mathbb{P}^1 := \mathbb{C} \sqcup \{\infty\}$, where "$\infty$" denotes its point at infinity.

§ 3.1. A weak form of the Köthe duality

Let L be a compact set in $\mathbb{C}$ and consider the space ${}^{E}\mathscr{O}(L) := \varinjlim_{V \ni L} {}^{E}\mathscr{O}(V)$ of E-valued holomorphic functions defined in a neighborhood of L, where V runs through open neighborhoods of L in $\mathbb{C}$. We endowed the space ${}^{E}\mathscr{O}(L)$ with the locally convex inductive limit topology, and give a weak form of the Köthe duality.

Definition 3.1. For open neighborhoods V, $W \subset \mathbb{C}$ of L, we take a compact neighborhood M of L in $W \cap V$ whose boundary $\gamma := \partial M$ consists of finite piecewise smooth simple closed curves, and define a bilinear form

$$\langle \cdot, \cdot \rangle_L : {}^{E'}\mathscr{O}(W \setminus L) \times {}^{E}\mathscr{O}(V) \to \mathbb{C}$$

by

$$\langle F, f \rangle_L := \int_\gamma F(w)(f(w))dw \tag{3.1}$$

for $F \in {}^{E'}\mathscr{O}(W \setminus L)$ and $f \in {}^{E}\mathscr{O}(V)$. Here $F(w)(f(w))$ is a value of the continuous linear functional $F(w) \in E'$ evaluated at $f(w) \in E$.

Let us explain in the followings the fact that $\langle \cdot, \cdot \rangle_L$ is well-defined and that it induces the duality between ${}^{E'}\mathscr{O}(W \setminus L)/{}^{E'}\mathscr{O}(W)$ and ${}^{E}\mathscr{O}(L)$.

The first remark is on the existence of M satisfying the requirements in Definition 3.1, that we may take as M a union of finite closed disks with $L \Subset M \Subset V \cap W$.

Lemma 3.2. *$F(w)(f(w))$ is holomorphic in $w \in (W \cap V) \setminus L$. Therefore the integral* (3.1) *does not depend on the choice of γ, and induces a bilinear form on ${}^{E'}\mathscr{O}(W \setminus L) \times {}^{E}\mathscr{O}(L)$.*

Proof. For the former statement, we shall show that $\frac{F(z)(f(z)) - F(w)(f(w))}{(z-w)}$ converges as $z \to w$. This quotient is equal to

$$F'(w)(f(z)) + F(w)\Big(\frac{f(z) - f(w)}{z - w}\Big) + \Big(\frac{F(z) - F(w)}{z - w} - F'(w)\Big)(f(z)),$$

whose first two terms converge to $F'(w)(f(w)) + F(w)(f'(w))$. To see that the third term converges to 0, note that $f(z)$ belongs to a bounded set in E when z belongs to a compact neighborhood of w, and that a convergence in E' is nothing but a uniform convergence as functionals on bounded sets in E.

The latter statement directly follows from the former. □

Definition 3.3. Let L be a compact set in $\mathbb{C}$ and W an open neighborhood. We define linear maps $\alpha : ({}^{E}\mathscr{O}(L))' \to {}^{E'}\mathscr{O}(W \setminus L)$ and $\beta : {}^{E'}\mathscr{O}(W \setminus L) \to ({}^{E}\mathscr{O}(L))'$ by

$$\alpha(\varphi)(w)(x) := \varphi\Big(\frac{1}{2\pi i}\frac{1}{w - \cdot}x\Big) \in \mathbb{C}, \tag{3.2}$$

for $\varphi \in ({}^{E}\mathscr{O}(L))'$, $x \in E$ and $w \in W \setminus L$, and by

$$\beta(F)(f) := \langle F, f \rangle_L, \tag{3.3}$$

for $F \in {}^{E'}\mathscr{O}(W \setminus L)$ and $f \in {}^{E}\mathscr{O}(L)$. Here we regard $\frac{1}{2\pi i}\frac{1}{w-\cdot}x$ as an element of ${}^{E}\mathscr{O}(L)$ in the right hand side of (3.2).

The linearity of the functional $\alpha(\varphi)(w) : E \to \mathbb{C}$ is trivial, and that of the map α is also trivial provided it is well-defined. Let us show the well-definedness of α.

