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# On polar varieties, logarithmic vector fields and holonomic D-modules

By

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## § 1. Introduction

This is a survey on our recent results on logarithmic vector fields and on holonomic D-modules for algebraic local cohomology classes supported on hypersurfaces with isolated singularities. In [18], we have developed a new framework to study logarithmic vector fields along a hypersurface with an isolated singular point. The key of this approach is the concept of a polar variety. By using the Grothendieck local duality on residues we have derived an algorithmic method to compute logarithmic vector fields. Further, we have adopted the same approach to investigate certain holonomic D-modules supported on hypersurfaces in question and obtained an effective method to construct first order partial differential operators that annihilate a local cohomology class supported on the hypersurface. We also have obtained a formula on the characteristic cycles that can be interpreted as a refinement of a special case of a result of M. Kashiwara [9].

In this paper we present main results of [18] and try to describe the key ideas. In section 2, we briefly recall the notion of logarithmic vector field and present first main result. We also give some examples for illustration. In section 3, we consider a local cohomology class supported on a hypersurface and introduce a holonomic D-module by using first order annihilators. We give second main result on its characteristic cycle. In section 4, we introduce a finite-dimensional vector space, $W_{\Gamma_f}$, consisting of local cohomology classes that describe a way of intersection of the hypersurface with the polar variety. We present a framework to deal logarithmic vector fields and give a sketch of proofs of the main results. In section 5, we give some examples of computation.

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## § 2. Logarithmic vector fields

Let $S = \{x \in X \mid f(x) = 0\}$ be a hypersurface with an isolated singularity at the origin $O \in \mathbb{C}^n$, where $X$ is an open neighbourhood in $\mathbb{C}^n$ of the origin and $f$ is a holomorphic defining function. Let $\mathcal{O}_X$ be the sheaf of holomorphic functions and let $(f)$ denote the ideal generated by $f$ in $\mathcal{O}_X$

**Definition 2.1** ([14]). A holomorphic vector field

$$v = a_1(x)\frac{\partial}{\partial x_1} + a_2(x)\frac{\partial}{\partial x_2} + \cdots + a_n(x)\frac{\partial}{\partial x_n}, \quad a_i(x) \in \mathcal{O}_X,\ i = 1, 2, ..., n$$

is logarithmic along $S$, if $v(f) \in (f)$ holds.

Let $\mathcal{D}er_X(-\log S)$ be the sheaf on X of logarithmic vector fields along S and let $\mathcal{D}er_{X,0}(-\log S)$ be the stalk at the origin $O$ of the sheaf $\mathcal{D}er_X(-\log S)$ .

A logarithmic vector field $v$ generated over $\mathcal{O}_{X,O}$ by

$$f\frac{\partial}{\partial x_i}, i = 1, 2, ..., n,\ \frac{\partial f}{\partial x_j}\frac{\partial}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}, 1 \le i \le j \le n,$$

is called trivial.

**Example 2.2** ([15]). Let $C_0 = \{(x, y) \in \mathbb{C}^2 \mid y^3 - x^8 = 0\}$. Then the logarithmic vector field

$$\frac{\partial f}{\partial y}\frac{\partial}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial}{\partial y} = 3y^2\frac{\partial}{\partial x} + 8x^7\frac{\partial}{\partial y}$$

is trivial. The Euler vector field $3x\frac{\partial}{\partial x} + 8y\frac{\partial}{\partial y}$ is a non-trivial logarithmic vector field along $C_0$.

