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The non-integrability of some system of fifth-order partial differential equations describing surfaces containing 6 families of circles

By

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Abstract

We prove that a surface germ $z = f(x, y)$ at the origin in $\mathbb{R}^3$ is a cyclide, if it contains some six continuous families of circular arcs through every point and it satisfies some conditions on the third order derivatives of f at the origin. This result was conjectured by Takeuchi in [6]: **A surface in $\mathbb{R}^3$ is a cyclide if it contains two circles through almost every point.** Our proof is based on the non-integrability of the corresponding system of fifth-order partial differential equations introduced in [3], and so we do not use any global information on the surfaces, for example, our result is free from the genus and the closedness of surfaces. The detailed results with proofs will be published in [4].

A general cyclide (Darboux cyclide [2]) is defined by a quartic equation

$$\alpha(x_1^2 + x_2^2 + x_3^2)^2 + 2(x_1^2 + x_2^2 + x_3^2)\sum_{i=1}^{3}\beta_i x_i + \sum_{i,j=1}^{3}\gamma_{ij}x_i x_j + 2\sum_{i=1}^{3}\delta_i x_i + \epsilon = 0 \tag{0.1}$$

with real numbers $\alpha, \beta_i, \gamma_{ij}, \delta_i, \epsilon$. Then a usual torus and a 6-circle Blum cyclide ([1]) correspond to the case $\alpha = 1, \beta_* = 0, \delta_* = 0, \gamma_{ij} = -2a_i\delta_{ij}, \epsilon = \ell^2$ with $0 < \ell < a_1 = a_2$, $a_3 = -\ell$, and to that with $0 < \ell < a_2 < a_1$, $-\ell \neq a_3 < \ell$, respectively.

We consider a surface germ $M : z = f(x, y)$ at the origin of $\mathbb{R}^3$, where f is a C^5-class function defined in a neighborhood of $x = y = 0$ satisfying the following conditions:

$$f(0,0) = f_x(0,0) = f_y(0,0) = f_{xy}(0,0) = 0, \tag{0.2}$$

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and

$$f_{xx}(0,0) - f_{yy}(0,0) \neq 0. \tag{0.3}$$

Indeed, if the origin is not an umbilical point of a C^2-surface M, we can take such a Euclidean coordinate system x, y, z of $\mathbb{R}^3$.

Before stating our main result, we introduce conformal transformations, which have a great importance in our proof of the main result.

Definition 0.1. A conformal transformation in $\mathbb{R}^3$ is a finite composition of translations, rotations, and the inversions $\vec{x} = \lambda\vec{y}/|\vec{y}|^2$ $(\lambda > 0)$ in $\mathbb{R}^3$. As a result, reflections and dilations are conformal transformations. Two surface germs M, M' at $p \in \mathbb{R}^3$ are said to be conformally equivalent to each other if there is a conformal transformation F with $F(p) = p$ such that $F(M) = M'$ as a surface germ.

Remark. A conformal transformation maps a sphere (or (a plane)$\cup\{\infty\}$) onto a sphere or (a plane)$\cup\{\infty\}$: As a result, a circle (or (a line)$\cup\{\infty\}$) onto a circle or (a line)$\cup\{\infty\}$. Hence the total number of circles and lines passing through a point on a surface is preserved under a conformal transformation. Further it is well-known that a general cyclide is transformed into another general cyclide by any conformal transformation. N. Takeuchi [6] proved that any general cyclide is conformally equivalent to a cyclide of the following type:

$$\alpha(x_1^2 + x_2^2 + x_3^2)^2 + \sum_{i}^{3} \gamma_{ii} x_i^2 + \epsilon = 0. \tag{0.4}$$

We denote by a, b, c_*, d_*, e_* the higher order derivatives of $f(x, y)$ and by $P(t)$ the characteristic polynomial for a surface germ $z = f(x, y)$, which will be given in Definition 1.1. Then, our main result is the following:

Theorem 0.2. *Let $z = f(x, y)$ be a C^5-class function defined in a neighborhood of $x = y = 0$ satisfying conditions* (0.2), (0.3)*. Let $P(t)$ be the characteristic polynomial at the origin for the surface germ $M : z = f(x, y)$. Suppose that $P(t) = 0$ has 6 distinct non-zero real roots $\{t_k\}_{k=1}^6$, and that M includes 6 continuous families of circular arcs corresponding to $\{t_k\}_{k=1}^6$. Then, f is analytic at $x = y = 0$, and f is completely determined by the* 11 *higher order derivatives c_0, c_2, d_*, e_* of f at $(0, 0)$. Furthermore, suppose the following additional conditions*

$$d_0 = d_1 = d_2 = d_3 = 0 \quad \textit{at}\ (0,0)$$

with some generic conditions on c_0, c_2, e_0, e_2, e_4. Then we obtain that $e_1 = e_3 = 0$ at $(0, 0)$, and that the surface germ $z = f(x, y)$ at $x = y = 0$ is a general cyclide. More *precisely, M is conformally equivalent to a germ at $(0,0,*)$ of the following 6-circle Blum cyclide:*

$$(x^2+y^2+z^2)^2 - 2a_1x^2 - 2a_2y^2 - 2a_3z^2 + a_4^2 = 0, \tag{0.5}$$

where $a_1 > a_3 > a_4 > 0$, $-a_2 > a_4$. Further this surface (0.5) *has the same characteristic roots $\{t_k\}_{k=1}^6$.*