Lemma 3.4. *$\alpha(\varphi)(w) : E \to \mathbb{C}$ is continuous.*

Proof. For any $w \in W \setminus L$, we take $\varepsilon > 0$ and a neighborhood V of L with $\mathrm{dist}(w, V) > \varepsilon$. For any compact set $M \subset V$ and continuous seminorm $p \in \mathcal{N}(E)$, we have

$$\|\frac{1}{2\pi i}\frac{1}{w - \cdot}x\|_{M,p} = \|\frac{1}{2\pi i}\frac{1}{w - \cdot}\|_M \cdot p(x) \leq \frac{p(x)}{2\pi\varepsilon},$$

which implies the continuity of $E \ni x \mapsto \frac{1}{2\pi i}\frac{1}{w-\cdot}x \in {}^E\mathscr{O}(V)$. Since the restriction ${}^E\mathscr{O}(V) \to {}^E\mathscr{O}(L)$ is also continuous, so is the composition $\alpha(\varphi)(w)$ of these maps. $\square$

Lemma 3.4 implies that $\alpha(\varphi)(w) \in E'$.

Lemma 3.5. $\alpha(\varphi) \in {}^{E'}\mathscr{O}(W \setminus L)$.

Proof. Thanks to [1, Theorem 3.1], it suffices to show that $W \setminus L \ni w \mapsto x^*(\alpha(\varphi)(w)) \in \mathbb{C}$ is holomorphic for any $x^* \in E''$. Since E is reflexive, there exists $x \in E$ such that $\imath(x) = x^*$, that is, $x^*(y) = y(x)$ for any $y \in E'$. Therefore, we shall prove that $\alpha(\varphi)(w)(x)$ is holomorphic in w. By a direct calculation, we have

$$\frac{\alpha(\varphi)(z)(x) - \alpha(\varphi)(w)(x)}{z-w} = \varphi\Big(\frac{1}{2\pi i}\frac{-1}{(z-\cdot)(w-\cdot)}x\Big),$$

for w, $z \in W \setminus L$. Now the conclusion follows from the fact that $\frac{-1}{(z-\cdot)(w-\cdot)}x \to \frac{-1}{(w-\cdot)^2}x$ as $z \to w$ in ${}^E\mathscr{O}(L)$. $\square$

Lemma 3.6. *$\beta(F) \in ({}^E\mathscr{O}(L))'$, i.e., $\beta(F) : {}^E\mathscr{O}(L) \to \mathbb{C}$ is continuous.*

Proof. By the definition of the locally convex inductive limit topology, it suffices to show that $\beta(F)$ is continuous as ${}^E\mathscr{O}(V) \to \mathbb{C}$ for any open neighborhood V of L. Once we fix V and W, then we can fix a contour γ in the calculation of $\langle F, f\rangle_L$ for any $F \in {}^{E'}\mathscr{O}(W \setminus L)$ and $f \in {}^E\mathscr{O}(V)$.

Since the subset $\mathcal{M} := \{F(w)\}_{w\in\gamma} \subset E'$ is compact and therefore bounded, and also since $\imath : E \to E''$ is a topological isomorphism, $p_{\mathcal{M}}(x) := \sup_{y\in\mathcal{M}}|\imath(x)(y)| = \sup_{y\in\mathcal{M}}|y(x)|$ for $x \in E$ defines a continuous seminorm on E. Now we have

$$\begin{aligned}|\beta(F)(f)| &\le \int_\gamma |F(w)(f(w))| \cdot |dw| \le \int_\gamma p_{\mathcal{M}}(f(w)) \cdot |dw| \\ &\le |\gamma| \cdot \sup_{w\in\gamma} p_{\mathcal{M}}(f(w)) = |\gamma| \cdot \|f\|_{\gamma,p_{\mathcal{M}}},\end{aligned}$$

which concludes the proof. $\square$

Lemma 3.7. *If $F \in {}^{E'}\mathscr{O}(W)$, then $\beta(F) = 0$.*

Proof. Under the same terminologies as in Definition 3.1, we have $\beta(F)(f) = \int_{\partial M} F(w)(f(w))dw$ for any $f \in {}^E\mathscr{O}(V)$. By the same proof of Lemma 3.2, we can show $F(w)(f(w)) \in \mathscr{O}(M)$. Therefore $\beta(F)(f) = 0$. $\square$

Lemma 3.8. $\beta \circ \alpha = \mathrm{id}_{({}^E\mathscr{O}(L))'}$.