Set $\tilde{C}_0 = \{t \mid t \in \mathbb{C}\}$ and define a map $\pi_0 : \tilde{C}_0 \longrightarrow \mathbb{C}^2$ by a Puiseux expansion $x = t^3, y = t^8$ of the curve $C_0$. Then, the conductor is equal to 14 and the Rosenlicht differential form([12]), or the Poincare residue of the meromorphic form $\dfrac{dx \wedge dy}{y^3 - x^8}$, is given by

$$\pi_0^{-1}(\frac{dx}{\frac{\partial f}{\partial y}})(t) = \frac{dt}{t^{14}}.$$

Since $\dfrac{d}{dt}g(x(t), y(t)) = 3t^2\dfrac{\partial g}{\partial x}(x(t), y(t)) + 8t^7\dfrac{\partial g}{\partial y}(x(t), y(t))$ for any $g(x, y) \in \mathcal{O}_{X,O}$, we have

$$t\frac{d}{dt}g(\pi_0(t)) = 3x(t)\frac{\partial g}{\partial x}(\pi_0(t)) + 8y(t)\frac{\partial g}{\partial y}(\pi_0(t))$$

and

$$t^{14}\frac{d}{dt}g(\pi_0(t)) = 3y(t)^2\frac{\partial g}{\partial x}(\pi_0(t)) + 8x(t)^7\frac{\partial g}{\partial y}(\pi_0(t)),$$

which implies that the trivial logarithmic vector field $3y^2\frac{\partial}{\partial x} + 8x^7\frac{\partial}{\partial y}$ corresponds to $t^{14}\frac{d}{dt}$ and the Euler vector field $3x\frac{\partial}{\partial x} + 8y\frac{\partial}{\partial y}$ corresponds to $t\frac{d}{dt}$, the Euler vector field on $\tilde{C}_0$.

Note that M. Kersken showed that, for quasihomogeneous cases, $\mathcal{D}er_{X,O}(-\log S)$ is generated as an $\mathcal{O}_{X,O}$ module by the Euler vector field and trivial logarithmic vector fields (see [20] for more precise statement for complete intersection cases).

We say that germs of two logarithmic vector fields $v, v' \in \mathcal{D}er_{X,0}(-\log S)$ are equivalent, denoted by $v \sim v'$, if $v - v'$ is trivial. Let $\mathcal{D}er_{X,O}(-\log S)/\sim$ be the quotient by the equivalence relation $\sim$.

Since every logarithmic vector field $v \in \mathcal{D}er_X(-\log S)$ is trivial outside the singular point, the sheaf $\mathcal{D}er_X(-\log S))/\sim$ is supported at the origin.

Let $\tau_f$ be the Tjurina number of the hypersurface $S$ at the origin defined by

$$\tau_f = \dim_C(\mathcal{O}_{X,O}/(f, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_n})),$$

where $\mathcal{O}_{X,O}$ is the stalk at the origin of the sheaf $\mathcal{O}_X$ of holomorphic functions.

**Theorem 2.3** ([18])**.**

$$\dim_{\mathbb{C}}(\mathcal{D}er_{X,O}(-\log S)/\sim) = \tau_f.$$

**Remark** Since logarithmic vector fields are directly related to torsion differential forms for plane curve cases, the theorem above can be interpreted as an generalization of a result of O. Zariski [21] on torsion differential forms.

**Example 2.4.** Let $C = \{(x, y) \in \mathbb{C}^2 \mid y^3 - 3x^6y - x^8 - x^{10} = 0\}$. The principal part of the defining function of the curve $C$ is $y^3 - x^8$, the defining weighted homogeneous polynomial of the curve $C_0$. The curve $C$ is not quasihomogeneous. In fact, the Milnor number which is equal to the conductor, is equal to 14 whereas the Tjurina number is equal to 13. We have therefore $\dim_{\mathbb{C}}(\mathcal{D}er_{X,O}(-\log C)/\sim) = 13$. The number of linearly independent non-trivial logarithmic vector fields diminishes by one as compared with that of the curve $C_0$ in Example 1. Let $\tilde{C} = \{t \mid t \in \mathbb{C}\}$ and set $\pi : \tilde{C} \longrightarrow \mathbb{C}^2$ by $x = t^3, y = t^8 + t^{10}$. Since

$$\frac{d}{dt}g(x(t), y(t)) = 3t^2\frac{\partial g}{\partial x}(x(t), y(t)) + (8t^7 + 10t^9)\frac{\partial g}{\partial y}(x(t), y(t)),$$

