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Boundary values of ultradistribution solutions to regular-specializable systems

By

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Abstract

For any hyperfunction solutions to the regular-specializable system, a boundary value morphism is defined by Laurent-Monteiro Fernandes. We announce that this morphism induces a boundary value morphism for extensible ultradistribution solutions to the regular-specializable system under an irregularity condition due to Tahara.

Introduction

In this article, we announce of result about boundary value problems for extensible ultradistribution solutions along an initial boundary to the *regular-specializable system* of analytic linear differential equations in the framework of *Algebraic Analysis*.

The regular-specializable system is defined by Kashiwara [3], and constitutes a special class of Fuchsian systems in the sense of Laurent-Monteiro Fernandes [14]. In a single equation case, this corresponds to a Fuchsian operator in the sense of Baouendi-Goulaouic [1] with constant characteristic exponents, or equivalently, a regular-singular operator with weak sense due to Kashiwara-Oshima [7] (cf. Oshima [21]). For any regular-specializable system, its vanishing cycle and nearby cycle in the $\mathscr{D}$-Module theory are defined (see Kashiwara [3], Laurent [13], Maisonobe-Mebkhout [16]). After the results by Kashiwara-Oshima [7] and Oshima [21], for any hyperfunction solutions to a regular-specializable system, Laurent-Monteiro Fernandes ([15], [17], [18]) defined an injective boundary value morphism which takes values in hyperfunction solutions to the nearby cycle of the system. This morphism extends the non-characteristic boundary value morphism due to Komatsu-Kawai and Schapira. Note that the solvability is

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discussed in [15] under a kind of hyperbolicity condition (see Yamazaki [25] for a microlocal version). Moreover if we replace hyperfunctions with distributions, then, we can prove that the boundary value morphism due to Laurent-Monteiro Fernandes induces the boundary value morphism for temperate (i.e. extensible) distribution solutions to any regular-specializable system (see Yamazaki [26]). Hence, we shall consider boundary value problems in the framework of (Gevrey) ultradistributions, and announce that the boundary value morphism above induces a boundary value morphism for extensible ultradistribution solutions to the regular-specializable system under an irregularity condition due to Tahara.

Details of this article will be appeared in a forthcoming paper [27].

§ 1. Notation

We fix the notation used in this paper. Our main references are Kashiwara [5] and Kashiwara-Schapira [8]: We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the sets of all the integers, real numbers and complex numbers respectively. Moreover we set

$$\mathbb{N} := \{n \in \mathbb{Z};\, n \geqslant 1\} \subset \mathbb{N}_0 := \mathbb{N} \cup \{0\},\ \mathbb{R}_{>0} := \{t \in \mathbb{R};\, r > 0\} \subset \mathbb{R}_{\geqslant 0} := \{t \in \mathbb{R};\, r \geqslant 0\}.$$

For a topological space Z and $A \subset Z$, we denote by $\operatorname{Int} A$ and $\operatorname{Cl} A$ the interior and the closure of A respectively. In this paper, we shall write *Module* or *Ring* with capital letters, instead of *sheaf of left modules* or *sheaf of rings* respectively. Let $\mathcal{A}$ be a Ring on Z. We denote by $\mathcal{A}^{\mathrm{op}}$ the opposed Ring, and we regard right $\mathcal{A}$-Modules as (left) $\mathcal{A}^{\mathrm{op}}$-Modules. We denote by $\mathfrak{Mod}(\mathcal{A})$ the category of $\mathcal{A}$-Modules, and by $\mathfrak{Coh}(\mathcal{A})$ the full subcategory of $\mathfrak{Mod}(\mathcal{A})$ consisting of coherent $\mathcal{A}$-Modules. Further we denote by $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ the bounded derived category of complexes of $\mathcal{A}$-Modules, and by $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A})$ the full subcategory of $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ consisting of objects with coherent cohomologies. In this paper, all the manifolds are assumed to be *smooth* and *paracompact*. Let M be a real analytic manifold, and X a complexification of M. Let N be an analytic hypersurface of M, and Y a complexification of N in X. Let $\iota\colon Y \hookrightarrow X$ be the natural embedding. Since the problem is local, we fix the following coordinates:

$$\begin{array}{ccc} N = \mathbb{R}^n_x \times \{0\} & \hookrightarrow & M = \mathbb{R}^n_x \times \mathbb{R}_t \\ \cap & & \cap \\ Y = \mathbb{C}^n_z \times \{0\} & \xhookrightarrow{\iota} & X = \mathbb{C}^n_z \times \mathbb{C}_\tau \end{array} \tag{1.1}$$

We set $\partial_{z_\nu} := \dfrac{\partial}{\partial z_\nu}$, $\partial_\tau := \dfrac{\partial}{\partial \tau}$ etc. and $\vartheta := \tau\partial_\tau$ (or $t\partial_t$ on real cases). For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, we set $|\alpha| := \sum\limits_{\nu=1}^{n} \alpha_\nu$ and $\partial_z^\alpha := \partial_{z_1}^{\alpha_1} \cdots \partial_{z_n}^{\alpha_n}$. We denote by $\mathscr{O}_X$ the *Ring of holomorphic functions*, by $\mathscr{D}_X$ the *Ring of holomorphic linear differential* *operators*, and by Ω_X the sheaf of the *holomorphic differential forms with maximal degree* on X respectively. Set $\mathscr{D}_{Y\to X} := \mathscr{O}_Y \underset{\mathscr{O}_X}{\otimes} \mathscr{D}_X$ and $\mathscr{D}_{X\leftarrow Y} := \Omega_Y \underset{\mathscr{O}_Y}{\otimes} \mathscr{D}_{Y\to X} \underset{\mathscr{O}_X}{\otimes} \Omega_X^{\otimes -1}$; that is, the *transfer* $(\mathscr{D}_Y \otimes \mathscr{D}_X^{\mathrm{op}}|_Y)$ and $(\mathscr{D}_X|_Y \otimes \mathscr{D}_Y^{\mathrm{op}})$-*Modules* associated with $\iota\colon Y \hookrightarrow X$ respectively. For a $\mathscr{D}_X$-Module $\mathscr{M}$ defined on a neighborhood of Y, we denote by

$$\boldsymbol{D}\iota^*\mathscr{M} := \mathscr{D}_{Y\to X} \overset{\boldsymbol{L}}{\underset{\mathscr{D}_X}{\otimes}} \mathscr{M},$$

$$\boldsymbol{D}\iota^!\mathscr{M} := \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y^{\mathrm{op}}}(\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X) \overset{\boldsymbol{L}}{\underset{\mathscr{D}_X}{\otimes}} \mathscr{D}_{X\leftarrow Y}, \mathscr{D}_Y)[-1],$$

