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Singularity propagation of compressible perfect fluid in a complex domain

By

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Abstract

This is a report of an approach to the following problem: Considering a compressible perfect fluid, how do the singularities propagate in a complex domain?

§ 1. Introduction

We consider a 2-D compressible perfect fluid in a complex domain. Let $x = (x_0, x_1, x_2) \in \mathbf{C}^3$. We denote the velocity of the fluid by $u(x) = (u_1(x), u_2(x))$ and the density by $\rho(x)$. We assume that the pressure p is determined by the density ρ due to some physical law. Therefore we assume $p = p(\rho(x))$. Denoting $Lf(x) = \partial_{x_0} f + u_1 \partial_{x_1} f + u_2 \partial_{x_2} f$, we consider the following Cauchy problem for the Euler system:

$$
(1.1) \qquad \begin{cases} L\rho &= -\rho \operatorname{div} u, & \rho(0, x_1, x_2) &= \rho^0(x_1, x_2), \\ Lu_1 &= -\dfrac{p'(\rho)}{\rho} \partial_{x_1} \rho, & u_1(0, x_1, x_2) &= u_1^0(x_1, x_2), \\ Lu_2 &= -\dfrac{p'(\rho)}{\rho} \partial_{x_2} \rho, & u_2(0, x_1, x_2) &= u_2^0(x_1, x_2). \end{cases}
$$

Let $\omega \subset \mathbf{C}^3$ be a small neighborhood of the origin, and let

$$
\begin{aligned} \omega^0 &= \omega \cap \{x_0 = 0\} \subset \omega , \\ Z &= \omega^0 \cap \{x_1 = 0\} \subset \omega^0. \end{aligned}
$$

We assume that the initial values ρ^0, u_1^0, u_2^0 are holomorphic on the universal covering space $\mathcal{R}(\omega^0 \setminus Z)$ of $\omega^0 \setminus Z$, and they have some regularity up to Z.

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Precisely speaking, we define

$$\begin{aligned}
\mathcal{O}(\mathcal{R}(\omega^0 \setminus Z)) &= \{\text{holomorphic functions on } \mathcal{R}(\omega^0 \setminus Z)\},\\
\mathcal{O}^{j-q}(\mathcal{R}(\omega^0 \setminus Z)) &= \{f(x) \in \mathcal{O}(\mathcal{R}(\omega^0 \setminus Z));\\
&\qquad \partial_x^\alpha f(x) \text{ is bounded if } |\alpha| \le j-1,\\
&\qquad x_1^q \partial_x^\alpha f(x) \text{ is bounded if } |\alpha| = j\}
\end{aligned}$$

for $j \in \mathbf{Z}_+,\ 0 < q < 1$. We remark that if $f(x_1, x_2) \in \mathcal{O}^{1-q}(\mathcal{R}(\omega^0 \setminus Z))$, then we can naturally define the trace $f(0, x_2)$ on Z. We assume the following:

A1 We have $\rho^0, u_1^0, u_2^0 \in \mathcal{O}^{3-q}(\mathcal{R}(\omega^0 \setminus Z))$.

A2 $p(\rho)$ is holomorphic near $\rho = \rho(0)$, and we have

$$\rho(0)p'(\rho(0)) \neq 0.$$

We want to show that the singularities of the solution to (1.1) appear on three complex hypersurfaces $Z_-, Z_0, Z_+ \subset \omega$, which start from Z at $x_0 = 0$. The propagation velocity of Z_0 coincides with the fluid velocity, and those of $Z_\pm$ coincide with (fluid velocity) $\pm$ (sonic velocity).

Remark. (a) J.-M. Delort [3] and the author [4] studied the case of an incompressible perfect fluid, and proved that the singularities are contained in Z_0, propagating at the fluid velocity.

(b) The author [5] considered the case for irrotational perfect fluid (i.e. the case of $\operatorname{rot} u = 0$), and proved that the singularities are contained in $Z_+ \cup Z_-$, propagating with the sonic velocity $\sqrt{p'(\rho)}$ forwards and backwards (relative to the fluid movement).

(c) J.-Y. Chemin [1] considered a compressible perfect fluid in a real space, and proved that if the initial values are smooth outside of the origin, then the singularities of the solution propagates on $Y \cup Y'$. Here Y is the orbit of the fluid issuing from the origin at the initial time, and Y' is a cone spreading around Y at the sonic velocity.