the trivial vector field $\frac{\partial f}{\partial y}\frac{\partial}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial}{\partial y}$ given by

$$(3y^2 - 3x^6)\frac{\partial}{\partial x} + (-18x^5y - 8x^7 - 10x^9)\frac{\partial}{\partial y}$$

corresponds to $(t^{14}+t^{16}+t^{18})\frac{d}{dt}$. Note that the coefficient $t^{14}+t^{16}+t^{18}$ coincides with the denominator of the Rosenlicht differential form

$$\frac{dt}{t^{14}+t^{16}+t^{18}}.$$

Since the curve $C$ is not quasihomogeneous, there are no logarithmic vector fields that correspond to $c(t)\frac{d}{dt}$, if the valuation of $c(t)$ is equal to one.

## § 3. Holonomic D-modules

Let $\Omega_X^n$ be the sheaf on $X$ of holomorphic differential $n$-forms and let $\mathcal{H}^1_{[S]}(\Omega_X^n)$ be the sheaf oflocal cohomology supported on the hypersurface $S$.

Let $\mathcal{D}_X$ be the sheaf of linear partial differential operators with holomorphic coefficients. Then, $\mathcal{H}^1_{[S]}(\Omega_X^n)$ naturally has a structure of a right $\mathcal{D}_X$-module. Let

$$\omega = [\frac{dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n}{f}]$$

denote the modulo class in $\mathcal{H}^1_{[S]}(\Omega_X^n)$ of a meromorphic form

$$\frac{dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n}{f} \in \quad \Omega_X^n(*S)$$

and let $\mathcal{A}nn^{(1)}_{\mathcal{D}_{X,O}}(\omega)$ be the right ideal in $\mathcal{D}_{X,O}$ generated by partial differential operators of order at most one that annihilate the given cohomology class $\omega$:

$$\mathcal{A}nn^{(1)}_{\mathcal{D}_{X,O}}(\omega) = \{P \in \mathcal{D}_{X,O} \mid \omega P = 0, \mathrm{ord}P \leq 1\},$$

where $\mathcal{D}_{X,O}$ is the stalk at $O$ of the sheaf $\mathcal{D}_X$.

Let $v$ be a logarithmic vector field along $S$, i.e., $v \in \mathcal{D}er_{X,O}(-\log S)$. Then, there exists a first order partial differential operator $P$ of the form $P = v + h$, with $h \in \mathcal{O}_{X,O}$ that annihilates the local cohomology class $w$. Conversely, one can readily see as in [3] that if $P$ is a partial differential operator of order one that annihilates $w$, then the first order part of $P$ is a logarithmic vector field along $S$.

The characteristic variety of the holonomic $\mathcal{D}_{X,O}$-module $\mathcal{D}_X/\mathcal{A}nn^{(1)}_{\mathcal{D}_{X,O}}(\omega)$ consists of two Lagrangian varieties, $T^*_{reg(S)}X$ and $T^*_O X$, where $reg(S)$ is the non-singular part of the hypersurface $S$.

Now let us denote by $\mu_f^{(n-1)}$ the Milnor number of $f|_{H_0}$, where $H_0$ is a generic hyperplane passing through the origin $O$. That is

$$\mu_f^{(n-1)} = \min\{\mu(f|_H) \mid H : \mathit{hyperplane} \subset \mathbb{C}^n \ \mathit{s.t.}\ O \in H\}.$$

In [18], we have obtaind the following result.

**Theorem 3.1** ([18]). *The multiplicity of the Lagrangian $T^*_O X$ of the characteristic cycle $CC(\mathcal{D}_X/\mathcal{A}nn^{(1)}_{\mathcal{D}_{X,O}}(\omega))$ is equal to $\mu_f - \tau_f + \mu_f^{(n-1)}$. That is*

$$CC(\mathcal{D}_{X,O}/\mathcal{A}nn^{(1)}_{\mathcal{D}_{X,O}}(\omega)) = T^*_{regS}X + (\mu_f - \tau_f + \mu_f^{(n-1)})T^*_O X.$$

A sketch of the proof of the theorem will be given in section 4.