Proof. We shall show $\beta(\alpha(\varphi))(f) = \varphi(f)$ for any $\varphi \in ({}^E\mathscr{O}(L))'$ and any $f \in {}^E\mathscr{O}(L)$. By the definition, we have

$$\beta(\alpha(\varphi))(f) = \int_\gamma \alpha(\varphi)(w)(f(w))dw = \int_\gamma \varphi\Big(\frac{1}{2\pi i}\frac{1}{w-\cdot}f(w)\Big)dw, \tag{3.4}$$

where $\gamma := \partial M$ with $L \Subset M \Subset W$.

Now we claim that the map $\gamma \ni w \mapsto \frac{1}{2\pi i}\frac{1}{w-\cdot}f(w) \in {}^E\mathscr{O}(L)$ is continuous. In fact, for any choice of L_1 with $L \Subset L_1 \Subset M$ ($\partial M = \gamma$), we have the estimate

$$\begin{aligned}\|\frac{1}{z-\cdot}f(z) - \frac{1}{w-\cdot}f(w)\|_{L_1,p} &\le \|\frac{w-z}{(z-\cdot)(w-\cdot)}f(z)\|_{L_1,p} + \|\frac{f(z)-f(w)}{w-\cdot}\|_{L_1,p} \\ &\le \frac{|w-z|}{\operatorname{dist}(\gamma,L_1)^2}\|f\|_{\gamma,p} + \frac{1}{\operatorname{dist}(\gamma,L_1)}p(f(z)-f(w)),\end{aligned}$$

for $p \in \mathcal{N}(E)$ and w, $z \in \gamma$. The right hand side converges to 0 as $z \to w$.

Therefore the calculation (3.4) can be continued to

$$\beta(\alpha(\varphi))(f) = \varphi\Big(\int_\gamma \frac{1}{2\pi i}\frac{1}{w-\cdot}f(w)dw\Big) = \varphi(f).$$

□

Lemma 3.9. *For any $F \in {}^{E'}\mathscr{O}(W \setminus L)$, $\alpha(\beta(F)) - F \in {}^{E'}\mathscr{O}(W)$.*

Proof. We take an arbitrary relatively compact open set U with piecewise smooth boundary $\Gamma = \partial U$ satisfying $L \Subset U \Subset W$, and we shall show that $\big(\alpha(\beta(F)) - F\big)|_{U\setminus L} \in {}^{E'}\mathscr{O}(U \setminus L)$ can be extended to a section in ${}^{E'}\mathscr{O}(U)$.

For any $w \in U \setminus L$, we choose M as in Definition 3.1 satisfying $w \notin M$ and $L \Subset M \Subset U$, and define $\gamma := \partial M$. We can show $\alpha(\beta(F))(w) = \frac{1}{2\pi i}\int_\gamma \frac{F(z)}{w-z}dz$ using a test with an arbitrary $x \in E$ as

$$\begin{aligned}\alpha(\beta(F))(w)(x) &= \beta(F)\Big(\frac{1}{2\pi i}\frac{1}{w-\cdot}x\Big) = \int_\gamma F(z)\Big(\frac{1}{2\pi i}\frac{1}{w-z}x\Big)dz = \frac{1}{2\pi i}\int_\gamma \frac{F(z)(x)}{w-z}dz \\ &= \Big(\frac{1}{2\pi i}\int_\gamma \frac{F(z)}{w-z}dz\Big)(x).\end{aligned}$$

On the other hand, we have $F(w) = \frac{1}{2\pi i}\int_{\Gamma-\gamma}\frac{F(z)}{z-w}dz$, since $w \in U \setminus M$ and $\partial(U \setminus M) = \Gamma - \gamma$. It follows from these equalities that

$$\big(\alpha(\beta(F)) - F\big)(w) = \frac{1}{2\pi i}\int_\Gamma \frac{F(z)}{w-z}dz,$$

which can be extended to U. □

By these preparations, we give

Theorem 3.10. *The maps α and β induce the isomorphism between vector spaces*

$$({}^{E}\mathscr{O}(L))' \xrightarrow{\sim} {}^{E'}\mathscr{O}(W \setminus L)/{}^{E'}\mathscr{O}(W).$$

Consider the case $W = \mathbb{C}$.