the *inverse image* and the *extraordinary inverse image* respectively. Let $\mathscr{or}_M$ denote the *orientation sheaf* on M, and set $\mathscr{or}_{N/M} := \mathscr{or}_N \otimes \mathscr{or}_M|_N$ (the *relative orientation sheaf* attached to $N \to M$). We set $\omega_{N/M} := \mathscr{or}_{N/M}[-1]$ and $\omega_{N/M}^{\otimes -1} := \mathscr{or}_{N/M}[1]$ (the *relative dualizing complex* and its dual). We set $\mathscr{D}_N^A := \mathscr{D}_Y|_N$, $\mathscr{D}_{M\leftarrow N}^A := \mathscr{D}_{X\leftarrow Y}|_N \otimes \mathscr{or}_{N/M}$ etc. (we add the superscript A in order to avoid the confusion with holomorphic cases). Let $\mathscr{B}_M$ and $\mathscr{D}b_M$ be the sheaves on M of *Sato hyperfunctions* and of *Schwartz distributions* respectively. Further, Let $\mathscr{D}f_M^*$ and $\mathscr{D}b_M^*$ be the sheaves on M of *ultradifferentiable functions* and of *ultradistributions* of Gevrey class $*$, respectively. Here and in what follows, $*$ stands for $\{s\}$ with $1 < s < \infty$ or (s) with $1 < s \leqslant \infty$ to indicate the Gevrey growth order, and we understand that $\mathscr{D}f_M^{(\infty)} := \mathscr{C}_M^\infty$ is the sheaf on M of functions of class C^∞. In particular, $\mathscr{D}b_M^{(\infty)} = \mathscr{D}b_M$. We fix the coordinates in (1.1), and recall the definitions of $\mathscr{D}f_M^*$ and $\mathscr{D}b_M^*$ (see [9], [10]): Let $U \subset M$ be an open set. For $u(x) \in \Gamma(U; \mathscr{C}_M^\infty)$, compact set $K \Subset U$ and $h > 0$, we set

$$\mathfrak{p}_Z^{s,h}(u) := \sup_{\substack{(x,t)\in Z \\ (\alpha,\nu)\in\mathbb{N}_0^{n+1}}} \frac{|\partial_x^\alpha \partial_t^\nu u(x,t)|}{h^{|\alpha|+\nu}(|\alpha|+\nu)!^s}.$$

Then $u(x) \in \Gamma(U; \mathscr{D}f_M^{\{s\}})$ (resp. $\Gamma(U; \mathscr{D}f_M^{(s)})$) if for any $K \Subset U$, there exist $h > 0$ such that (resp. for any $K \Subset U$ and $h > 0$) $\mathfrak{p}_K^{s,h}(u) < \infty$. By the system of semi-norms $\{\mathfrak{p}_K^{s,h}(\cdot)\}_{h>0,\, K\Subset U}$, we can endow each $\Gamma(U; \mathscr{D}f_M^*)$ and $\Gamma_{\mathrm{c}}(U; \mathscr{D}f_M^*)$ with a natural locally convex topology (the subscript c means the sections with compact support), and consequently

(a) $\Gamma(U; \mathscr{D}f_M^{\{s\}})$ is a DLFS space, and $\Gamma_{\mathrm{c}}(U; \mathscr{D}f_M^{\{s\}})$ is a DFS space,
(b) $\Gamma(U; \mathscr{D}f_M^{(s)})$ is an FS space, and $\Gamma_{\mathrm{c}}(U; \mathscr{D}f_M^{(s)})$ is an LFS space.

These all spaces are reflexive.

Set $\mathscr{V}_M := \Omega_X|_M \otimes \mathscr{or}_M$, and $\mathscr{V}_M^* := \mathscr{D}f_M^* \underset{\mathscr{A}_M}{\otimes} \mathscr{V}_M$ for $* = \{s\}$ or (s) with $1 < s < \infty$; that is, $\mathscr{V}_M^*$ is the sheaf on M of volume elements with coefficients in $\mathscr{D}f_M^*$. Since we fix the coordinates, we have a (global) isomorphism

$$\mathscr{D}f_M^* \ni u(x,t) \mapsto u(x,t)\,dx\,dt \in \mathscr{V}_M^*, \tag{1.2}$$

where $dx\,dt$ denotes the standard Lebesgue measure on $M \simeq \mathbb{R}^{n+1}$. We endow $\mathscr{V}_M^*$ with the locally convex topology under which (1.2) is the topological isomorphism and set:

$$\Gamma(U;\mathscr{D}\!\ell_M^*) := \Gamma_{\mathrm{c}}(U;\mathscr{V}_M^*)'.$$

Here the prime means the strong dual of a topological vector space. The assignment $U \mapsto \Gamma(U;\mathscr{D}\!\ell_M^*)$ defines a sheaf $\mathscr{D}\!\ell_M^*$ on M. Then $\mathscr{D}\!\ell_M^{\{s\}}$ is called the sheaf of *ultradistributions of class* $\{s\}$ (*of Roumieu type*), and $\mathscr{D}\!\ell_M^{(s)}$ is called the sheaf of *ultradistributions of class* (s) (*of Beurling-Björck type*). It is known that $\Gamma_{\mathrm{c}}(U;\mathscr{D}\!\ell_M^*) = \Gamma(U;\mathscr{V}_M^*)'$. The global isomorphism (1.2) permits us the following identifications as usual:

$$\Gamma(U;\mathscr{D}\!f_M^*)' = \Gamma_{\mathrm{c}}(U;\mathscr{D}\!\ell_M^*) \subset \Gamma(U;\mathscr{D}\!\ell_M^*) = \Gamma_{\mathrm{c}}(U;\mathscr{D}\!f_M^*)'.$$

Further

(a) $\Gamma(U;\mathscr{D}\!\ell_M^{\{s\}})$ is an FS space, and $\Gamma_{\mathrm{c}}(U;\mathscr{D}\!\ell_M^{\{s\}})$ is an LFS space,
(b) $\Gamma(U;\mathscr{D}\!\ell_M^{(s)})$ is a DLFS space, and $\Gamma_{\mathrm{c}}(U;\mathscr{D}\!\ell_M^{(s)})$ is a DFS space.

If $1 < s < t$, then as $\mathscr{D}_M^A$ sub-Modules

$$\mathscr{D}\!\ell_M \subset \mathscr{D}\!\ell_M^{\{t\}} \subset \mathscr{D}\!\ell_M^{(t)} \subset \mathscr{D}\!\ell_M^{\{s\}} \subset \mathscr{D}\!\ell_M^{(s)} \subset \mathscr{B}_M\,.$$

For any $\mathscr{D}_N^A$ sub-Module $\mathscr{F} \subset \mathscr{B}_N$, set

$$\Gamma_{[N]}(\mathscr{F}) := \mathscr{D}_{M\leftarrow N}^A \underset{\mathscr{D}_N^A}{\otimes} \mathscr{F}.$$

We can represent any section (or any germ) of $\mathscr{D}_{M\leftarrow N}^A$ as $\sum\limits_r a_r(x,\partial_x)\,\partial_t^r \cdot \mathbb{1}_{M\leftarrow N}$, where $\mathbb{1}_{M\leftarrow N}$ is a canonical generator of $\mathscr{D}_{M\leftarrow N}^A$ over $\mathscr{D}_M^A$ associated with coordinates in (1.1). Let $\delta(t)$ be the delta function, and set $\delta^{(r)}(t) := \partial_t^r\delta(t)$. If $1 < s < t$, then the identification that $\mathscr{D}_{M\leftarrow N}^A \ni \partial_t^r \cdot \mathbb{1}_{M\leftarrow N} \leftrightarrow \delta^{(r)}(t)$ for any $r \in \mathbb{N}_0$ induces

$$\begin{array}{ccccccccccc}
\Gamma_{[N]}(\mathscr{D}\!\ell_M) & \subset & \Gamma_{[N]}(\mathscr{D}\!\ell_M^{\{t\}}) & \subset & \Gamma_{[N]}(\mathscr{D}\!\ell_M^{(t)}) & \subset & \Gamma_{[N]}(\mathscr{D}\!\ell_M^{\{s\}}) & \subset & \Gamma_{[N]}(\mathscr{D}\!\ell_M^{(s)}) & \subset & \Gamma_{[N]}(\mathscr{B}_M) \\
\| & & \cap & & \cap & & \cap & & \cap & & \cap \\
\Gamma_N(\mathscr{D}\!\ell_M) & \subset & \Gamma_N(\mathscr{D}\!\ell_M^{\{t\}}) & \subset & \Gamma_N(\mathscr{D}\!\ell_M^{(t)}) & \subset & \Gamma_N(\mathscr{D}\!\ell_M^{\{s\}}) & \subset & \Gamma_N(\mathscr{D}\!\ell_M^{(s)}) & \subset & \Gamma_N(\mathscr{B}_M)
\end{array}$$

Here we remark that $\Gamma_{[N]}(\mathscr{D}\!\ell_M) = \Gamma_N(\mathscr{D}\!\ell_M)$.