Remark. From (1.1) we can naturally define $\partial_{x_0}^i u_j(0, x_1, x_2),\ \partial_{x_0}^i \rho(0, x_1, x_2) \in \mathcal{O}^{3-i-q}(\mathcal{R}(\omega^0 \setminus Z))$ for $0 \le i \le 2$.

§ 2. Characteristics

Let $\nabla f(x) = (\partial^\alpha f(x); |\alpha| \le j)$. We denote $v(x) = (v_0, v_1, v_2) = (\rho, u_1, u_2)$. From (1.1) we have

$$\begin{cases} L(L^2 - p'(v_0)\Delta)v = Q(\nabla^2 v), \\ \partial_{x_0}^i v(0, x_1, x_2) = v^i(x_1, x_2), \qquad 0 \le i \le 2. \end{cases} \tag{2.1}$$

Here $Q = (Q_0, Q_1, Q_2)$ is determined by $\nabla^2 v$, and $v^i(x_1, x_2) = \partial^i_{x_0} v(0, x_1, x_2)$ is naturally determined by (1.1). The principal symbol of the above equation is $L(L^2 - p'(v_0)\Delta)$, and we define three characteristic functions $\varphi_i(x)$ for $i = -1, 0, 1$ by the following three characteristic equations:

$$
\tag{2.2}
\begin{cases}
L\varphi_i(x) = i\sqrt{p'(v_0)\{(\partial_{x_1}\varphi_i(x))^2 + (\partial_{x_2}\varphi_i(x))^2\}}, \\
\varphi_i(0, x_1, x_2) = x_1.
\end{cases}
$$

We define $Z_i = \{x \in \omega;\ \varphi_i(x) = 0\}$, and sometimes also denote as $Z_\pm = Z_{\pm 1}$. It follows that Z_0 moves at the fluid velocity. To the contrary, $L^2 - p'(v_0)\Delta$ is the wave operator of the sound, and thus the propagation velocities of $Z_\pm$ coincide with the sonic velocity.

Remark. It is well-known that the characteristic variety of the Euler system is $Z_- \cup Z_0 \cup Z_+$ (c.f. [2]).

Remark. Unfortunately, we cannot determine $\varphi_i(x) = 0$ until we know the solution v. On the other hand, the solution should be singular along $Z_- \cup Z_0 \cup Z_+$, and we cannot determine the domain of definition for v, until we know the characteristic function φ_i. To avoid such an circular reasoning, we shall use the characteristic coordinate system below.

At first, we make the following approximation. We have $\partial_{x_2}\varphi_i(0, x_1, x_2) = 0$, and thus the solution should satisfy

$$
\partial_{x_2}\varphi_i,\ v(x) - v(0) = \mathrm{O}(|x|).
$$

It follows that

$$
\begin{aligned}
L\varphi_i &\sim L'\varphi_i \stackrel{\text{def}}{=} (\partial_{x_0} + v_1(0)\partial_{x_1})\varphi_i, \\
\sqrt{p'(v_0)} &\sim a_0 \ \ \stackrel{\text{def}}{=} \sqrt{p'(v_0(0))} \\
&\qquad\quad \neq 0.
\end{aligned}
$$

We consider the following approximate characteristic equation:

$$
\tag{2.3}
\partial_{x_0}\varphi'_i(x) + v_1(0)\partial_{x_1}\varphi'_i(x) = ia_0\partial_{x_1}\varphi'_i(x).
$$

The solution is $\varphi'_i(x) = x_1 + (-v_1(0) + ia_0)x_0$. We introduce new complex coordinate system:

$$
\begin{cases}
y_0 = x_0, \\
y_1 = \varphi'_i(x)/a_0 = a_0^{-1}x_1 - v_1(0)x_0, \\
y_2 = x_2.
\end{cases}
$$

Then we can expect

$$Z_i = \{\varphi_i = 0\} \subset Z_i' = \{|y_1 + iy_0| \leq R|y_0|\}.$$

Our purpose is the following:

(a) We want to solve the Euler system at least on the universal covering space $\mathcal{R}(X)$ of $X = C^3 \setminus Z'_{-1} \setminus Z'_0 \setminus Z'_1 = \{|y_1 + iy_0| > R|y_0| \ (-1 \leq i \leq 1)\}$ (near the origin).