Corollary 3.11. *The maps α and β also induce the isomorphism between vector spaces*

$$({}^{E}\mathscr{O}(L))' \xrightarrow{\sim} {}^{E'}\mathscr{O}^{\circ}(\mathbb{P}^1 \setminus L).$$

Here ${}^{E'}\mathscr{O}^{\circ}(\mathbb{P}^1 \setminus L)$ denotes the subspace $\{F \in {}^{E'}\mathscr{O}(\mathbb{P}^1 \setminus L);\ F(\infty) = 0\}$ of ${}^{E'}\mathscr{O}(\mathbb{P}^1 \setminus L)$.

This corollary follows from Theorem 3.10 and the lemma below.

Lemma 3.12. *For any $\varphi \in ({}^{E}\mathscr{O}(L))'$, $\alpha(\varphi) \in {}^{E'}\mathscr{O}(\mathbb{C} \setminus L)$ extends holomorphically to $\mathbb{P}^1 \setminus L$ and satisfy $\alpha(\varphi)(\infty) = 0$.*

Proof. We define a map $F : \mathbb{P}^1 \setminus L \to E'$ by $F(w) = \alpha(\varphi)(w)$ for $w \in \mathbb{C} \setminus L$ and $F(\infty) = 0$. Since $\frac{1}{2\pi i}\frac{1}{w-\cdot}x \to 0$ in ${}^{E}\mathscr{O}(L)$ as $w \to \infty$ for any $x \in E$, we have $\alpha(\varphi)(w)(x) \to 0$ as $w \to \infty$. Then it follows from [1, Theorem 3.1] that $F \in {}^{E'}\mathscr{O}(\mathbb{P}^1 \setminus L)$. $\square$

By abuse, we denote the isomorphism in Corollary 3.11 again by α, and its inverse by β.

§ 3.2. Closedness of an operator of type K in weak topologies

We denote by E_{w} the space E endowed with the weak topology. Since E is reflexive, E_{w} is isomorphic to the dual space of the barrelled space E' endowed with the weak star topology. Therefore it follows from the Banach-Steinhaus theorem that E_{w} is also sequentially complete. Refer, for example, to Schaefer-Wolff [11, III.4.6 and IV.5.6].

In this subsection, we consider an operator $P = \{P_V : {}^{E}\mathscr{O}(V + K) \to {}^{E}\mathscr{O}(V)\}$ of type $K = [a, b] \subset \mathbb{R}$ for ${}^{E}\mathscr{O}$ on $U \subset \mathbb{C}$, and we shall show that each P_V becomes sequentially closed as a map ${}^{E_{\mathrm{w}}}\mathscr{O}(V + K) \to {}^{E_{\mathrm{w}}}\mathscr{O}(V)$. Note that ${}^{E}\mathscr{O}(V) = {}^{E_{\mathrm{w}}}\mathscr{O}(V)$ as vector spaces. In fact, the inclusion ${}^{E}\mathscr{O}(V) \subset {}^{E_{\mathrm{w}}}\mathscr{O}(V)$ follows from the very definition, and the equality follows again from [1, Theorem 3.1].

For a compact set L in U, we denote by $P_L : {}^{E}\mathscr{O}(L + K) \to {}^{E}\mathscr{O}(L)$ the inductive limit of P_V in V with $L \Subset V \subset U$, and by $P_L^* : {}^{E'}\mathscr{O}^{\circ}(\mathbb{P}^1 \setminus L) \to {}^{E'}\mathscr{O}^{\circ}(\mathbb{P}^1 \setminus (L + K))$ its adjoint given by Corollary 3.11. In other words, P_L^* is a linear map satisfying $\langle P_L^*(F), f\rangle_{L_1} = \langle F, P_L(f)\rangle_{L_2}$ for any $F \in {}^{E'}\mathscr{O}^{\circ}(\mathbb{P}^1 \setminus L)$ and $f \in {}^{E}\mathscr{O}(L + K)$.