In 1973, M. Kashiwara [9] gave a beautiful and deep result on the indices of holonomic D-modles. By applying Kashiwara's result to the holonomic $D_{X,O}$-module $\mathcal{H}^1_{[S]}(\Omega^n_X)$, one has the following ([5], [1])

**Theorem 3.2.**

*Let $S$ be a hypersurface with an isolated singularity at the origin $O$. Then the characteristic cylce of the holonomic $\mathcal{D}_{X,O}$-module $\mathcal{H}^1_{[S]}(\Omega^n_X)$ is given by*

$$CC(\mathcal{H}^1_{[S]}(\Omega^n_X)) = T^*_{regS}X + \mu_f^{(n-1)}T^*_O X.$$

The following result relevant to quasihomogeneity ([13]) can be obtained from Theorem 3.1 and Theorem 3.2.

**Corollary 3.3** ([3], [6]). *Let $S = \{x \in X \mid f(x) = 0\}$ be a hypersurface with an isolated singularity at the origin $O$. Then the following conditions are equivalent.*
*(i)* $\mu_f = \tau_f$.
*(ii)* $\mathcal{A}nn^{(1)}_{\mathcal{D}_{X,O}}(\omega) = \mathcal{A}nn_{\mathcal{D}_{X,O}}(\omega)$.

Note that the result above on the quasihomogeneity and D-modules was already proved by F. J. Castro-Jiménez and J. M. Ucha-Enríquez ([3]) for plane curves and was generalized by M. Granger and M. Schulze ([6]) to the case of hypersurface with isolated singularities.

## § 4. Polar varieties and local cohomology

Let $\Gamma_f$ be a polar variety ([10], [19]) of the hypersurface $S$ defined to be

$$\Gamma_f = \{x \in X \mid \frac{\partial f}{\partial x_2} = \frac{\partial f}{\partial x_3} = \cdots = \frac{\partial f}{\partial x_n} = 0\}.$$

We may assume that the coordinates $x_1, x_2, ..., x_n$ are generic so that $f, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, ..., \frac{\partial f}{\partial x_n}$ is a regular sequence. Let $\mathcal{H}^n_O(\Omega^n_X)$ be the local cohomology supported at $O$.

In order to extract information, relevant to logarithmic vector fields, from the polar variety we consider following sets of local cohomology classes.

$$W_{\Gamma_f} = \{\omega \in \mathcal{H}^n_O(\Omega^n_X) \mid f\omega = \frac{\partial f}{\partial x_2}\omega = \frac{\partial f}{\partial x_3}\omega = \cdots = \frac{\partial f}{\partial x_n}\omega = 0\}$$

$$W_{T_f} = \{\omega \in \mathcal{H}^n_O(\Omega^n_X) \mid f\omega = \frac{\partial f}{\partial x_1}\omega = \frac{\partial f}{\partial x_2}\omega = \cdots = \frac{\partial f}{\partial x_n}\omega = 0\}$$

It follows from the Grothendieck local duality theorem that the pairings

$$W_{\Gamma_f} \times \mathcal{O}_{X,O}/(f, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, .., \frac{\partial f}{\partial x_n}) \longrightarrow \mathbb{C}$$

and

$$W_{T_f} \times \mathcal{O}_{X,O}/(f, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, .., \frac{\partial f}{\partial x_n}) \longrightarrow \mathbb{C}$$

defined by local residues are non-degenerate.