Our plan of this paper is the following: In Section 1, we recall the results in [3] (and the complete version [4]) with some necessary definitions, $P(t), c_0, c_2, d_*, e_*$ for M. Section 2 is devoted to show some explicit calculations for the Blum 6-circle cyclides and to give a rough sketch of the proof of Theorem 0.2. In particular, the explicit calculations of $T(x,y)$ in Proposition 2.4 are the original results in this paper.

§ 1. The system of fifth-order partial differential equations which describes surfaces including several continuous families of circles

Definition 1.1. (The key polynomial $Z(T)$). Let $z = f(x,y)$ be a C^4-class function defined in a neighborhood of $(0,0) \in \mathbb{R}^2$. Put the Taylor coefficients of f at (x,y) as follows:

$$\begin{cases} a := f_x(x,y),\ b := f_y(x,y), \\ c_0 := f_{xx}(x,y)/2,\ c_1 := f_{xy}(x,y),\ c_2 := f_{yy}(x,y)/2, \\ d_0 := f_{xxx}(x,y)/3!,\ d_1 := f_{xxy}(x,y)/2!, \\ d_2 := f_{xyy}(x,y)/2!,\ d_3 := f_{yyy}(x,y)/3!, \\ e_0 := f_{xxxx}(x,y)/4!,\ e_1 := f_{xxxy}(x,y)/3!,\ e_2 := f_{xxyy}(x,y)/2!^2, \\ e_3 := f_{xyyy}(x,y)/3!,\ e_4 := f_{yyyy}(x,y)/4!. \end{cases}$$

We define some polynomials $C(T), D(T), E(T), R(T), S(T), K(T), W(T)$ and the key polynomial $Z(T)$ in T as follows:

$$\begin{aligned} C(T) &:= c_0 + c_1T + c_2T^2, \quad D(T) := d_0 + d_1T + d_2T^2 + d_3T^3, \\ E(T) &:= e_0 + e_1T + e_2T^2 + e_3T^3 + e_4T^4, \\ R(T) &:= (b^2+1)T^2 + 2abT + a^2 + 1, \\ S(T) &:= D(T)R(T) - 2(bT+a)C(T)^2, \\ K(T) &:= R'(T)C(T) - R(T)C'(T), \\ W(T) &:= bS(T) + C(T)K(T) = 2TC(T)^2 + (bD(T) - C'(T)C(T))R(T), \end{aligned}$$

where $C'(T) = \partial_T C(T), R'(T) = \partial_T R(T)$,...etc..

$$Z(T) \equiv Z(T;x,y) :=$$

$$
\begin{aligned}
&K(T)^2\big(R(T)E(T) - C(T)^3\big) + R(T)K(T)D(T)\big(D'(T)R(T) \\
&- 3(b^2+1)TD(T)\big) + D(T)^2R(T)[-ab\big(2K(T) + TK'(T)\big) \\
&- 2(a^2+1)(b^2+1)C(T) + \big((a^2+1)c_2 + (b^2+1)c_0\big)R(T)] \\
&+ 2R(T)C(T)[(bT+a)\{D(T)K'(T)C(T) + D(T)K(T)C'(T) \\
&- D'(T)K(T)C(T)\} - bD(T)C(T)K(T)] \\
&+ 4C(T)^4(bT+a)\{\big((a^2-1)c_2 + (b^2+1)c_0\big)(bT+a) \\
&- \frac{1}{2}ac_1R'(T) + 2a(c_2 - c_0) - bc_1\}.
\end{aligned}
$$