Let $U_0 \subset N$ be an open subset. Then

(a) $\Gamma(U_0;\Gamma_N(\mathscr{D}\!\ell_M^{\{s\}}))$ is an FS space, and $\Gamma_{\mathrm{c}}(U_0;\Gamma_N(\mathscr{D}\!\ell_M^{\{s\}}))$ is an LFS space,
(b) $\Gamma(U_0;\Gamma_N(\mathscr{D}\!\ell_M^{(s)}))$ is a DLFS space, and $\Gamma_{\mathrm{c}}(U_0;\Gamma_N(\mathscr{D}\!\ell_M^{(s)}))$ is a DFS space.

We set $\Omega_+ := \{(x,t) \in M;\, t > 0\} \subset M_+ := \{(x,t) \in M;\, t \geqslant 0\}$. Let $\mathcal{T}\!hom(*,\mathscr{D}\!\ell_M)$ be the *Schwartz functor* due to Kashiwara (see [4]), and set

$$\Gamma_{\Omega_+}^t(\mathscr{D}\!\ell_M) := \mathcal{T}\!hom(\mathbb{C}_{\Omega_+},\mathscr{D}\!\ell_M) = \mathscr{D}\!\ell_M / \Gamma_{M\smallsetminus\Omega_+}(\mathscr{D}\!\ell_M).$$

Let $U \subset M$ be an open subset with $U \cap N \neq \emptyset$. As in the case of distribution, we set

$$\Gamma^{\mathrm{ext}}_{\Omega_+}(\mathscr{D}\!\ell^*_M) := \mathscr{D}\!\ell^*_M / \Gamma_{M \smallsetminus \Omega_+}(\mathscr{D}\!\ell^*_M).$$

If $U \cap N \neq \emptyset$, then we have (see [12] and cf. [2])

$$\Gamma(U; \Gamma^{\mathrm{ext}}_{\Omega_+}(\mathscr{D}\!\ell^*_M)) = \Gamma(U; \mathscr{D}\!\ell^*_M)/\Gamma_{U \smallsetminus \Omega_+}(U; \mathscr{D}\!\ell^*_M) \subset \Gamma(U \cap \Omega_+; \mathscr{D}\!\ell^*_M).$$

By using a result of [11], we can prove

Proposition 1.1. *For any $k \in \mathbb{Z}$, there exists an isomorphism*

$$\Gamma^{\mathrm{ext}}_{\Omega_+}(\mathscr{D}\!\ell^*_M)|_N \ni u(x,t) \mapsto t^{-k} u(x,t) \in \Gamma^{\mathrm{ext}}_{\Omega_+}(\mathscr{D}\!\ell^*_M)|_N\,.$$

For a vector bundle $\tau \colon E \to Z$ over a manifold Z, we set $\dot\tau \colon \dot E := E \smallsetminus Z \to Z$ (the zero-section removed). Let $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}_{>0}}(\mathbb{C}_E) \subset \mathbf{D}^{\mathrm{b}}(\mathbb{C}_E)$ be the subcategory of the bounded derived category of $\mathbb{C}_E$-Modules such that each cohomology is conic. We set

$$P^+ := \{(v,\xi) \in \dot T_N M \underset{N}{\times} \dot T^*_N M;\ \langle v, \xi\rangle > 0\}$$

and denote by $p_1^+ \colon P^+ \to \dot T_N M$ and $p_2^+ \colon P^+ \to \dot T^*_N M$ the canonical projections. Then:

Proposition 1.2 ([24, Corollary A.2], cf. [22, Chapter I])**.** *There exists the following distinguished triangle for any $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}_{>0}}(\mathbb{C}_{T_N M})$:*

$$F \to \tau_N^{-1} \boldsymbol{R}\tau_{N!}\, F \otimes \omega^{\otimes -1}_{N/M} \to \boldsymbol{R} p^+_{1*}\, p_2^{+\,-1} F^\wedge \otimes \omega^{\otimes -1}_{N/M} \xrightarrow{+1}.$$

Here $F^\wedge$ denotes the Fourier-Sato transform of F.

Taking $F = \nu_N(\mathcal{F})$ (*specialization* along N) for any $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_M)$, we have the following distinguished triangle by Proposition 1.2:

$$\boldsymbol{R}\Gamma_N(\mathcal{F}) \otimes \mathscr{or}_{N/M} \to \boldsymbol{R}\Gamma_{M_+}(\mathcal{F})|_N \otimes \mathscr{or}_{N/M} \to \boldsymbol{R}\Gamma_{\Omega_+}(\mathcal{F})|_N \xrightarrow{+1},$$

and this induces the following:

$$\begin{array}{ccccccccc}
0 \to & \Gamma_N(\mathscr{D}\!\ell_M) \otimes \mathscr{or}_{N/M} & \to & \Gamma_{M_+}(\mathscr{D}\!\ell_M)|_N \otimes \mathscr{or}_{N/M} & \to & \Gamma^t_{\Omega_+}(\mathscr{D}\!\ell_M)|_N & \to 0 \\
& \cap & & \cap & & \cap & \\
0 \to & \Gamma_N(\mathscr{D}\!\ell^*_M) \otimes \mathscr{or}_{N/M} & \to & \Gamma_{M_+}(\mathscr{D}\!\ell^*_M)|_N \otimes \mathscr{or}_{N/M} & \to & \Gamma^{\mathrm{ext}}_{\Omega_+}(\mathscr{D}\!\ell^*_M)|_N & \to 0 \\
& \cap & & \cap & & \cap & \\
0 \to & \Gamma_N(\mathscr{B}_M) \otimes \mathscr{or}_{N/M} & \to & \Gamma_{M_+}(\mathscr{B}_M)|_N \otimes \mathscr{or}_{N/M} & \to & \Gamma_{\Omega_+}(\mathscr{B}_M)|_N & \to 0
\end{array} \tag{1.3}$$

§ 2. Regular-Specializable Systems

We recall the definitions of regular-specializable systems and vanishing and nearby cycle Modules (see [13], [16], [26, Appendix] and references cited therein). Since the problem is local, we fix the coordinates in (1.1). Let $\{\mathscr{D}_{X}(p)\}_{p\in\mathbb{N}_0}$ the usual *order filtration*. Further we denote by $\mathsf{V}_Y(\mathscr{D}_X) = \{\mathsf{V}^k_Y(\mathscr{D}_X)\}_{k\in\mathbb{Z}}$ the *V-filtration* (*along* Y). This filtration is given by

$$\mathsf{V}^k_Y(\mathscr{D}_X) := \bigcap_{j\in\mathbb{Z}}\{P \in \mathscr{D}_X|_Y;\ P\mathcal{I}^j_Y \subset \mathcal{I}^{j-k}_Y\} = \{\sum_{j-i\leqslant k} P_{ij}(z,\partial_z)\,\tau^i\partial^j_\tau \in \mathscr{D}_X\big|_Y\}\,.$$

Here $\mathcal{I}_Y$ be the defining Ideal of Y in $\mathscr{O}_X$ with a convention that $\mathcal{I}^j_Y = \mathscr{O}_X$ for $j \leqslant 0$. Then we have:

(i) $\mathsf{V}^k_Y(\mathscr{D}_X) \subset \mathsf{V}^{k+1}_Y(\mathscr{D}_X)$ holds, and $\bigcup\limits_k \mathsf{V}^k_Y(\mathscr{D}_X) = \mathscr{D}_X$;

(ii) $\mathsf{V}^k_Y(\mathscr{D}_X)\,\mathsf{V}^l_Y(\mathscr{D}_X) \subset \mathsf{V}^{k+l}_Y(\mathscr{D}_X)$ holds for any $k,\ l \in \mathbb{Z}$.