Now, let $\alpha : W_{\Gamma_f} \longrightarrow W_{\Gamma_f}$ be the map defined by $\alpha(\omega) = \frac{\partial f}{\partial x_1}\omega$ and let $W_{\Delta_f}$ denote the image of the map $\alpha$:

$$W_{\Delta_f} = \{\frac{\partial f}{\partial x_1}\omega \mid \omega \in W_{\Gamma_f}\}.$$

Since the kernel of the map $\alpha$ is $W_{T_f}$, we have the following exact sequance.

$$0 \longrightarrow W_{T_f} \longrightarrow W_{\Gamma_f} \longrightarrow W_{\Delta_f} \longrightarrow 0.$$

We let $\mathcal{A}_{X,O}$ denote the annihilator ideal in $\mathcal{O}_{X,O}$ of $W_{\Delta_f}$ defined to be

$$\mathcal{A}_{X,O} = \{a \in \mathcal{O}_{X,O} \mid a(W_{\Delta_f}) = 0\}.$$

Note that the Grothendieck local duality implies the non-degeneracy of the pairing:

$$W_{\Delta_f} \times \mathcal{O}_{X,O}/\mathcal{A}_{X,O} \longrightarrow \mathbb{C}.$$

We have the following lemma, which amounts to saying that $\mathcal{A}_{X,O}$ is the ideal quotient $(f, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, .., \frac{\partial f}{\partial x_n}) : \frac{\partial f}{\partial x_1}$.

**Lemma 4.1.**
*(i)* $(f, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, .., \frac{\partial f}{\partial x_n}) \subset \mathcal{A}_{X,O}$
*(ii)* $\mathcal{A}_{X,O} = \{a \mid a\frac{\partial f}{\partial x_1} \in (f, \frac{\partial f}{\partial x_2}, .., \frac{\partial f}{\partial x_n})\}$

The ideal $\mathcal{A}_{X,O}$ enjoys the following.

**Lemma 4.2.**
*(i)* $\mathcal{A}_{X,O} = \{a \mid a(W_{\Gamma_f}) \subset W_{T_f}\}$
*(ii)* $\{a\omega \mid a \in \mathcal{A}_{X,O}, \omega \in W_{\Gamma_f}\} = W_{T_f}$

**Proposition 4.3.** *Let $a(x) \in \mathcal{O}_{X,O}$. Then the following conditions are equivalent.*

*(i)* $a(x) \in \mathcal{A}_{X,O}$

*(ii) There is a logarithmic vector field $v$ along $S$ ( $v \in \mathcal{D}er_{X,O}(-\log S)$), so that*

$$v = a(x)\frac{\partial}{\partial x_1} + a_2(x)\frac{\partial}{\partial x_2} + \cdots + a_n(x)\frac{\partial}{\partial x_n}, \quad a_i(x) \in \mathcal{O}_X, \ i = 2, ..., n$$

Since $f, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, ..., \frac{\partial f}{\partial x_n}$ is a regular sequence, we have the following.

**Lemma 4.4.** *Let $v' = a_2(x)\frac{\partial}{\partial x_2} + \cdots + a_n(x)\frac{\partial}{\partial x_n}$ be a germ of holomorphic vector field. If $v'$ is a logarithmic vector field along $S$, then $v'$ is trivial.*

This yields the following result.

**Proposition 4.5.** *Let $v = a(x)\frac{\partial}{\partial x_1} + a_2(x)\frac{\partial}{\partial x_2} + \cdots + a_n(x)\frac{\partial}{\partial x_n}$ be a holomorphic vector field. Then the following conditions are equivalent.*

*(i) $v$ is trivial.*

*(ii) $a(x) \in (f, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, .., \frac{\partial f}{\partial x_n})$*

Now, we are ready to give a skech of the main result.

**Theorem 4.6** ([18])**.**

$$\dim_{\mathbb{C}}(\mathcal{D}er_{X,O}(-\log S)/\sim) = \tau_f.$$

*Proof.* Proposition 4.3 and Proposition 4.4 imply the following isomorphism :

$$\mathcal{D}er_{X,O}(-\log S)/\sim \cong \mathcal{A}_{X,O}/(f, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, .., \frac{\partial f}{\partial x_n}).$$