It is easy to verify that (the degree of $Z(T)$ in T) ≤ 10. Consider a C^4-surface germ $M = \{z = f(x,y)\}$ at $(0,0,f(0,0))$. We call

$$P(t) := Z(t;0,0)/(c_0(0,0) - c_2(0,0))$$

the characteristic polynomial of the surface germ M if conditions (0.2), (0.3) are satisfied. Since $a = b = c_1 = 0$ at $(0,0)$, we have a more simplified form:

$$
\begin{aligned}
P(t) =&(t^2+1)\mathring{D}(t)\{2t(t^2+1)\mathring{D}'(t) - (5t^2+1)\mathring{D}(t)\} \\
&+ 4(\mathring{c}_0 - \mathring{c}_2)t^2\{(t^2+1)\mathring{E}(t) - \mathring{C}(t)^3\},
\end{aligned}
\tag{1.1}
$$

where $\mathring{c}_j := c_j(0,0), \mathring{C}(t) := C(t;0,0)$, etc.. Moreover, under the additional condition

$$d_0(0,0) = d_1(0,0) = d_2(0,0) = d_3(0,0) = 0,$$

we have $P(t) = 4(\mathring{c}_0 - \mathring{c}_2)t^2Q(t)$ with a polynomial $Q(t)$ of degree 6, which we call *the reduced characteristic polynomial*:

$$Q(t) := (t^2+1)\mathring{E}(t) - \mathring{C}(t)^3. \tag{1.2}$$

Then we can introduce the main theorems of [3], [4]:

Theorem 1.2. *Let $z = f(x,y)$ be a C^4-function defined in $U_{\delta_0} = \{x^2 + y^2 < \delta_0^2\}$ $(\delta_0 > 0)$ satisfying (0.2), (0.3). Then we have the following (i), (ii), (iii).*

(*i*) *Let $t_0, s_0 \in \mathbb{R}$. If*

$$M \cap \{y = t_0x + s_0z\} \tag{1.3}$$

is a circular arc or a line segment in a neighborhood of the origin, then

$$Z(t_0;0,0) = 0.$$

Further, if it is a circular arc, then $C(t_0;0,0) \neq 0$ *and under an additional condition* $t_0 \neq 0$ *we have*

$$s_0 = \frac{(t_0^2+1)D(t_0;0,0)}{2(c_0(0,0)-c_2(0,0))t_0C(t_0;0,0)}.$$

If it is a line segment, then $C(t_0;0,0) = D(t_0;0,0) = E(t_0;0,0) = 0$.

(*ii*) *Let* $t(x,y), s(x,y)$ *be real-valued continuous functions defined in a neighborhood of* $(0,0)$ *such that, for some* $\delta > 0$ *and any* $(x_0,y_0) \in U_\delta$*, the set*

$$M \cap \{y - y_0 = t(x_0,y_0)(x-x_0) + s(x_0,y_0)(z - f(x_0,y_0))\} \tag{1.4}$$

coincides with a circle in a neighborhood of $(x_0,y_0,f(x_0,y_0))$. *Assume that* $t(0,0) \neq 0$. *Consider a continuous function*

$$T(x,y) := \frac{t(x,y) + f_x(x,y)s(x,y)}{1 - f_y(x,y)s(x,y)} \tag{1.5}$$

defined in a neighborhood of $(0,0)$. *Then,* $T(x,y), s(x,y), t(x,y)$ *satisfy the following equations:*

$$Z(T(x,y);x,y) = 0, \tag{1.6}$$

$$s(x,y) = \frac{S(T)}{W(T)}, \quad t(x,y) = \frac{TK(T)C(T) - aS(T)}{W(T)}. \tag{1.7}$$

Moreover, if $t(x,y), s(x,y)$ *are constant on each circular arc* (1.4)*,* f *is a* C^5*-function in* U_{δ_0} *and* $Z'(t(0,0);0,0) \neq 0$*, then* $T(x,y)$ *is a* C^1*-function in a neighborhood of the origin satisfying the following equation:*

$$(\partial_x + T(x,y)\partial_y)T(x,y) = \frac{2S(T)}{K(T)}. \tag{1.8}$$

(*iii*) *Conversely, let* $f(x,y) \in C^5(U_{\delta_0})$*, and let* $T(x,y)$ *be a real-valued* C^1*-function defined in a neighborhood of* $(0,0)$ *satisfying* $T(0,0) \neq 0$*, equations* (1.6),(1.8)*. Then,* $t(x,y)$*,* $s(x,y)$ *defined by* (1.7) *belong to* $C^1(U_\delta)$ *for a small* $\delta > 0$*, and satisfy that, for any* $(x_0,y_0) \in U_\delta$*, the set*

$$M \cap \{y - y_0 = t(x_0,y_0)(x-x_0) + s(x_0,y_0)(z - f(x_0,y_0))\}$$

coincides with a circle in a neighborhood of $(x_0,y_0,f(x_0,y_0))$*, and that* $t(x,y), s(x,y)$ *are constant on this circular arc.*

Though equation (1.8) looks like a first order PDE, this is a fifth-order PDE for $f(x,y)$:

$$\sum_{j=0}^{5}\binom{5}{j}T^j\partial_x^{5-j}\partial_y^j f(x,y)=\frac{24N(T)}{R(T)K(T)^3}. \tag{1.9}$$

