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A Banach algebra and Cauchy problems with small analytic data

By

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Abstract

We announce our recent results about nonlinear Cauchy problems with small analytic data. In each problem, the nonlinear term is analytic in the derivatives of the unknown function (and their complex conjugates). By using Banach algebra techniques, it can be shown that a solution exists for a large time if the data are small. Direction of future research is discussed.

§ 1. Introduction

When a Cauchy problem has a solution locally in time, one is interested in how far it can be extended. A well-known sufficient condition for a long lifespan of the solution is smallness (defined in various ways) of the initial data. In particular, there is a significant amount of results about nonlinear wave equations or nonlinear Schrödinger equations in the $\mathcal{C}^\infty$-category. On the other hand, there are some results in the real-analytic category with different formulations of smallness. The Kirchhoff equation was studied in [1] and [2], and some m-th order equations were solved in the Gevrey class in [2]. In the present paper, we announce some results about second-order fully nonlinear Cauchy problems with small data in the real- and complex-analytic categories *without hyperbolicity assumption* in the spirit of the Cauchy-Kowalevsky theorem. The nonlinear term is an analytic function in the unknown function u, ∇u, $\nabla^2 u$, $\partial_t u$ and $\nabla \partial_t u$ (and their complex conjugates).

Some more results and complete proofs are given in [9]; in [10] the author gives the proofs without technical details and with greater emphasis on essential ideas. In the present article, we state some results with a short sketch of the proof and discuss possible directions of future research.

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§ 2. Results

Let Ω be an open set of $\mathbb{R}^n_x$, $x = (x_1, \ldots, x_n)$. A smooth function $\varphi(x)$ on Ω is said to be *uniformly analytic* on Ω if there exists $C > 0$ such that for any multi-index $\alpha \in \mathbb{N}^n = \{0, 1, 2, \ldots\}^n$ one has

$$\sup_{x\in\Omega} |\partial^\alpha \varphi(x)| \le C^{|\alpha|+1}|\alpha|!,$$

where $\partial^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}$. The totality of uniformly analytic functions on Ω is denoted by $A(\Omega)$.

For $T > 0$, let I_T be the open interval $]-T, T[$ and set $\Omega_T = I_T \times \Omega \subset \mathbb{R}_t \times \mathbb{R}^n_x$. For $k \in \mathbb{N}$, we say that a continuous function $u(t, x)$ on Ω_T belongs to $\mathcal{C}^k(T; A(\Omega))$ if it satisfies the following two conditions:

(i) $\partial_t^j \partial^\alpha u$ exists and is continuous in Ω_T for any $j \in \{0, \ldots, k\}$ and any $\alpha \in \mathbb{N}^n$.

(ii) For any $T' \in]0, T[$, there exists a positive constant $C = C_{T'}$ such that for any $j \in \{0, \ldots, k\}$ and any $\alpha \in \mathbb{N}^n$, we have $\sup_{|t|\le T', x\in\Omega} |\partial_t^j \partial^\alpha u(t, x)| \le C^{|\alpha|+1}|\alpha|!$.

Notice that the estimate in (ii) is uniform in x but is only locally uniform in t. This formulation has been chosen so that the Banach algebra defined below is a subspace of $\mathcal{C}^k(T; A(\Omega))$.

Let $P(\partial_t, \partial_x) = \sum_{j=1}^n p_j \partial_t \partial_j + \sum_{k=1}^n \sum_{j=1}^k p_{jk} \partial_j \partial_k$ be a second-order linear partial differential operator with constant coefficients, where $\partial_j = \partial/\partial x_j$ and $p_j, p_{jk} \in \mathbb{C}$. We consider the following fully nonlinear Cauchy problem:

$$(\mathrm{CP1}) \begin{cases} \left(\partial_t^2 - P(\partial_t, \partial_x)\right) u = f_1(t; u;\ \partial_t u, \nabla u;\ \nabla \partial_t u, \nabla^2 u), \\ u(0, x) = \varphi(x),\ \partial_t u(0, x) = \psi(x), \end{cases}$$

where $\partial_t = \partial/\partial t, \nabla u = (\partial_j u)_{1\le j\le n}$ and $\nabla^2 u = (\partial_j \partial_k u)_{1\le j\le k\le n}$. Here $\varphi(x)$ and $\psi(x)$ are uniformly analytic in an open subset Ω of $\mathbb{R}^n$. We assume that $f_1(t; X; Y; Z)$ is a bounded continuous function on $\mathbb{R}_t \times \mathcal{U}$, where $\mathcal{U}$ is an open neighborhood of $(X, Y, Z) = 0 \in \mathbb{C} \times \mathbb{C}^{n+1} \times \mathbb{C}^N$, $N = n(n+3)/2$. Moreover we assume that it is complex-analytic in $\mathcal{U}$ for each fixed $t \in \mathbb{R}$ and has an expansion of the form

$$f_1(t; X; Y; Z) = \sum_{L\ge 4} a_{\alpha\beta\gamma}(t) X^\alpha Y^\beta Z^\gamma, \quad L = \alpha + 2|\beta| + 3|\gamma|. \tag{1}$$

We shall study the lifespan of a solution when the data are small in a certain sense.