From

$$0 \longrightarrow Hom_{\mathbb{C}}(\mathrm{Im}(\alpha), \mathbb{C}) \longrightarrow Hom_{\mathbb{C}}(W_{\Gamma_f}, \mathbb{C}) \longrightarrow Hom_{\mathbb{C}}(W_{T_f}, \mathbb{C}) \longrightarrow 0,$$

we have the following exact sequence

$$0 \to \mathcal{O}_{X,O}/\mathcal{A}_{X,O} \longrightarrow \mathcal{O}_{X,O}/(f, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, .., \frac{\partial f}{\partial x_n}) \longrightarrow \mathcal{O}_{X,O}/(f, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, .., \frac{\partial f}{\partial x_n}) \to 0,$$

which imediately yields

$$\mathcal{A}_{X,O}/(f, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, .., \frac{\partial f}{\partial x_n}) \cong \mathcal{O}_{X,O}/(f, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, .., \frac{\partial f}{\partial x_n}).$$

We thus have

$$\dim_{\mathbb{C}}(\mathcal{D}er_{X,O}(-\log S)/\sim) = \dim_{\mathbb{C}}(\mathcal{O}_{X,O}/(f, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, .., \frac{\partial f}{\partial x_n}) = \tau_f.$$

□

Note that the theorem also follows from Lemma 4.2.

Now, let $P = v + h \in \mathcal{D}_{X,O}$ be a first order annihilator of the local cohomology class $\omega = [\frac{dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n}{f}]$. Then, as we have seen in the previous section, the first

order part $v$ is a logarithmic vector field along $S$. It follows from this observation and the genericity of the coordinates system $x_1, x_2, ..., x_n$ that the multiplicity of the Lagrangian $T^*_O X$ of the characteristic cycle of the holonomic $\mathcal{D}_{X,O}$-module $\mathcal{D}_{X,O}/\mathcal{A}nn^{(1)}_{\mathcal{D}_{X,O}}(\omega)$ is equal to $\dim_{\mathbb{C}}(\mathcal{O}_{X,O}/\mathcal{A}_{X,O})$.

Since

$$0 \longrightarrow W_{T_f} \longrightarrow W_{\Gamma_f} \longrightarrow W_{\Delta_f} \longrightarrow 0$$

is exact, we have

$$\dim_{\mathbb{C}}(\mathcal{O}_{X,O}/\mathcal{A}_{X,O}) = \dim_{\mathbb{C}}(W_{\Delta_f}) = \dim_{\mathbb{C}}(W_{\Gamma_f}) - \dim_{\mathbb{C}}(W_{T_f}).$$

Therefore, the Lê-Teissier formula ([10], [19]) implies the following result on the characteristic cycle $CC(\mathcal{D}_X/\mathcal{A}nn^{(1)}_{\mathcal{D}_{X,O}}(\omega))$ of the holonomic $\mathcal{D}_{X,O}$-module $\mathcal{D}_{X,O}/\mathcal{A}nn^{(1)}_{\mathcal{D}_{X,O}}(\omega)$.

**Theorem 4.7** ([18])**.**

$$CC(\mathcal{D}_{X,O}/\mathcal{A}nn^{(1)}_{\mathcal{D}_{X,O}}(\omega)) = T^*_{reg_S}X + (\mu_f - \tau_f + \mu_f^{(n-1)})T^*_O X.$$

## § 5. Examples

In this section, we consider $E_{12}$ isolated singularity. We explicitely compute a standard basis of logarithmic vector fields by adopting algorithms given by [16], [17].

**Example 5.1.** Let $f_0(x,y) = x^3 + y^7$ and $S_0 = \{(x,y) \in X \mid f_0(x,y) = 0\}$, where $X = \mathbb{C}^2$. Since $f_0|_{y=0} = x^3$, we have $\mu_f = \tau_f = 12$ and $\mu_f^{(1)} = 2$.