Lemma 3.13. *Consider $L \Subset V$, $F \in {}^{E'}\mathscr{O}^{\circ}(\mathbb{P}^1 \setminus L)$ and a convergent sequence $\{f_n\}_n$ in ${}^{E_{\mathrm{w}}}\mathscr{O}(V)$ with limit f. Then $\langle F, f_n\rangle_L \to \langle F, f\rangle_L$.*

Proof. We can take $L \Subset M \Subset V$ and $\gamma := \partial M$ such that $\langle F, g\rangle_L = \int_\gamma F(w)(g(w))dw$ for any $g \in {}^{E_{\rm w}}\mathscr{O}(V)$. Since $\{f_n\}$ is a convergent sequence, $\mathcal{L} := \{f_n(w);\, w \in \gamma,\, n \in \mathbb{N}\}$ is bounded in $E_{\rm w}$, and therefore bounded also in E by virtue of Mackey's theorem. The seminorm $q_{\mathcal{L}}$ on E' defined by $q_{\mathcal{L}}(y) := \sup_{x\in\mathcal{L}}|y(x)|$ is continuous and we have $C := \sup_{w\in\gamma} q_{\mathcal{L}}(F(w)) < +\infty$. Therefore $|F(w)(f_n(w))| \leq C$ for any $w \in \gamma$ and any $n \in \mathbb{N}$. On the other hand, we have $F(w)(f_n(w)) \to F(w)(f(w))$ for each fixed $w \in \gamma$ since $f_n(w) \to f(w)$ in $E_{\rm w}$. Now the conclusion follows from Lebesgue's bounded convergence theorem. □

Lemma 3.14. *Let $P = \{P_V : {}^E\mathscr{O}(V+K) \to {}^E\mathscr{O}(V)\}_{V\subset U}$ be an operator of type $K = [a, b]$ for ${}^E\mathscr{O}$ on U. Then, each $P_V : {}^{E_{\rm w}}\mathscr{O}(V+K) \to {}^{E_{\rm w}}\mathscr{O}(V)$ is sequentially closed.*

Proof. Consider a sequence $\{f_n\}_n \subset {}^E\mathscr{O}(V+K)$ convergent in ${}^{E_{\rm w}}\mathscr{O}(V+K)$ with limit $f \in {}^E\mathscr{O}(V+K)$, such that $P_V(f_n)$ converges to $g \in {}^E\mathscr{O}(V)$ in ${}^{E_{\rm w}}\mathscr{O}(V)$. We shall show that $P_V(f) = g$.

The equality $P_V(f) = g$ in ${}^E\mathscr{O}(V)$ is reduced to the equalities $P_V(f)|_L = g|_L$ for compact subsets $L \Subset V$, which can be checked by duality. Therefore, thanks to Corollary 3.11 and the definition of P_L^*, it suffices to prove that $\langle P_L^*(F), f\rangle_L = \langle F, g\rangle_L$ for any $L \Subset V$ and any $F \in {}^{E'}\mathscr{O}^\circ(\mathbb{P}^1 \setminus L)$. We get this equality from the equalities $\langle P_L^*(F), f_n\rangle_{L+K} = \langle F, P_V(f_n)\rangle_L$ for $n \in \mathbb{N}$, by applying Lemma 3.13 to the both sides. □

§ 4. Main result

We recall the notion of the sequential Montel property (M) for locally convex spaces. Refer to section 4 of [7]. (See also Zubelevich [13].)

Definition 4.1. Let E be a sequentially complete Hausdorff locally convex space. E is said to admit the Montel property if it satisfies the condition:

(M) Any bounded sequence in E has a convergent subsequence.

When E admits the Montel property, we have the following weak variant of the Montel type theorem for ${}^E\mathscr{O}_{L^\infty}(U)$.

Theorem 4.2 ([7, Theorem 4.1]). *Assume that E satisfies the Montel property* (M)*. Then for any bounded sequence $(f_j)_j$ in ${}^E\mathscr{O}_{L^\infty}(U)$, we can take a subsequence $(f_{j_k})_k$ which converges in ${}^E\mathscr{O}(U \cap \mathbb{C})$. The limit $f \in {}^E\mathscr{O}(U \cap \mathbb{C})$ of such a convergent subsequence belongs to ${}^E\mathscr{O}_{L^\infty}(U)$.*

As we mentioned in the introduction, infinite dimensional Banach spaces never admit the Montel property. But it is well-known that the weak topology of a reflexive Banach space does.