Theorem 2.1. *There exist $\delta > 0$ and $\varepsilon_0 > 0$ such that the following holds for all ε with $0 < \varepsilon \le \varepsilon_0$:*

If $\sup_{x\in\Omega} |\partial^\alpha \varphi| \le \varepsilon^{|\alpha|+1}|\alpha|!$ and $\sup_{x\in\Omega} |\partial^\alpha \psi| \le \varepsilon^{|\alpha|+2}|\alpha|!$ for all $\alpha \in \mathbb{N}^n$, then (CP1) *has a solution $u(t, x) \in \mathcal{C}^2(T; A(\Omega))$ for $T = \delta/\varepsilon$.*

Remark. The order ε^{-1} is the best possible result even in the case of the $1+1$ dimensional linear wave equation on a finite interval. The solution of $(\partial_t^2 - \partial_x^2)u = 0$, $\varphi(x) = \varepsilon/(1+\varepsilon x)$, $\psi(x) = \varepsilon^2/(1+\varepsilon x)^2$ is $u(t,x) = \varepsilon/(1-\varepsilon t+\varepsilon x)$. If Ω is a relatively compact open subinterval of $\{-r < x < r\}$, u belongs to $\mathcal{C}^2(-r+1/\varepsilon; A(\Omega))$ for $0 < \varepsilon < 1/r$. Another illuminating example is a Cauchy problem for a nonlinear ODE. The solution of $d^2u/dt^2 = 6u^2$, $u(0) = \varepsilon^2$, $u'(0) = 2\varepsilon^3$ is $u(t) = \varepsilon^2/(1-\varepsilon t)^2$.

Let $\varphi(x)$ and $\psi(x)$ be *complex*-analytic functions on an open set U of $\mathbb{C}_x^n$. We assume that f_1 is independent of t (a bounded entire function in t). Set $B_T = \{t \in \mathbb{C}; |t| < T\}$ for $T > 0$. Then we can formulate the complex version of (CP1) and refer to it as (CP1c).

Theorem 2.2. *There exist $\delta > 0$ and $\varepsilon_0 > 0$ such that the following holds for all ε with $0 < \varepsilon \leq \varepsilon_0$:*

If $\sup_{x\in U} |\partial^\alpha \varphi| \leq \varepsilon^{|\alpha|+1}|\alpha|!$ and $\sup_{x\in U} |\partial^\alpha \psi| \leq \varepsilon^{|\alpha|+2}|\alpha|!$ for all $\alpha \in \mathbb{N}^n$, then (CP1c) *has a unique solution $u(t,x)$ which is complex-analytic on $B_T \times U$ for $T = \delta/\varepsilon$ and satisfies the following estimate: for all T' with $0 < T' < T = \delta/\varepsilon$, there exists $C = C_{T'} > 0$ such that*

$$\sup_{|t|\leq T', x\in U} |\partial^\alpha u(t,x)| \leq C^{|\alpha|+1}|\alpha|!$$

holds for any $\alpha \in \mathbb{N}^n$.

We can relax the condition on ψ in exchange for posing stronger conditions on P and the nonlinear term. Let us assume that $P = P(\partial_x) = \sum_{k=1}^n \sum_{j=1}^k p_{jk}\partial_j\partial_k$ is free from ∂_t and consider

$$\text{(CP2)} \begin{cases} \big(\partial_t^2 - P(\partial_x)\big)u = f_2(t, u, \partial_t u, \nabla u, \nabla\partial_t u, \nabla^2 u), \\ u(0,x) = \varphi(x),\ \partial_t u(0,x) = \psi(x), \end{cases}$$

where $f_2(t, X, Y, Z, \Theta, \Xi)$ is a bounded continuous function on $\mathbb{R}_t \times \mathcal{V}$, where $\mathcal{V}$ is an open neighborhood of $(X, Y, Z, \Theta, \Xi) = 0 \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^{n(n+1)/2}$. Moreover we assume that it is complex-analytic in $\mathcal{V}$ for each fixed $t \in \mathbb{R}$ and is expanded as follows:

$$(2) \qquad \begin{aligned} f_2(t, X, Y, Z, \Theta, \Xi) &= \sum_{L_1\geq 2, L_2\geq 2} a_{\alpha\beta\gamma\lambda\mu}(t) X^\alpha Y^\beta Z^\gamma \Theta^\lambda \Xi^\mu, \\ L_1 = \alpha + |\gamma| + |\mu|,&\ L_2 = \beta + |\gamma| + 2|\lambda| + 2|\mu|. \end{aligned}$$

Notice that the combination of $L_1 \geq 2$ and $L_2 \geq 2$ implies $L \geq 4$.