Set $\Gamma_{f_0} = \{(x,y) \in X \mid \frac{\partial f_0}{\partial x}(x,y) = 0\}$. Then,

$$W_{\Gamma_{f_0}} = \mathrm{Span}_{\mathbb{C}}\{\begin{bmatrix} dx \wedge dy \\ x^i y^j \end{bmatrix} \mid 1 \leq i \leq 2,\ 1 \leq j \leq 7\}.$$

The vector space $W_{\Gamma_{f_0}}$ is generated over $\mathcal{O}_{X,O}$ by $\begin{bmatrix} dx \wedge dy \\ x^2 y^7 \end{bmatrix}$. Since

$\frac{\partial f_0}{\partial y}\begin{bmatrix} dx \wedge dy \\ x^2 y^7 \end{bmatrix} = 7\begin{bmatrix} dx \wedge dy \\ x^2 y \end{bmatrix}$, we have

$$W_{\Delta_{f_0}} = \mathrm{Span}_{\mathbb{C}}\{\begin{bmatrix} dx \wedge dy \\ xy \end{bmatrix}, \begin{bmatrix} dx \wedge dy \\ x^2 y \end{bmatrix}\}.$$

Thus, $\dim_{rmC}(W_{\Delta_{f_0}}) = 2$ and $\mathcal{A}_{X,O} = (x^2, y)$. One find therefore that the set of germs of logarithmic vector fields $\mathcal{D}er_{X,O}(-\log S_0)$ is generated over $\mathcal{O}_{X,O}$ by

$$-7y^6\frac{\partial}{\partial x} + 3x^2\frac{\partial}{\partial y},\ 7x\frac{\partial}{\partial x} + 3y\frac{\partial}{\partial y}.$$

**Example 5.2.** Let $f(x,y) = x^3 + y^7 + xy^5$ and $S = \{(x,y) \in X \mid f(x,y) = 0\}$. We have $\mu_f = 12, \mu_f^{(1)} = 2$ as in previous Example 5.1, whereas $\tau_f = 11$.

Set $\Gamma_f = \{(x,y) \in X \mid \frac{\partial f}{\partial x}(x,y) = 0\}$. Then, $W_{\Gamma_f}$ is generated over $\mathcal{O}_{X,O}$ by

$$\omega = (\begin{bmatrix} 1 \\ x^2y^7 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 \\ x^4y^2 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 1 \\ xy^9 \end{bmatrix} + \frac{2}{9}\begin{bmatrix} 1 \\ x^3y^4 \end{bmatrix})dx \wedge dy.$$

Since

$$\frac{\partial f}{\partial y}\omega = (7\begin{bmatrix} 1 \\ x^2y \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 \\ xy^3 \end{bmatrix})dx \wedge dy,$$

we have

$$W_{\Delta_f} = \mathrm{Span}_{\mathbb{C}}\{\begin{bmatrix} dx \wedge dy \\ xy \end{bmatrix}, \begin{bmatrix} dx \wedge dy \\ x^2y \end{bmatrix}, 21\begin{bmatrix} dx \wedge dy \\ x^2y \end{bmatrix} + \begin{bmatrix} dx \wedge dy \\ xy^3 \end{bmatrix}\}.$$

Thus, $\dim_{\mathbb{C}}(W_{\Delta_f}) = 3$, which is equal to $\mu_f - \tau_f + \mu_f^{(1)}$ and

$$\mathcal{A}_{X,O} = (x^2, xy, 21y^2 - x).$$

The corresponding standard basis of germs of logarithmic vector fields $\mathcal{D}er_{X,O}(-\log S)$ is given by

$$(-10y^7 - 10xy^5 - 63y^6 - 66xy^4 - x^2y^2)\frac{\partial}{\partial x} + (4y + 27)x^2\frac{\partial}{\partial y},$$

$$(-3y^5 + xy^3 + 10x^2y + 63x^2)\frac{\partial}{\partial x} + (4y + 27)xy\frac{\partial}{\partial y},$$

$$(45y^4 + 211xy^2 - 10x^2 + 1323xy)\frac{\partial}{\partial x} + (4y + 27)(21y^2 - x)\frac{\partial}{\partial y}.$$