In what follows we omit the phrase “along Y” since Y is fixed.

Definition 2.1. Let $\mathscr{M} \in \mathfrak{Coh}(\mathscr{D}_X|_Y)$. Then a V-filtration $\mathsf{V}(\mathscr{M})$ on $\mathscr{M}$ is a family $\{\mathsf{V}^k(\mathscr{M})\}_{k\in\mathbb{Z}}$ of sub-Groups such that

(i) $\mathsf{V}^k(\mathscr{M}) \subset \mathsf{V}^{k+1}(\mathscr{M})$ holds, and $\bigcup\limits_k \mathsf{V}^k(\mathscr{M}) = \mathscr{M}$;

(ii) $\mathsf{V}^k_Y(\mathscr{D}_X)\,\mathsf{V}^l(\mathscr{M}) \subset \mathsf{V}^{k+l}(\mathscr{M})$ holds for any $k,\ l \in \mathbb{Z}$.

Moreover a V-filtration $\mathsf{V}(\mathscr{M})$ is said to be *good* if (locally) there exist $I \in \mathbb{N}$, $\{u_i\}_{i=1}^I \subset \mathscr{M}$, and $\{k_i\}_{i=1}^I \in \mathbb{Z}$ such that for any k, the following holds:

$$\mathsf{V}^k(\mathscr{M}) = \sum_{i=1}^{I} \mathsf{V}^{k-k_i}_Y(\mathscr{D}_X)\,u_i\,.$$

We set $\mathsf{F}^p\mathsf{V}^k_Y(\mathscr{D}_X) := \mathsf{V}^k_Y(\mathscr{D}_X(p))$, and call $\mathsf{F}\mathsf{V}_Y(\mathscr{D}_X) := \{\mathsf{F}^p\mathsf{V}^k_Y(\mathscr{D}_X)\}_{(p,k)\in\mathbb{N}_0\times\mathbb{Z}}$ the *bi-filtration*. This enjoys the following properties:

(i) $\mathsf{F}^p\mathsf{V}^k_Y(\mathscr{D}_X) \subset \mathsf{F}^{p+q}\mathsf{V}^{k+l}_Y(\mathscr{D}_X)$ holds if $q,\ l \in \mathbb{N}_0$, and $\bigcup\limits_{p,k} \mathsf{F}^p\mathsf{V}^k_Y(\mathscr{D}_X) = \mathscr{D}_X$;

(ii) $\mathsf{F}^p\mathsf{V}^k_Y(\mathscr{D}_X)\,\mathsf{F}^q\mathsf{V}^l_Y(\mathscr{D}_X) \subset \mathsf{F}^{p+q}\mathsf{V}^{k+l}_Y(\mathscr{D}_X)$ holds if $p,\ q,\ k,\ l \in \mathbb{Z}$.

Definition 2.2. Let $\mathscr{M} \in \mathfrak{Coh}(\mathscr{D}_X|_Y)$. Then a *bi-filtration* $\mathsf{F}\mathsf{V}(\mathscr{M})$ on $\mathscr{M}$ is a family $\{\mathsf{F}^p\mathsf{V}^k(\mathscr{M})\}_{p,k\in\mathbb{Z}}$ of sub-Groups such that

(i) $\mathsf{F}^p\mathsf{V}^k(\mathscr{M}) \subset \mathsf{F}^{p+q}\mathsf{V}^{k+l}(\mathscr{M})$ holds if $q,\ l \in \mathbb{N}_0$, and $\bigcup\limits_{p,k} \mathsf{F}^p\mathsf{V}^k(\mathscr{M}) = \mathscr{M}$;

(ii) $\mathsf{F}^p\mathsf{V}^k_Y(\mathscr{D}_X)\,\mathsf{F}^q\mathsf{V}^l(\mathscr{M}) \subset \mathsf{F}^{p+q}\mathsf{V}^{k+l}(\mathscr{M})$ holds if $p,\ q,\ k,\ l \in \mathbb{Z}$.

Moreover, a bi-filtration $\mathsf{FV}(\mathscr{M})$ is said to be *good* if locally there exist $I \in \mathbb{N}$, $\{u_i\}_{i=1}^{I} \subset \mathscr{M}$, $\{p_i\}_{i=1}^{I}$, $\{k_i\}_{i=1}^{I} \subset \mathbb{Z}$ such that for any p, k, we have

$$\mathsf{F}^p\mathsf{V}^k(\mathscr{M}) = \sum_{i=1}^{I} \mathsf{F}^{p-p_i}\mathsf{V}_Y^{k-k_i}(\mathscr{D}_X)\, u_i\,.$$

Definition 2.3. $\mathscr{M} \in \mathfrak{Coh}(\mathscr{D}_X|_Y)$ is said to be *regular-specializable* if the following equivalent conditions are satisfied:

(1) there exist (locally) a coherent $\mathscr{O}_X$ sub-Module $\mathcal{L}$ of $\mathscr{M}$ and a non-zero polynomial $b(\alpha) \in \mathbb{C}[\alpha]$ such that $\mathscr{M} = \mathscr{D}_X\mathcal{L}$ and $b(\vartheta)\mathcal{L} \subset \mathsf{F}^{\deg b}\mathsf{V}_Y^{-1}(\mathscr{D}_X)\mathcal{L}$.

(2) there exist (locally) a good bi-filtration $\mathsf{FV}(\mathscr{M})$ and a non-zero polynomial $b(\alpha) \in \mathbb{C}[\alpha]$ such that for any p, $k \in \mathbb{Z}$, the following holds:

$$b(\vartheta + k)\, \mathsf{F}^p\mathsf{V}^k(\mathscr{M}) \subset \mathsf{F}^{p+\deg b}\mathsf{V}^{k-1}(\mathscr{M}).$$

(3) for any (local) section $u \in \mathscr{M}$, there exist a non-zero polynomial $b_u(\alpha) \in \mathbb{C}[\alpha]$ and $Q_u \in \mathsf{F}^{\deg b_u}\mathsf{V}_Y^{-1}(\mathscr{D}_X)$ such that $(b_u(\vartheta) - Q_u)\, u = 0$.

We denote by $\mathcal{R}_Y(\mathscr{D}_X) \subset \mathfrak{Coh}(\mathscr{D}_X|_Y)$ the subcategory consisting of regular-specializable $\mathscr{D}_X|_Y$-Modules.

Remark 2.4. (1) If Y is non-characteristic for $\mathscr{M} \in \mathfrak{Coh}(\mathscr{D}_X|_Y)$, then $\mathscr{M} \in \mathcal{R}_Y(\mathscr{D}_X)$.

(2) Every regular-holonomic $\mathscr{D}_X|_Y$-Module are regular-specializable (Kashiwara-Kawai [6]).