(b) For each $i_0 \in \{-1, 0, 1\}$ and for each $\tilde{x} \in \mathcal{R}(X)$ near Z'_{i_0}, we want to determine the characteristic hypersurface $Z_{i_0} \subset Z'_{i_0}$, and solve the equation outside of Z_{i_0}.

In the rest of this report, we explain an approach as follows: We fix an arbitrary number $i_0 \in \{-1, 0, 1\}$. We want to consider another approximation which makes no errors according to the singularities on Z_{i_0}. Afterwards, we prepare a discussion about the universal covering space, which will be necessary to study many branches of the solution. For the sake of simplicity, from now on we explain our approach for the case $i_0 = 0$.

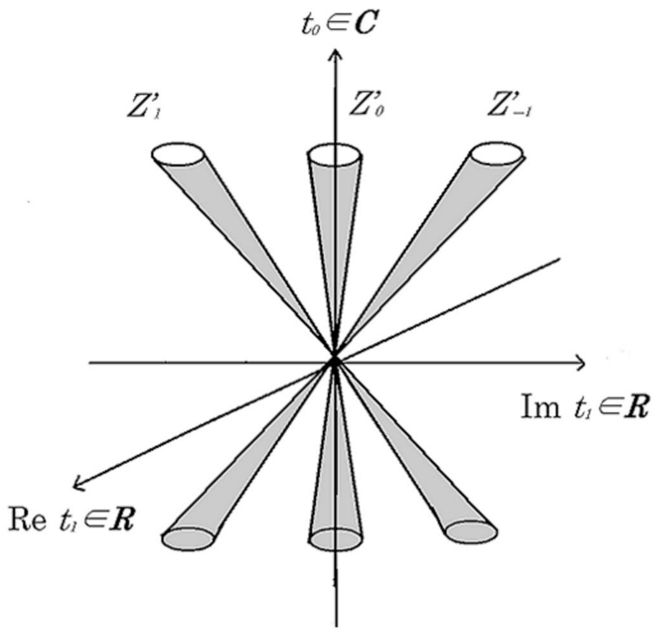

Figure 1. Domain X

§ 3. An approach of approximation

We introduce another coordinate system:

$$\begin{cases} t_0 = x_0, \\ t_1 = \varphi_0/a_0, \\ t_2 = x_2. \end{cases}$$

Then we have

$$\begin{cases} \partial_{x_0} = \partial_{t_0} + \partial_{x_0}\varphi_0 \cdot \partial_{t_1}, \\ \partial_{x_1} = \partial_{x_1}\varphi_0 \cdot \partial_{t_1}, \\ \partial_{x_2} = \partial_{t_2} + \partial_{x_2}\varphi_0 \cdot \partial_{t_1}, \end{cases} \tag{3.1}$$

Here $\varphi_0(x)$ is an unknown function, but if it exists, then we should have

$$\frac{\partial t}{\partial x} = \begin{pmatrix} 1 & 0 & 0 \\ \partial_{x_0}\varphi_0 & \partial_{x_1}\varphi_0 & \partial_{x_2}\varphi_0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since $\partial_{x_1}\varphi_0 \sim 1$, this matrix is invertible and we have

$$\frac{\partial x}{\partial t} = \frac{1}{\partial_{x_1}\varphi_0} \begin{pmatrix} \partial_{x_1}\varphi_0 & 0 & 0 \\ -\partial_{x_0}\varphi_0 & 1 & -\partial_{x_2}\varphi_0 \\ 0 & 0 & \partial_{x_1}\varphi_0 \end{pmatrix}.$$

If we can express x as a function $x(t)$ of t, then we should have

$$\begin{cases} \partial_{t_0}x_1 = -\partial_{x_0}\varphi_0/\partial_{x_1}\varphi_0, \\ \partial_{t_1}x_1 = 1/\partial_{x_1}\varphi_0, \\ \partial_{t_2}x_1 = -\partial_{x_2}\varphi_0/\partial_{x_1}\varphi_0. \end{cases}$$

Therefore we have

$$\begin{cases} \partial_{x_0}\varphi_0 = -\partial_{t_0}x_1/\partial_{t_1}x_1, \\ \partial_{x_1}\varphi_0 = 1/\partial_{t_1}x_1, \\ \partial_{x_2}\varphi_0 = -\partial_{t_2}x_1/\partial_{t_1}x_1. \end{cases}$$