Theorem 2.3. *There exist $\delta > 0$ and $\varepsilon_0 > 0$ such that the following holds for all ε with $0 < \varepsilon \leq \varepsilon_0$:*

If $\sup_{x\in\Omega} |\partial^\alpha \varphi| \le \varepsilon^{|\alpha|+1}|\alpha|!$ *and* $\sup_{x\in\Omega} |\partial^\alpha \psi| \le \varepsilon^{|\alpha|+1}|\alpha|!$ *for all* $\alpha \in \mathbb{N}^n$*, then* (CP2) *has a solution* $u(t,x) \in \mathcal{C}^2(T; A(\Omega))$ *for* $T = \delta/\varepsilon$.

Remark 2.4. *The complex-analytic version of Theorem 2.3 can be formulated in an obvious way. We omit the details.*

We can generalize Theorems 2.1 and 2.3 so that the nonlinear terms involve the complex conjugates of the derivatives of the unknown function. Here we state the generalization of Theorem 2.3. Let us consider

$$(\mathrm{CP3})\begin{cases} \big(\partial_t^2 - P(\partial_x)\big)u \\ \qquad = f_3(t; u, \bar{u};\, \partial_t u, \partial_t \bar{u};\, \nabla u, \nabla \bar{u};\, \nabla\partial_t u, \nabla\partial_t \bar{u};\, \nabla^2 u, \nabla^2 \bar{u}), \\ u(0,x) = \varphi(x),\ \partial_t u(0,x) = \psi(x), \end{cases}$$

where $f_3(t; \tilde{X}; \tilde{Y}; \tilde{Z}; \tilde{\Theta}; \tilde{\Xi})$ is a bounded continuous function on $\mathbb{R}_t \times \widetilde{\mathcal{V}}$, where $\widetilde{\mathcal{V}}$ is an open neighborhood of $(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{\Theta}, \tilde{\Xi}) = 0 \in \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^{2n} \times \mathbb{C}^{2n} \times \mathbb{C}^{n(n+1)}$. Moreover we assume that it is complex-analytic in $\widetilde{\mathcal{V}}$ for each fixed $t \in \mathbb{R}$ and is expanded in the form

$$\begin{aligned} f_3(t; \tilde{X}; \tilde{Y}; \tilde{Z}; \tilde{\Theta}; \tilde{\Xi}) &= \sum_{\tilde{L}_1 \ge 2, \tilde{L}_2 \ge 2} a_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\lambda}\tilde{\mu}}(t) \tilde{X}^{\tilde{\alpha}} \tilde{Y}^{\tilde{\beta}} \tilde{Z}^{\tilde{\gamma}} \tilde{\Theta}^{\tilde{\lambda}} \tilde{\Xi}^{\tilde{\mu}}, \\ \tilde{L}_1 = |\tilde{\alpha}| + |\tilde{\gamma}| + |\tilde{\mu}|,\ \tilde{L}_2 &= |\tilde{\beta}| + |\tilde{\gamma}| + 2|\tilde{\lambda}| + 2|\tilde{\mu}|. \end{aligned}$$

Theorem 2.5. *There exist* $\delta > 0$ *and* $\varepsilon_0 > 0$ *such that the following holds for all* ε *with* $0 < \varepsilon \le \varepsilon_0$*:*

If $\sup_{x\in\Omega} |\partial^\alpha \varphi| \le \varepsilon^{|\alpha|+1}|\alpha|!$ *and* $\sup_{x\in\Omega} |\partial^\alpha \psi| \le \varepsilon^{|\alpha|+1}|\alpha|!$ *for all* $\alpha \in \mathbb{N}^n$*, then* (CP3) *has a solution* $u(t,x) \in \mathcal{C}^2(T; A(\Omega))$ *for* $T = \delta/\varepsilon$.

Example 2.6. The theorem above can be applied to nonlinearities such as $|\nabla u|^2 = \sum_{j=1}^n \partial_j u\, \partial_j \bar{u}$.