Proposition 2.5. *For any $\mathscr{M} \in \mathcal{R}_Y(\mathscr{D}_X)$, there exist a non-zero polynomial $b_Y(\alpha) \in \mathbb{C}[\alpha]$ (unique under the assumption that the degree is minimum) and a unique good V-filtration $\mathsf{V}_Y(\mathscr{M}) = \{\mathsf{V}_Y^k(\mathscr{M})\}_{k\in\mathbb{Z}}$ such that $b_Y^{-1}(0) \subset \{\sigma \in \mathbb{C};\, 0 \preccurlyeq \sigma \prec 1\}$ and for any $k \in \mathbb{Z}$*

$$b_Y(\vartheta + k)\, \mathsf{V}_Y^k(\mathscr{M}) \subset \mathsf{V}_Y^{k-1}(\mathscr{M}).$$

Here $\prec$ stands for the lexicographical order on $\mathbb{C} = \mathbb{R} + \sqrt{-1}\,\mathbb{R}$.

Definition 2.6. For any $\mathscr{M} \in \mathcal{R}_Y(\mathscr{D}_X)$ and $\mathsf{V}_Y(\mathscr{M})$ in Proposition 2.5, we set $\mathrm{Gr}_Y^k(\mathscr{M}) := \mathsf{V}_Y^k(\mathscr{M})/\mathsf{V}_Y^{k-1}(\mathscr{M})$. Then the *vanishing cycle* $\varPhi_Y(\mathscr{M})$ and *nearby cycle* $\varPsi_Y(\mathscr{M})$ are defined respectively by

$$\varPhi_Y(\mathscr{M}) := \mathrm{Gr}_Y^1(\mathscr{M}), \quad \varPsi_Y(\mathscr{M}) := \mathrm{Gr}_Y^0(\mathscr{M}).$$

It is known that $\mathrm{Gr}_Y^k(\mathscr{M}) \in \mathfrak{Coh}(\mathscr{D}_Y)$ for any $k \in \mathbb{Z}$, and for any $k \in \mathbb{N}_0$

$$\mathrm{Gr}_Y^{k+1}(\mathscr{M}) = \partial_\tau^k\, \varPhi_Y(\mathscr{M}), \quad \mathrm{Gr}_Y^{-k}(\mathscr{M}) = \tau^k\, \varPsi_Y(\mathscr{M}).$$

Proposition 2.7. *Let*

$$0 \to \mathscr{M}' \to \mathscr{M} \to \mathscr{M}'' \to 0 \tag{2.1}$$

be an exact sequence in $\mathfrak{Coh}(\mathscr{D}_X|_Y)$. *Then* $\mathscr{M} \in \mathcal{R}_Y(\mathscr{D}_X)$ *if and only if* $\mathscr{M}'$, $\mathscr{M}'' \in \mathcal{R}_Y(\mathscr{D}_X)$. *Further if* (2.1) *is an exact sequence in* $\mathcal{R}_Y(\mathscr{D}_X)$, *then for any* $k \in \mathbb{Z}$, *there exists an exact sequence in* $\mathfrak{Coh}(\mathscr{D}_Y)$:

$$0 \to \mathrm{Gr}_Y^k(\mathscr{M}') \to \mathrm{Gr}_Y^k(\mathscr{M}) \to \mathrm{Gr}_Y^k(\mathscr{M}'') \to 0.$$

Theorem 2.8. *If* $\mathscr{M} \in \mathcal{R}_Y(\mathscr{D}_X)$, *then* $\boldsymbol{D}\iota^*\mathscr{M}$, $\boldsymbol{D}\iota^!\mathscr{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_Y)$, *and there exist the following distinguished triangles*:

$$\Phi_Y(\mathscr{M}) \xrightarrow{\tau} \Psi_Y(\mathscr{M}) \longrightarrow \boldsymbol{D}\iota^*\mathscr{M} \xrightarrow{+1},$$
$$\boldsymbol{D}\iota^!\mathscr{M} \longrightarrow \Psi_Y(\mathscr{M}) \xrightarrow{\partial_\tau} \Phi_Y(\mathscr{M}) \xrightarrow{+1}.$$

Proposition 2.9. *Let* $\mathscr{M} \in \mathfrak{Coh}(\mathscr{D}_X|_Y)$, *and assume that* Y *is non characteristic for* $\mathscr{M}$. *Set* $D\iota^*\mathscr{M} := H^0\boldsymbol{D}\iota^*\mathscr{M}$. *Then* $\Phi_Y(\mathscr{M}) = 0$ *and*

$$\boldsymbol{D}\iota^!\mathscr{M} \simeq \Psi_Y(\mathscr{M}) \simeq \boldsymbol{D}\iota^*\mathscr{M} \simeq D\iota^*\mathscr{M}. \tag{2.2}$$

§ 3. Boundary Values for Solutions to Regular-Specializable System

We recall (1.3) and Theorem 2.8. Then:

Theorem 3.1 ([18], [26], cf. [25]). *For any* $\mathscr{M} \in \mathcal{R}_Y(\mathscr{D}_X)$, *there exists the following morphism of distinguished triangles*:

$$\begin{CD}
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_N(\mathscr{B}_M)) \otimes \mathscr{or}_{N/M} @>{\sim}>{\text{division}}> \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}\iota^!\mathscr{M}, \mathscr{B}_N)[-1] \\
@VVV @VVV \\
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_{M_+}(\mathscr{B}_M))|_N \otimes \mathscr{or}_{N/M} @>{\overline{\beta}}>> \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\Phi_Y(\mathscr{M}), \mathscr{B}_N) \\
@VVV @VVV \\
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_{\Omega_+}(\mathscr{B}_M))|_N @>{\beta}>> \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\Psi_Y(\mathscr{M}), \mathscr{B}_N) \\
@V{+1}VV @V{+1}VV
\end{CD} \tag{3.1}$$

and (3.1) *induces the following morphism of distinguished triangles*:

$$\begin{CD}
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_N(\mathscr{D}\!\mathscr{b}_M)) \otimes \mathscr{or}_{N/M} @>{\sim}>{\text{division}}> \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}\iota^!\mathscr{M}, \mathscr{D}\!\mathscr{b}_N)[-1] \\
@VVV @VVV \\
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_{M_+}(\mathscr{D}\!\mathscr{b}_M))|_N \otimes \mathscr{or}_{N/M} @>{\overline{\beta}^{\,t}}>> \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\Phi_Y(\mathscr{M}), \mathscr{D}\!\mathscr{b}_N) \\
@VVV @VVV \\
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma^t_{\Omega_+}(\mathscr{D}\!\mathscr{b}_M))|_N @>{\beta^t}>> \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\Psi_Y(\mathscr{M}), \mathscr{D}\!\mathscr{b}_N) \\
@V{+1}VV @V{+1}VV
\end{CD} \tag{3.2}$$

that is, (3.2) *is compatible with* (3.1) *under* (1.3). *Moreover,* $\overline{\beta}$ *and* β *induce monomorphisms*:

$$\begin{array}{ccc}
\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_{M_+}(\mathscr{D}b_M))|_N \otimes \mathit{or}_{N/M} & \overset{\overline{\beta}^{\,t,0}}{\rightarrowtail} & \mathscr{H}om_{\mathscr{D}_Y}(\Phi_Y(\mathscr{M}), \mathscr{D}b_N) \\
\cap & & \cap \\
\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_{M_+}(\mathscr{B}_M))|_N \otimes \mathit{or}_{N/M} & \overset{\overline{\beta}^{0}}{\rightarrowtail} & \mathscr{H}om_{\mathscr{D}_Y}(\Phi_Y(\mathscr{M}), \mathscr{B}_N),
\end{array}$$

$$\begin{array}{ccc}
\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma^t_{\Omega_+}(\mathscr{D}b_M))|_N & \overset{\beta^{t,0}}{\rightarrowtail} & \mathscr{H}om_{\mathscr{D}_Y}(\Psi_Y(\mathscr{M}), \mathscr{D}b_N) \\
\cap & & \cap \\
\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_{\Omega_+}(\mathscr{B}_M))|_N & \overset{\beta^{0}}{\rightarrowtail} & \mathscr{H}om_{\mathscr{D}_Y}(\Psi_Y(\mathscr{M}), \mathscr{B}_N).
\end{array}$$

Here, $\beta^i := H^i(\beta)$ *etc.*

Remark 3.2. (1) In (3.1), both $\overline{\beta}$ and β are isomorphisms under the near-hyperbolicity condition in the sense of Laurent-Monteiro Fernandes [15, Definition 1.3.1], and a microlocal counterpart of β is defined in Yamazaki [25] along the line of [20].