This means that we should rewrite (2.2) (for $i = 0$) by

$$\begin{cases} -\partial_{t_0}x_1 + v_1 - v_2\partial_{t_2}x_1 = 0, \\ x_1(0, t_1, t_2) = t_1. \end{cases}$$

From (2.2), (3.1) we have

$$\begin{aligned} L &= \partial_{t_0} + v_2\partial_{t_2} + (\partial_{x_0}\varphi_0 + v_1\partial_{x_1}\varphi_0 + v_2\partial_{x_2}\varphi_0)\partial_{t_1} \\ &= \partial_{t_0} + v_2\partial_{t_2}, \end{aligned}$$

By a direct calculation we can prove

$$\begin{aligned} L(L^2 - p'(v_0)\Delta)f = \partial_{t_0}(\partial_{t_0}^2 - \partial_{t_1}^2)f \\ + \sum_{|\alpha|=3}(A_\alpha\partial_t^\alpha f + B_\alpha\partial_t^\alpha x_1) + C. \end{aligned}$$

Here we can neglect $A_\alpha,\ B_\alpha,\ C$ in the following sense: A_α is determined by $v(t), x_1(t)$ and the first order derivatives of $x_1(t)$:

$$A_\alpha = A_\alpha(v, x_1) \quad = A_\alpha(v(t), \nabla_t^1 x_1(t)).$$

B_α is determined by $v(t), x_1(t), f(t)$, and the first order derivatives of $x_1(t)$:

$$B_\alpha = B_\alpha(v, x_1, f) = B_\alpha(v(t), \nabla_t^1 x_1(t), f(t)).$$

C is determine by $v(t), x_1(t), f(t)$, and their derivatives of order at most 2:

$$C \ = C(v, x_1, f) \quad = C(\nabla_t^2 v(t), \nabla_t^2 x_1(t), \nabla_t^2 f(t)).$$

Furthermore, we have

$$\begin{cases} A_\alpha = B_\alpha = 0, \alpha = (3,0,0) \text{ or } (0,3,0), \\ [A_\alpha]_{x=0} = 0, \quad \alpha = (2,1,0) \text{ or } (1,2,0) \end{cases} \tag{3.2}$$

This means the following. We can regard t_2 as a holomorphic parameter, and it does not affect the singularity propagation. In this sense we can say

$$L(L^2 - p'(v_0)\Delta)f \sim \partial_{t_0}(\partial_{t_0}^2 - \partial_{t_1}^2)f + \sum_{\substack{|\alpha|=3 \\ \alpha_2=0}}(A_\alpha\partial_t^\alpha f + B_\alpha\partial_t^\alpha x_1)$$

without affecting the singularity propagation. Sometimes $x_1(t)$ is more temperate than $f(t)$, and from (3.2) we can say

$$L(L^2 - p'(v_0)\Delta)f \sim \partial_{t_0}(\partial_{t_0}^2 - \partial_{t_1}^2)f$$

without affecting the singularity propagation along $\{t_1 = 0\}$ at all, and along other singularity sets too much. Let $\Lambda_j = \partial_{t_0} - j\partial_{t_1}$ for $j = -1, 0, 1$. Therefore we have

$\Lambda_{-1}\Lambda_0\Lambda_1 = \partial_{t_0}(\partial_{t_0}^2 - \partial_{t_1}^2)$. We rewrite (2.1) in the following form:

$$
\begin{cases}
\Lambda_{-1}\Lambda_0\Lambda_1 v_j & = -\sum\limits_{|\alpha|=3} A_\alpha(v, x_1)\partial_t^\alpha v_j \\
& \qquad + B_\alpha(v, x_1, v_j)\partial_t^\alpha x_1 - C(v, x_1, v_j), \\
\Lambda_0 x_1 & = v_1 - v_2 \partial_{t_2} x_1, \\
\partial_{x_0}^i v(0, x_1, x_2) & = v^i(x_1, x_2), \qquad 0 \le i \le 2 \\
x_1(0, t_1, t_2) & = t_1.
\end{cases} \tag{3.3}
$$

We regard the right hand side as an error term, and solve the equation by iteration.

Remark. In (3.3) we can replace the initial condition for v by

$$
\begin{aligned}
v(0, t_1, t_2) &= v^0(t_1, t_2), \\
\Lambda_{-1} v(0, t_1, t_2) &= v^1(t_1, t_2), \\
\Lambda_1 \Lambda_{-1} v(0, t_1, t_2) &= v^2(t_1, t_2)
\end{aligned}
$$

changing v^1 and v^2 if necessary.