We can deal with operators with first-order terms. (As will be clear later, our present result leaves some room for improvement.) Let $P'(\partial_x) = \sum_{j=1}^n p'_j \partial_j \ \ (p'_j \in \mathbb{C})$ be a vector field. Let us consider

$$(\mathrm{CP4})\begin{cases} \big(\partial_t^2 - P(\partial_t, \partial_x) - P'(\partial_x)\big)u = f_4(t, u, \partial_t u, \nabla u, \nabla\partial_t u, \nabla^2 u), \\ u(0,x) = \varphi(x),\ \partial_t u(0,x) = \psi(x), \end{cases}$$

where $f_4(t, X, Y, Z, \Theta, \Xi)$ is a bounded continuous function on $\mathbb{R}_t \times \mathcal{V}$, where $\mathcal{V}$ is as in (CP2). Moreover we assume that it is complex-analytic in $\mathcal{V}$ for each fixed $t \in \mathbb{R}$ and is expanded in the form

$$f_4(t, X, Y, Z, \Theta, \Xi) = \sum_{\ell \geq 5/2} a_{\alpha\beta\gamma\lambda\mu}(t) X^\alpha Y^\beta Z^\gamma \Theta^\lambda \Xi^\mu,$$
$$\ell = \alpha + \frac{3}{2}\beta + 2|\gamma| + \frac{5}{2}|\lambda| + \frac{5}{2}|\mu|.$$

Theorem 2.7. *There exist $\delta > 0$ and $\varepsilon_0 > 0$ such that the following holds for all ε with $0 < \varepsilon \leq \varepsilon_0$:*

If $\sup_{x\in\Omega} |\partial^\alpha \varphi| \leq \varepsilon^{|\alpha|+1}|\alpha|!$ and $\sup_{x\in\Omega} |\partial^\alpha \psi| \leq \varepsilon^{|\alpha|+3/2}|\alpha|!$ for all $\alpha \in \mathbb{N}^n$, then (CP4) *has a solution $u(t, x) \in \mathcal{C}^2(T; A(\Omega))$ for $T = \delta/\sqrt{\varepsilon}$.*

§ 3. The Banach algebra $\mathcal{G}_{T,\zeta}(\Omega)$

We recall some known facts about a Banach algebra defined in terms of a power series. Set $\theta(X) = K^{-1} \sum_{k=0}^\infty X^k/(k + 1)^2$, $K = 4\pi^2/3$ (a series due to Peter Lax) and let $D^j\theta(X)$ be its j-th derivative. For $\zeta > 0$, a continuous function $u(t, x)$ on Ω_T is an element of $\mathcal{G}_{T,\zeta}(\Omega)$ if it is infinitely differentiable in x and there exists an constant $C > 0$ such that for any $\alpha \in \mathbb{N}^n$ and any $t \in I_T$ one has

$$\sup_{x\in\Omega} |\partial^\alpha u(t, x)| \leq C\zeta^{|\alpha|} D^{|\alpha|}\theta(|t|/T). \tag{3}$$

Let the norm $\|u\|$ be the infimum of such C's. Then $\mathcal{G}_{T,\zeta}(\Omega)$ is a Banach algebra and is a subspace of $\mathcal{C}^0(T; A(\Omega))$.

Proposition 3.8. *Set $\partial_t^{-1}u(t, x) = \int_0^t u(s, x)ds$. For all $(k, \alpha) \in (-\mathbb{N}) \times \mathbb{N}^n$ with $k + |\alpha| \leq 0$, there exists a constant $C_{k,|\alpha|} > 0$ such that $\partial_t^k \partial^\alpha$ is an endomorphism of the Banach space $\mathcal{G}_{T,\zeta}(\Omega)$ and its norm is not larger than $C_{k,|\alpha|}T^{-k}\zeta^{|\alpha|}$.*

For proofs of the facts in this section, see [11], [2] or [8].

§ 4. Sketch of the Proof of Theorem 2.1

We sketch the proof of Theorem 2.1. The other theorems are shown basically in the same way.

Set $w(t, x) = \partial_t^2 u(t, x)$, then $u = \partial_t^{-2}w + \varphi + t\psi$. We define Q and $\mathcal{L}_1$ by

$$Qu = (u; \partial_t u, \nabla u; \nabla\partial_t u, \nabla^2 u),$$
$$\mathcal{L}(w) = P(\partial_t^{-2}w + \varphi + t\psi) + f_1\big(t; Q(\partial_t^{-2}w + \varphi + t\psi)\big).$$

Then (CP1) is equivalent to $w = \mathcal{L}(w)$ (i.e. w is a fixed point of $\mathcal{L}$). We have only to find $w \in \mathcal{G}_{T,\zeta}(\Omega) \subset \mathcal{C}^0(T; A(\Omega))$ for $T = \delta/\varepsilon, \zeta = 2e^2\varepsilon$, because $u = \partial_t^{-2}w + \varphi + t\psi$

becomes an element of $\mathcal{C}^2(T; A(\Omega))$. By using Proposition 3.8, we can show that $\mathcal{L}$ is a contraction from a (carefully chosen) closed ball of $\mathcal{G}_{T,\zeta}(\Omega)$ to itself. Then it has a unique fixed point w in the ball.

§ 5. Further developments

The following problems with small initial data could be studied by using similar techniques,

1. Third or higher order equations.
2. Fuchsian equations.
3. Systems.
4. Problems in the Gevrey classes.

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