(2) For non-characteristic cases, see Remark 4.3.

Example 3.3. Let $b(\alpha) \in \mathbb{C}[\alpha]$ be a monic polynomial of degree m, and $Q \in \mathsf{F}^m\mathsf{V}_Y^{-1}(\mathscr{D}_X)$. Assume that $b(\alpha) = \prod_{i=1}^{\mu}(\alpha - \alpha_i)^{m_i}$ with $\alpha_i - \alpha_j \notin \mathbb{Z}$ $(i \neq j)$ and $\sum_{i=1}^{\mu} m_i = m$. We set $\mathscr{M} := \mathscr{D}_X/\mathscr{D}_X(b(\vartheta) - Q)$. Then we see that $\Psi_Y(\mathscr{M}) \simeq \mathscr{D}_Y^m$. We take the following local coordinates:

$$\begin{tikzcd}
N = \mathbb{R}^n_x \times \{0\} \arrow[r, hook] \arrow[d, hook] & M = \mathbb{R}^n_x \times \mathbb{R}_t \arrow[r, hook] \arrow[d, hook] & X = \mathbb{C}^n_z \times \mathbb{C}_\tau \\
Y = \mathbb{C}^n_z \times \{0\} \arrow[r, hook] & L = \mathbb{C}^n_z \times \mathbb{R}_t \arrow[ur, hook] &
\end{tikzcd}$$

and set $\widetilde{\mathscr{B}}_{N|M} := H^n_{T_N M}(\nu_Y(H^1_L(\mathscr{O}_X))) \otimes \mathit{or}_{N/L}$ (see [19], [20], [25]). Then there exists a monomorphism $\nu_N(\mathscr{B}_M) \rightarrowtail \widetilde{\mathscr{B}}_{N|M}$. Let us take any $v^* = (x_0; 1\,\dfrac{d}{dt}) \in \dot{T}_N M$ and $u(x,t) \in \mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \nu_N(\mathscr{B}_M))_{v^*}$. Then it is known (see [25], cf. [19]) that as a section of $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \widetilde{\mathscr{B}}_{N|M})_{v^*}$, $u(x,t)$ can be written as

$$u(x,t) = \sum_{i=1}^{\mu}\sum_{j=1}^{m_i}\sum_{\nu=1}^{\nu_{ij}} F^{\nu}_{ij}(x + \sqrt{-1}\,\Gamma^{\nu}_{ij}\,0, t)\, t^{\alpha_i}(\log t)^{j-1}.$$

Here each $F^{\nu}_{ij}(z,\tau)$ is holomorphic on a neighborhood of $\{(z,0) \in X;\ |x_0 - z| < \varepsilon,\ \operatorname{Im} z \in \Gamma^{\nu}_{ij}\}$ with a positive constant ε and an open convex cone $\Gamma^{\nu}_{ij} \subset \mathbb{R}^n$. Hence $u_{ij}(x) := \sum_{\nu=1}^{\nu_{ij}} F^{\nu}_{ij}(x + \sqrt{-1}\,\Gamma^{\nu}_{ij}\,0, 0) \in \mathscr{B}_{N,x_0}$ are well defined, and $\beta^0(u)$ is equivalent to

$$\{u_{ij}(x,0);\ 1 \leqslant i \leqslant \mu,\ 1 \leqslant j \leqslant m_i\} \subset \mathscr{B}^{\,m}_{N,x_0}.$$

We shall consider ultradistribution solutions. Locally we can write

$$\mathsf{F}^m\mathsf{V}_Y^{-1}(\mathscr{D}_X) = \{\sum_{|\alpha|+r\leqslant m} \tau^{v_{\alpha r}} q_{\alpha r}(z,\tau)\,\partial_z^\alpha\vartheta^r;\ v_{\alpha r}\in\mathbb{N}\}.$$

We need several notions due to Tahara [23]:

Definition 3.4. For $Q = \sum_{|\alpha|+r\leqslant m} \tau^{v_{\alpha r}} q_{\alpha r}(z,\tau)\,\partial_z^\alpha\vartheta^r \in \mathsf{F}^m\mathsf{V}_Y^{-1}(\mathscr{D}_X)$ ($v_{\alpha r}\in\mathbb{N}$, and $q_{\alpha r}(z,0)\neq 0$ if $q_{\alpha r}(z,\tau)\not\equiv 0$), we set

$$\mathcal{S}_T(Q) := \{v_{\alpha r}\in\mathbb{N};\ q_{\alpha r}(z,\tau)\not\equiv 0 \text{ and } |\alpha| - v_{\alpha r}\geqslant 1\},$$

$$\mathcal{I}_T(Q) := \begin{cases} \displaystyle\min_{v_{\alpha r}\in\mathcal{S}_T(Q)} \Big\{\frac{m-r-v_{\alpha r}}{|\alpha|-v_{\alpha r}}\Big\} & (\mathcal{S}_T(Q)\neq\emptyset),\\ \infty & (\mathcal{S}_T(Q)=\emptyset). \end{cases}$$

Let $m,\ d\in\mathbb{N}$. We shall consider the following square matrix of size d whose components belong to $\mathscr{D}_{X^{(m)}}$:

$$P(z,\tau,\partial_z,\partial_\tau) = P(z,\tau,\partial_z,\vartheta) = b(\vartheta)\mathbb{1}_d - Q(z,\tau,\partial_z,\vartheta). \tag{3.3}$$

Here $b(\vartheta)\in\mathbb{C}[\vartheta]$ with degree m, $\mathbb{1}_d$ stands for the identity matrix of size d, and each component Q_{ij} of $Q=(Q_{ij})_{i,j=1}^d$ belongs to $\mathsf{F}^m\mathsf{V}_Y^{-1}(\mathscr{D}_X)$. Set $\mathscr{M}_P := \mathscr{D}_X^d/\mathscr{D}_X^dP$. Then $\mathscr{M}_P\in\mathcal{R}_Y(\mathscr{D}_X)$.

Definition 3.5. *Tahara's index* for P is defined by

$$\mathcal{I}_T(P) := \min\{\mathcal{I}_T(Q_{ij});\ 1\leqslant i,j\leqslant d\}.$$

Remark 3.6. (1) Tahara [23] defined his index for Fuchsian operators in the sense of Baouendi-Goulaouic [1], and this index measures the difference of Fuchsian operators from operators with regular singularities due to Kashiwara-Oshima [7].

(2) Let ${}^{\mathsf{t}}P$ be the formal adjoint of P. Then we see that $\mathcal{I}_T(P)=\mathcal{I}_T({}^{\mathsf{t}}P)$.