§ 4. An approach from the geometry

We define

$$Y = \{|t_1 \pm t_0| > R|t_0|/2,\ t_1 = 0\}.$$

We remind the reader that

$$
\begin{aligned}
&X = \{|y_1 + iy_0| > R|y_0| \ (-1 \le i \le 1)\}, \\
&y = (x_0, \varphi_0'/a_0, x_2), \\
&t = (x_0, \varphi_0/a_0, x_2).
\end{aligned}
$$

From (2.2) and (2.3) we have $\varphi_0 - \varphi_0' = O(|x|^2)$. Therefore we can expect

$$X \cap \omega \subset Y \cap \omega,$$

if $\omega \subset \mathbf{C}^3$ is a small neighborhood of the origin. Therefore we want to solve the Cauchy problem on $\mathcal{R}(Y)$, instead of $\mathcal{R}(X)$. Furthermore, Y is the complement of two approximate singularity sets $\{|t_1 \pm t_0| \le R|t_0|/2\}$ and one true singularity set $\{t_1 = 0\}$.

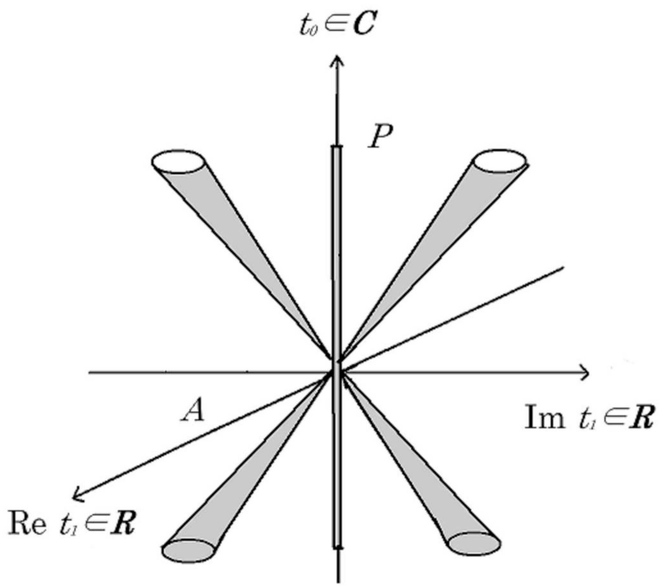

Figure 2. Domain Y

A point $\tilde{t}^* \in \mathcal{R}(Y)$ corresponds to the homotopy class $[\gamma]$ in Y of a continuous curve $\gamma \subset Y$ from a base point $A(0, \varepsilon, 0)$ to $P(t_0^*, t_1^*, t_2^*)$.

We can show that an arbitrary curve γ from A to P is homptopically equivalent to $\gamma_0 + \gamma'$ in Y. Here γ_0 is a curve in the initial hyperplane $\{t \in Y;\ t_0 = 0\}$ from $A(0, \varepsilon, 0)$ to $P'(0, t_1^*, t_2^*)$, and γ_1 is a curve in $\{t \in X;\ t_1 = t_1^*, t_2 = t_2^*\}$ from $P'(0, t_1^*, t_2^*)$ to $P(t_0^*, t_1^*, t_2^*)$.

In order to calculate the value of the solution at $\tilde{t}^*$, we need to continue the solution along $\gamma \sim \gamma_0 + \gamma'$, from A to P. But the value of the solution is given along γ_0 from the beginning as the initial value. Therefore we need to consider how we can continue the solution along γ', from P' to P.

As before, we denote $\Lambda = \Lambda_{-1}\Lambda_0\Lambda_1$, where $\Lambda_j = \partial_{t_0} - j\partial_{t_1}$ for $j = -1, 0, 1$. Therefore we have $\Lambda = \partial_{t_0}(\partial_{t_0}^2 - \partial_{t_1}^2)$. W consider some given functions $f(t) = (f_1(t), f_2(t))$