Theorem 4.3. *Let E be a reflexive Banach space. Then any bounded sequence in E has a subsequence which converges in the weak topology. In particular, E_{w} admits the Montel property* (M)*.*

Refer to Yosida [12, V. §2 Theorem 1] for the former, and recall again that a subset in E is bounded if and only if it is bounded in E_{w}.

Now we give our main result.

Theorem 4.4. *Let E be a reflexive Banach space, K a closed interval in $\mathbb{R}$, and ω a positive number. Consider an ω-periodic operator P of type K for ${}^E\mathscr{O}_{L^\infty}$ on a strip domain $\mathbb{D}^1 + i]{-d}, d[$ with some $d > 0$ and an ω-periodic E-valued hyperfunction f. The equation $Pu = f$ has an ω-periodic E-valued hyperfunction solution if and only if it has an $({}^E\mathscr{B}_{L^\infty})_{+\infty}$-solution.*

Proof. The necessity follows from Corollary 2.6, and we shall prove the sufficiency.

Assume that $Pu = f$ has an $({}^E\mathscr{B}_{L^\infty})_{+\infty}$-solution u. Then, under the notations

$$\Omega :=]a, +\infty], \quad U :=]a, +\infty] + iB_{d'}, \quad \dot{U} :=]a, +\infty] + i\dot{B}_{d'} = U \setminus \mathbb{D}^1,$$

we can take $\tilde{u} \in {}^E\mathscr{O}_{L^\infty}(\dot{U} + K)$, $\tilde{f} \in {}^E\mathscr{O}_{L^\infty}(\mathbb{D}^1 + i\dot{B}_{d'})$ satisfying $(T_\omega - 1)\tilde{f} = 0$ and $g \in {}^E\mathscr{O}_{L^\infty}(U)$ for some $a \in \mathbb{R}$ and $0 < d' < d$, such that

$$[\tilde{u}] = u \text{ on } \Omega, \quad [\tilde{f}] = f \text{ on } \mathbb{D}^1, \quad P_{\dot{U}}\tilde{u} - g = \tilde{f} \text{ on } \dot{U}.$$

In fact, we can first choose, for some choice of a and d', a local defining function $\tilde{u} \in {}^E\mathscr{O}_{L^\infty}(\dot{U} + K)$ of u by Proposition 2.3 and an ω-periodic defining function $\tilde{f} \in {}^E\mathscr{O}_{L^\infty}(\mathbb{D}^1 + i\dot{B}_{d'})$ of f by Proposition 2.5. Next, since $P_{\dot{U}}\tilde{u} - \tilde{f}$ represents 0 in $({}^E\mathscr{B}_{L^\infty})_{+\infty}$, it extends to a germ $g \in ({}^E\mathscr{O}_{L^\infty})_{+\infty}$. Finally we shrink U (i.e., increase a and decrease d') if necessary, so that g belongs to ${}^E\mathscr{O}_{L^\infty}(U)$. Here we used the commutativity of P with restrictions.

Now, in the same way as the proof of Theorem 4.3 of [7], we define

$$S_k\tilde{u} := \frac{1}{k}\sum_{j=0}^{k-1} T_{j\omega}\tilde{u}|_{\dot{U}+K} \in {}^E\mathscr{O}_{L^\infty}(\dot{U} + K), \quad S_k g := \frac{1}{k}\sum_{j=0}^{k-1} T_{j\omega} g|_U \in {}^E\mathscr{O}_{L^\infty}(U),$$

for $k \in \mathbb{N}$. It follows, again from the commutativity of P with restrictions and from the ω-periodicity of P and $\tilde{f}$, that

$$P_{\dot{U}} S_k\tilde{u} - S_k g = \tilde{f} \text{ on } \dot{U} \text{ for any } k \in \mathbb{N}. \tag{4.1}$$

Moreover, $\tilde{u} \in {}^E\mathscr{O}_{L^\infty}(\dot{U}+K)$ and $g \in {}^E\mathscr{O}_{L^\infty}(U)$ imply that $\{S_k\tilde{u}\}_{k\in\mathbb{N}} \subset {}^E\mathscr{O}_{L^\infty}(\dot{U}+K)$ and $\{S_k g\}_{k\in\mathbb{N}} \subset {}^E\mathscr{O}_{L^\infty}(U)$ are bounded.