Now let $\Gamma'_f = \{(x,y) \in X \mid \frac{\partial f}{\partial y}(x,y) = 0\}$. Then, the vector space $W_{\Gamma'_f}$ defined by

$$W_{\Gamma'_f} = \{\omega \in \mathcal{H}^2_O(\Omega^2_X) \mid f\omega = \frac{\partial f}{\partial x}\omega = 0\}$$

is generated over $\mathcal{O}_{X,O}$ by

$$\omega' = (\begin{bmatrix} 1 \\ x^3y^6 \end{bmatrix} - \frac{2}{7}\begin{bmatrix} 1 \\ x^5y \end{bmatrix} - \frac{5}{7}\begin{bmatrix} 1 \\ x^2y^8 \end{bmatrix} + \frac{2 \times 5}{7^2}\begin{bmatrix} 1 \\ x^4y^3 \end{bmatrix} + \frac{5^2}{7^2}\begin{bmatrix} 1 \\ xy^{10} \end{bmatrix})dx \wedge dy.$$

Since, $\mu(f|_{x=0}) = 6$, we have $\dim_{\mathbb{C}}(W_{\Gamma'_f}) = 12 + 6 = 18$. Let $W_{\Delta'_f}$ denote the image of the map $\alpha' : W_{\Gamma'_f} \longrightarrow W_{\Gamma'_f}$ defined by the multiplication with $\frac{\partial f}{\partial x}$. From

$$\frac{\partial f}{\partial x}\omega' = (3\begin{bmatrix} 1 \\ xy^6 \end{bmatrix} + \frac{1}{7}\begin{bmatrix} 1 \\ x^3y \end{bmatrix} - \frac{5}{7^2}\begin{bmatrix} 1 \\ x^2y^3 \end{bmatrix} + \frac{5^2}{7^2}\begin{bmatrix} 1 \\ xy^5 \end{bmatrix})dx \wedge dy,$$

we find that

$$\begin{bmatrix} dx\wedge dy \\ xy \end{bmatrix}, \begin{bmatrix} dx\wedge dy \\ xy^2 \end{bmatrix}, \begin{bmatrix} dx\wedge dy \\ xy^3 \end{bmatrix}, \begin{bmatrix} dx\wedge dy \\ x^2y \end{bmatrix}, 21\begin{bmatrix} dx\wedge dy \\ xy^5 \end{bmatrix} - \frac{5}{7}\begin{bmatrix} dx\wedge dy \\ x^2y^2 \end{bmatrix}$$

$$21\begin{bmatrix} dx\wedge dy \\ xy^6 \end{bmatrix} + \begin{bmatrix} dx\wedge dy \\ x^3y \end{bmatrix} - \frac{5}{7}\begin{bmatrix} dx\wedge dy \\ x^2y^3 \end{bmatrix} + \frac{5^2}{7}\begin{bmatrix} dx\wedge dy \\ xy^5 \end{bmatrix}$$

constitutes a basis of the vector space $W_{\Delta'_f}$. Note that $\dim_{\mathbf{C}} W_{\Delta'_f} = 7$ is consistent with $\mu_f + \mu(f|_{x=0}) - \tau_f = 18 - 11 = 7$.

The standard basis of the annihilator

$$\mathcal{A}'_{X,O} = \{a \in \mathcal{O}_{X,O} \mid a(W_{\Delta'_f}) = 0\}$$

with respect to a term order that is compatible with the weight vector $(7, 3)$ is given by

$$7203xy + 245y^4 - 125y^5, 21x^2 - y^5, y^6.$$

The corresponding logarithmic vector fields are

$$3(50y + 343)(7203xy + 245y^4 - 125y^5)\tfrac{\partial}{\partial x} +$$

$$(3125y^5 + 875y^4 - 625xy^2 + 427035y^3 + 3125x^2 + 3675xy + 3176523y^2 - 151263x)\frac{\partial}{\partial y},$$

$$(50y + 343)(21x^2 - y^5)\frac{\partial}{\partial x} + (25y^5 + 49y^4 + 415xy^2 + 25x^2 + 3087xy)\frac{\partial}{\partial y},$$

$$(50y + 343)y^6\frac{\partial}{\partial x} + (-25y^6 - 154y^5 + 5xy^3 - 25x^2y - 147x^2)\frac{\partial}{\partial y}.$$

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