Then we state our main theorem:

Theorem 3.7. *Assume that* $1<s\leqslant\mathcal{I}_T(P)$. *Then* (3.1) *induces the following morphism of distinguished triangles for* $* = \{s\}$ *or* (s):

$$\begin{array}{ccc} \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}_P, \Gamma_N(\mathscr{D}b_M^*))\otimes or_{N/M} & \xrightarrow[\text{division}]{\sim} & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}\iota^!\mathscr{M}_P, \mathscr{D}b_N^*)[-1] \\ \downarrow & & \downarrow \\ \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}_P, \Gamma_{M_+}(\mathscr{D}b_M^*))|_N\otimes or_{N/M} & \xrightarrow{\overline{\beta}^*} & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\Phi_Y(\mathscr{M}_P), \mathscr{D}b_N^*) \\ \downarrow & & \downarrow \\ \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}_P, \Gamma^{\mathrm{ext}}_{\Omega_+}(\mathscr{D}b_M^*))|_N & \xrightarrow{\beta^*} & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\Psi_Y(\mathscr{M}_P), \mathscr{D}b_N^*) \\ \downarrow{\scriptstyle +1} & & \downarrow{\scriptstyle +1} \end{array} \tag{3.4}$$

that is, (3.4) *is compatible with* (3.1) *under* (1.3).

By virtue of division theorem, the construction of morphisms in (3.4) are same as in [17], [26].

Remark 3.8. (1) We see that $\Psi_Y(\mathscr{M}_P) \simeq \mathscr{D}_Y^{md}$ and $\Phi_Y(\mathscr{M}_P) \simeq \mathscr{D}_Y^{md}$. Thus there exist the following commutative diagrams:

$$\begin{array}{ccc}
\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}_P, \Gamma_{M_+}(\mathscr{D}\ell_M))|_N \otimes or_{N/M} & \xrightarrow{\overline{\beta}^{\,t,0}} & \mathscr{D}\ell_N^{\oplus md} \\
\cap & & \cap \\
\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}_P, \Gamma_{M_+}(\mathscr{D}\ell_M^*))|_N \otimes or_{N/M} & \xrightarrow{\overline{\beta}^{*,0}} & (\mathscr{D}\ell_N^*)^{\oplus md} \\
\cap & & \cap \\
\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}_P, \Gamma_{M_+}(\mathscr{B}_M))|_N \otimes or_{N/M} & \xrightarrow{\overline{\beta}^{0}} & \mathscr{B}_N^{\oplus md},
\end{array}$$

$$\begin{array}{ccc}
\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}_P, \Gamma^t_{\Omega_+}(\mathscr{D}\ell_M))|_N & \xrightarrow{\beta^{t,0}} & \mathscr{D}\ell_N^{\oplus md} \\
\cap & & \cap \\
\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}_P, \Gamma^{\mathrm{ext}}_{\Omega_+}(\mathscr{D}\ell_M^*))|_N & \xrightarrow{\beta^{*,0}} & (\mathscr{D}\ell_N^*)^{\oplus md} \\
\cap & & \cap \\
\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}_P, \Gamma_{\Omega_+}(\mathscr{B}_M))|_N & \xrightarrow{\beta^{0}} & \mathscr{B}_N^{\oplus md},
\end{array}$$

and if $i \neq 0$, then $\overline{\beta}^{\,t,i} = \overline{\beta}^{*,i} = \overline{\beta}^{i} = \beta^{t,i} = \beta^{*,i} = \beta^i = 0$.

(2) If $1 < s \leqslant \mathcal{I}_T(P) < \infty$, then this condition can be written as

$$\max\Big\{1, \max\big\{\frac{m - v^{ij}_{\alpha r} - r}{m - |\alpha| - r};\ v^{ij}_{\alpha r} \in \mathcal{S}_T(Q_{ij}),\ 1 \leqslant i, j \leqslant d\big\}\Big\} \leqslant \frac{s}{s-1}.$$

Therefore, we can regard this condition as a counterpart of an irregularity condition for ordinary differential equation.

§ 4. Case of General Regular Specializable Systems

Let $\mathscr{M} \in \mathcal{R}_Y(\mathscr{D}_X)$. Then there exists locally an epimorphism

$$\mathscr{L}_0 \to \mathscr{M} \to 0. \tag{4.1}$$

Here $\mathscr{L}_0 = \bigoplus_{\nu=1}^{\nu_0} \mathscr{M}_{P_0^\nu}$ and each $P_0^\nu = b_0^\nu(\vartheta)\mathbb{1}_{d_0^\nu} - Q_0^\nu(z, \tau, \partial_z, \vartheta)$ is of the type (3.3). We set $I := \sum_{\nu=1}^{\nu_0} d_0^\nu \deg b_0^\nu$. Then first we have the following partial result:

Proposition 4.1. *Under* (4.1), *assume that* $1 < s \leqslant \min\{\mathcal{I}_T(P_0^\nu);\ 1 \leqslant \nu \leqslant \nu_0\}$. *Then* $\overline{\beta}^{*,0}$ *and* $\beta^{*,0}$ *induce the following monomorphisms respectively for* $* = \{s\}$ *or* (s):

$$\overline{\beta}^{*,0}\colon \mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_{M_+}(\mathscr{D}\ell_M^*))|_N \otimes or_{N/M} \rightarrowtail \mathscr{H}om_{\mathscr{D}_Y}(\Phi_Y(\mathscr{M}), \mathscr{D}\ell_N^*),$$

$$\beta^{*,0}\colon \mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma^{\mathrm{ext}}_{\Omega_+}(\mathscr{D}\ell_M^*))|_N \rightarrowtail \mathscr{H}om_{\mathscr{D}_Y}(\Psi_Y(\mathscr{M}), \mathscr{D}\ell_N^*).$$

In particular, there exist the following:

$$\begin{array}{ccc}
\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \varGamma_{M_+}(\mathscr{D}b_M))|_N \otimes or_{N/M} & \xrightarrow{\overline{\beta}^{\,t,0}} & \mathscr{H}om_{\mathscr{D}_Y}(\varPhi_Y(\mathscr{M}), \mathscr{D}b_N) \\
\cap & & \cap \\
\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \varGamma_{M_+}(\mathscr{D}b_M^*))|_N \otimes or_{N/M} & \xrightarrow{\overline{\beta}^{*,0}} & \mathscr{H}om_{\mathscr{D}_Y}(\varPhi_Y(\mathscr{M}), \mathscr{D}b_N^*) \\
\cap & & \cap \\
\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \varGamma_{M_+}(\mathscr{B}_M))|_N \otimes or_{N/M} & \xrightarrow{\overline{\beta}^{0}} & \mathscr{H}om_{\mathscr{D}_Y}(\varPhi_Y(\mathscr{M}), \mathscr{B}_N),
\end{array}$$

$$\begin{array}{ccc}
\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \varGamma^t_{\Omega_+}(\mathscr{D}b_M))|_N & \xrightarrow{\beta^{t,0}} & \mathscr{H}om_{\mathscr{D}_Y}(\varPsi_Y(\mathscr{M}), \mathscr{D}b_N) \\
\cap & & \cap \\
\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \varGamma^{\mathrm{ext}}_{\Omega_+}(\mathscr{D}b_M^*))|_N & \xrightarrow{\beta^{*,0}} & \mathscr{H}om_{\mathscr{D}_Y}(\varPsi_Y(\mathscr{M}), \mathscr{D}b_N^*) \\
\cap & & \cap \\
\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \varGamma_{\Omega_+}(\mathscr{B}_M))|_N & \xrightarrow{\beta^{0}} & \mathscr{H}om_{\mathscr{D}_Y}(\varPsi_Y(\mathscr{M}), \mathscr{B}_N).
\end{array}$$

Next, for any $\mathscr{M} \in \mathcal{R}_Y(\mathscr{D}_X)$, there exists locally an exact sequence

$$0 \to \mathscr{K} \to \mathscr{L}_{n+2} \to \cdots \to \mathscr{L}_1 \to \mathscr{L}_0 \to \mathscr{M} \to 0. \tag{4.2}$$