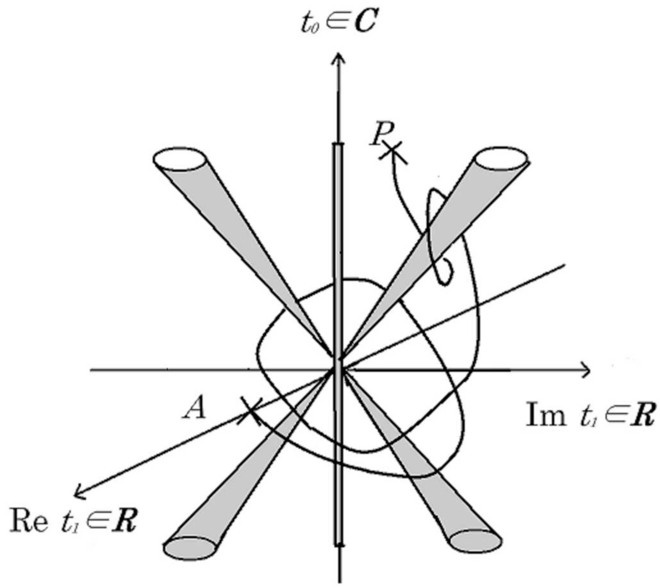

Figure 3. path γ

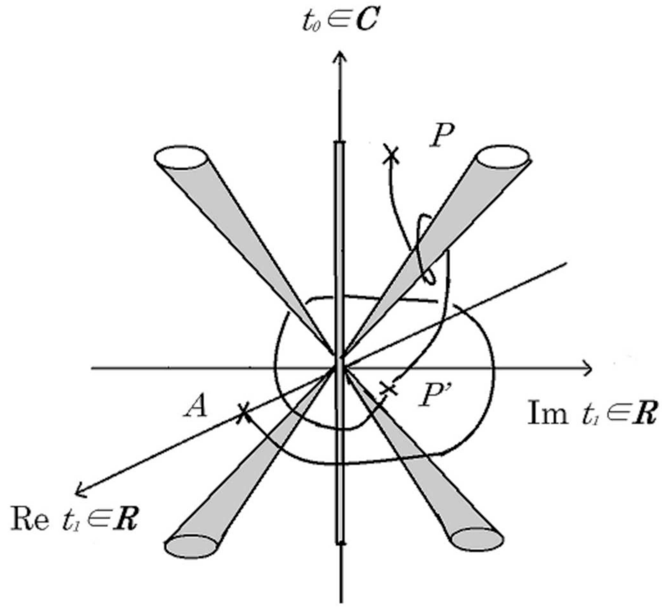

Figure 4. path $\gamma_0 + \gamma'$

and $g(t)$ on $\mathcal{R}(Y \cap \omega)$, and consider the following Cauchy problem there:

$$
\begin{cases}
\Lambda v & = f, \\
\Lambda_0 x_1 & = g, \\
v(0, t_1, t_2) & = v^0(t_1, t_2), \\
\Lambda_{-1} v(0, t_1, t_2) & = v^1(t_1, t_2), \\
\Lambda_1 \Lambda_{-1} v(0, t_1, t_2) & = v^2(t_1, t_2), \\
t_1(0, t_1, t_2) & = t_1.
\end{cases}
\tag{4.1}
$$

We have

$$\Lambda_1 \Lambda_{-1} v(t^*) = \int_0^{t_0^*} f(t_0, t_1^*, t_2^*) dt_0 + v^0(t_1^*, t_2^*).$$

Precisely speaking, the branch corresponding to the above $\tilde{t}^*$ is given by

$$\Lambda_1 \Lambda_{-1} v(\tilde{t}^*) = \int_{\gamma'} f(s, t_1^*, t_2^*) ds + v^0(P').$$

Here γ' and P' was previously defined for $\tilde{t}^*$. Similarly we have

$$
\begin{aligned}
x_1(\tilde{t}^*) \quad & = \textstyle\int_{\gamma'} g(s, t_1^*, t_2^*) ds + t_1, \\
\Lambda_{-1} v(\tilde{t}^*) \quad & = \textstyle\int_{\gamma''} f(s, t_1^* - s, t_2^*) ds + v^0(P''), \\
v(\tilde{t}^*) \quad & = \textstyle\int_{\gamma'''} f(s, t_1^* + s, t_2^*) ds + v^0(P''')
\end{aligned}
$$

Here we can determine $\gamma'', \gamma''', P'', P'''$ in a similar way as before. The author believes that if the lengths of these paths are sufficiently small, then the our iteration method works and we can solve the Cauchy problem for the Euler system.

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