Recall that ${}^E\mathscr{O}(V) = {}^{E_{\mathrm{w}}}\mathscr{O}(V)$ for any open $V \subset \mathbb{C}$, and that the boundedness in E and that in E_{w} coincide. These two properties imply that ${}^E\mathscr{O}_{L^\infty}(V) = {}^{E_{\mathrm{w}}}\mathscr{O}_{L^\infty}(V)$ as vector spaces for any open $V \subset \mathbb{D}^1 + i\mathbb{R}$, and that the two notions of the boundedness in both topologies coincide. Therefore, we have the inclusions $\{S_k\tilde{u}\}_{k\in\mathbb{N}} \subset {}^{E_{\mathrm{w}}}\mathscr{O}_{L^\infty}(\dot{U}+K)$ and $\{S_k g\}_{k\in\mathbb{N}} \subset {}^{E_{\mathrm{w}}}\mathscr{O}_{L^\infty}(U)$, and the left hand sides of both are bounded. Now, thanks to Theorem 4.3, we can apply Theorem 4.2 with E replaced by E_{w} to these sequences, and get a subsequence $\{k(l)\}_{l\in\mathbb{N}}$, $v \in {}^E\mathscr{O}_{L^\infty}(\dot{U}+K)$ and $h \in {}^E\mathscr{O}_{L^\infty}(U)$ such that

$$S_{k(l)}\tilde{u} \to v \text{ as } l \to \infty \text{ in } {}^{E_{\mathrm{w}}}\mathscr{O}_{L^\infty}((\dot{U}+K)\cap\mathbb{C}), \tag{4.2}$$

$$S_{k(l)}g \to h \text{ as } l \to \infty \text{ in } {}^{E_{\mathrm{w}}}\mathscr{O}_{L^\infty}(U\cap\mathbb{C}). \tag{4.3}$$

Let us show the equality $P_{\dot{U}}v - h = \tilde{f}$ in ${}^E\mathscr{O}_{L^\infty}(\dot{U})$, and the ω-periodicity of v.

For the equality above, it suffices to prove it in ${}^E\mathscr{O}_{L^\infty}(\dot{U}\cap\mathbb{C})$, since ${}^E\mathscr{O}_{L^\infty}(\dot{U}) \subset {}^E\mathscr{O}_{L^\infty}(\dot{U}\cap\mathbb{C})$. By restricting (4.1) to $\dot{U}\cap\mathbb{C}$ with $k = k(l)$, we get

$$P_{\dot{U}\cap\mathbb{C}}S_{k(l)}\tilde{u} = S_{k(l)}g + \tilde{f} \text{ for } l \in \mathbb{N}.$$

Applying Lemma 3.14 with $V = \dot{U}\cap\mathbb{C}$, the desired equality follows from (4.2) and (4.3).

In order to show the ω-periodicity of v, note that $(T_\omega - 1)S_k\tilde{u} \to 0$ as $k \to \infty$ in ${}^E\mathscr{O}(\dot{U}+K)$ since $(T_\omega - 1)S_k\tilde{u} = k^{-1}(T_{k\omega} - 1)\tilde{u}$. Therefore $(T_\omega - 1)S_k\tilde{u} \to 0$ holds also in ${}^{E_{\mathrm{w}}}\mathscr{O}(\dot{U}+K)$, and by applying Lemma 3.14 to the sequence $\{(T_\omega - 1)S_{k(l)}\tilde{u}\}_{l\in\mathbb{N}}$ for operator $T_\omega - 1$ on the set $(\dot{U}+K)\cap\mathbb{C}$, we have $(T_\omega - 1)v = 0$ on that set, and also on $\dot{U}+K$.

Due to the ω-periodicity, v has a unique ω-periodic extension in ${}^E\mathscr{O}_{L^\infty}(\mathbb{D}^1 + i\dot{B}'_d)$. Moreover h has a unique ω-periodic extension in ${}^E\mathscr{O}_{L^\infty}(\mathbb{D}^1 + iB'_d)$. In fact, since $h = P_{\dot{U}}v - \tilde{f}$ is ω-periodic on $\dot{U}$, it is ω-periodic also in U, and can be extended.

Finally note that $[v] \in {}^E\mathscr{B}_{L^\infty}(\mathbb{D}^1)$ becomes a desired ω-periodic solution. $\square$

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