Here $\mathscr{K} \in \mathcal{R}_Y(\mathscr{D}_X)$ and $\mathscr{L}_i = \bigoplus_{\nu=1}^{\nu_i} \mathscr{M}_{P_i^\nu}$ and each P_i^ν is of the type (3.3). Then we have

$$\mathscr{K}[n+2] \to \mathscr{L}_\bullet \to \mathscr{M} \xrightarrow{+1} \quad \text{where} \quad \mathscr{L}_\bullet := (\mathscr{L}_{n+2} \to \cdots \to \mathscr{L}_0).$$

As a special case, if there exists locally an exact sequence instead of (4.2):

$$0 \to \mathscr{L}_\mu \to \cdots \to \mathscr{L}_1 \to \mathscr{L}_0 \to \mathscr{M} \to 0, \tag{4.3}$$

where $\mu \leqslant n+1$ and each $\mathscr{L}_i$ is same as in (4.2), then we have

$$\mathscr{L}_\bullet := (\mathscr{L}_\mu \to \cdots \to \mathscr{L}_0) = \mathscr{M}.$$

Under this notation we set

$$\mathcal{I}_T(\mathscr{M}) := \begin{cases} \min\{\mathcal{I}_T(P_i^\nu);\ 1 \leqslant \nu \leqslant \nu_i,\ 0 \leqslant i \leqslant n+2\} & \text{(the case of (4.2))}, \\ \min\{\mathcal{I}_T(P_i^\nu);\ 1 \leqslant \nu \leqslant \nu_i,\ 0 \leqslant i \leqslant \mu\} & \text{(the case of (4.3))}. \end{cases}$$

Then Theorem 3.7 is generalized as follows:

Theorem 4.2. *Assume that* $1 < s \leqslant \mathcal{I}_T(\mathscr{M})$. *Then* (3.1) *induces the following morphism of distinguished triangles for* $* = \{s\}$ *or* (s):

$$\begin{array}{ccc}
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \varGamma_N(\mathscr{D}b_M^*)) \otimes or_{N/M} & \xrightarrow[\textit{division}]{\sim} & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}\iota^!\mathscr{M}, \mathscr{D}b_N^*)[-1] \\
\downarrow & & \downarrow \\
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \varGamma_{M_+}(\mathscr{D}b_M^*))|_N \otimes or_{N/M} & \xrightarrow{\overline{\beta}^*} & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\varPhi_Y(\mathscr{M}), \mathscr{D}b_N^*) \\
\downarrow & & \downarrow \\
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \varGamma^{\mathrm{ext}}_{\Omega_+}(\mathscr{D}b_M^*))|_N & \xrightarrow{\beta^*} & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\varPsi_Y(\mathscr{M}), \mathscr{D}b_N^*) \\
\downarrow{\scriptstyle +1} & & \downarrow{\scriptstyle +1}
\end{array} \tag{4.4}$$

that is, (4.4) *is compatible with* (3.1) *under* (1.3). *In particular, the monomorphisms* $\overline{\beta}^0$ *and* β^0 *in Theorem* 3.1 *induce the following monomorphisms for* $* = \{s\}$ *or* (s):

$$
\begin{array}{ccc}
\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_{M_+}(\mathscr{D}b_M))|_N \otimes \mathscr{or}_{N/M} & \xrightarrow{\overline{\beta}^{\,t,0}} & \mathscr{H}om_{\mathscr{D}_Y}(\Phi_Y(\mathscr{M}), \mathscr{D}b_N) \\
\cap & & \cap \\
\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_{M_+}(\mathscr{D}b^*_M))|_N \otimes \mathscr{or}_{N/M} & \xrightarrow{\overline{\beta}^{*,0}} & \mathscr{H}om_{\mathscr{D}_Y}(\Phi_Y(\mathscr{M}), \mathscr{D}b^*_N) \\
\cap & & \cap \\
\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_{M_+}(\mathscr{B}_M))|_N \otimes \mathscr{or}_{N/M} & \xrightarrow{\overline{\beta}^{0}} & \mathscr{H}om_{\mathscr{D}_Y}(\Phi_Y(\mathscr{M}), \mathscr{B}_N),
\end{array}
$$

$$
\begin{array}{ccc}
\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma^t_{\Omega_+}(\mathscr{D}b_M))|_N & \xrightarrow{\beta^{t,0}} & \mathscr{H}om_{\mathscr{D}_Y}(\Psi_Y(\mathscr{M}), \mathscr{D}b_N) \\
\cap & & \cap \\
\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma^{\mathrm{ext}}_{\Omega_+}(\mathscr{D}b^*_M))|_N & \xrightarrow{\beta^{*,0}} & \mathscr{H}om_{\mathscr{D}_Y}(\Psi_Y(\mathscr{M}), \mathscr{D}b^*_N) \\
\cap & & \cap \\
\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_{\Omega_+}(\mathscr{B}_M))|_N & \xrightarrow{\beta^{0}} & \mathscr{H}om_{\mathscr{D}_Y}(\Psi_Y(\mathscr{M}), \mathscr{B}_N).
\end{array}
$$

Remark 4.3. Let $\mathscr{M} \in \mathfrak{Coh}(\mathscr{D}_X|_Y)$ and assume that Y is non-characteristic for $\mathscr{M}$. Then there exist the following (recall (2.2)):

(4.5)
$$
\begin{tikzcd}
& \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma^t_{\Omega_+}(\mathscr{D}b_M))|_N \arrow[dl] \arrow[dr, "\beta^t"] \arrow[dd] & \\
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_N(\mathscr{D}b_M)) \otimes \omega^{\otimes -1}_{N/M} \arrow[rr, equal] \arrow[dd] & & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(D\iota^*\mathscr{M}, \mathscr{D}b_N) \arrow[dd] \\
& \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma^{\mathrm{ext}}_{\Omega_+}(\mathscr{D}b^*_M))|_N \arrow[dl] \arrow[dr, "\beta^*"] \arrow[dd] & \\
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_N(\mathscr{D}b^*_M)) \otimes \omega^{\otimes -1}_{N/M} \arrow[rr, equal] \arrow[dd] & & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(D\iota^*\mathscr{M}, \mathscr{D}b^*_N) \arrow[dd] \\
& \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_{\Omega_+}(\mathscr{B}_M))|_N \arrow[dl] \arrow[dr, "\beta"] & \\
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_N(\mathscr{B}_N)) \otimes \omega^{\otimes -1}_{N/M} \arrow[rr, equal] & & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(D\iota^*\mathscr{M}, \mathscr{B}_N)
\end{tikzcd}
$$

and (4.5) induces the following monomorphisms:

$$
\begin{array}{ccc}
\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma^t_{\Omega_+}(\mathscr{D}b_M))|_N & \xrightarrow{\beta^{t,0}} & \mathscr{H}om_{\mathscr{D}_Y}(D\iota^*\mathscr{M}, \mathscr{D}b_N) \\
\cap & & \cap \\
\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma^{\mathrm{ext}}_{\Omega_+}(\mathscr{D}b^*_M))|_N & \xrightarrow{\beta^{*,0}} & \mathscr{H}om_{\mathscr{D}_Y}(D\iota^*\mathscr{M}, \mathscr{D}b^*_N) \\
\cap & & \cap \\
\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_{\Omega_+}(\mathscr{B}_M))|_N & \xrightarrow{\beta^{0}} & \mathscr{H}om_{\mathscr{D}_Y}(D\iota^*\mathscr{M}, \mathscr{B}_N)
\end{array}
$$

and β^0 coincides with boundary value morphism due to Komatsu-Kawai and Schapira in the single equation case (see Komatsu [12]).

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