This is because T is an analytic function of $\nabla f, \nabla^2 f, \nabla^3 f, \nabla^4 f$ through $Z(T)=0$. Here $N(T)$ is a polynomial in T of degree 14 defined by

$$\begin{aligned}N(T):=&-K(T)\left((\partial_x+T\partial_y)Z(T)-\frac{K(T)^2R(T)}{24}\sum_{j=0}^{5}\binom{5}{j}T^j\partial_x^{5-j}\partial_y^j f\right)\\&-2S(T)Z'(T),\end{aligned} \tag{1.10}$$

where $(\partial_x+T\partial_y)Z(T):=(\partial_x+T\partial_y)Z(T;x,y)$ means a differentiation for each coefficient of $Z(T)$. It is easy to see that the degree of $N(T)$ in T is at most 14. As we give an explicit form of $N(T)$ in Proposition 1.3, all the coefficients of $N(T)$ are polynomials of derivatives a,b,c_*,d_*,e_* of $f(x,y)$ introduced in Definition 1.1.

Proposition 1.3. *We have the following expression of polynomial $N(T)$ introduced at* (1.10)*. In particular, the degree of $N(T)$ in T is* 14.

$$\begin{aligned}N(T)=&-5R(T)K(T)^2E'(T)\big[R(T)D(T)-2(bT+a)C(T)^2\big]\\&+D(T)^3R(T)B_1(T)+2D(T)^2D'(T)R(T)^2B_2(T)\\&-D(T)^2R(T)^2K(T)\big[(3d_3T+d_2)\big(5R(T)-(b^2+1)T^2\big)\\&+(d_1T+3d_0)(b^2+1)\big]+D(T)^2B_3(T)\\&+2(bT+a)D(T)D'(T)R(T)C(T)B_4(T)\\&+10(bT+a)D(T)D''(T)R(T)^2K(T)C(T)^2+D(T)B_5(T)\\&-4(bT+a)D'(T)R(T)K(T)C(T)^3\big[5(bT+a)C'(T)+2bC(T)\big]\\&+4(bT+a)C(T)^4K(T)\Big[3d_0B_6(T)+d_1B_7(T)+d_2B_8(T)\\&-3d_3TB_9(T)\Big]+4C(T)^4B_{10}(T).\end{aligned}$$

Here, a,b,c_*,d_*,e_* are the higher-order derivatives of $f(x,y)$, and $C(T)$, $C'(T),D(T),D'(T),D''(T),E(T),E'(T),R(T),R'(T)$, $K(T),K'(T)$ are the polynomials in T (or their derivatives in T) with coefficients in polynomials in a,b,c_*,d_*,e_*, which are introduced in Definition 1.1. Further, $B_1(T),...,B_{10}(T)$ are the polynomials in T,a,b,c_* given by the following:

$$B_1(T):=K(T)\big\{42(b^2+1)R(T)-20(a^2+b^2+1)\big\}$$

$$
\begin{aligned}
&- 4c_2R(T)\big\{3abR(T) - 2(a^2+b^2+1-a^2b^2)T + 2ab(a^2+1)\big\}\\
&+ 2(b^2+1)c_1R(T)\{3R(T) + 4abT + 4a^2 + 4\}\\
&- 4(b^2+1)R(T)R'(T)c_0,\\
B_2(T) &:= -\big(4(b^2+1)T + 5ab\big)K(T) - \big((b^2+1)T^2 + a^2 + 1\big)K'(T)\\
&+ 4(a^2+1)(b^2+1)C(T) - 2\big((a^2+1)c_2 + (b^2+1)c_0\big)R(T),\\
B_3(T) &:= K(T)^2\big\{-4aR(T)C'(T) + 12(b^2+1)(bT+a)TC(T)\\
&+ 8ab(bT+a)C(T) + 12bR(T)C(T) + 2bR(T)(2c_0 + c_1T)\big\}\\
&+ K(T)K'(T)\big\{4abT(bT+a)C(T) - 12(bT+a)R(T)C(T)\\
&+ bTR(T)(2c_0 + c_1T) + aTR(T)C'(T)\big\}\\
&+ K(T)\big\{-4a\big(b^2+6\big)R(T)C(T)^2 - \big(18(b^2+1)T(bT+a)\\
&+ 16ab(bT+a) - 4b(a^2+1)\big)R(T)C(T)C'(T)\\
&+ 8(a^2+1)(b^2+1)(bT+a)C(T)^2 - 8(bT+a)\big((a^2+1)c_2\\
&+ (b^2+1)c_0\big)R(T)C(T) + 4abT(bc_1 - 2ac_2)R(T)C(T)\\
&+ 4a(b^2+1)(2c_0 + c_1T)R(T)C(T) - 2ac_2(2c_0 + c_1T)R(T)^2\\
&- 2bc_0C'(T)R(T)^2 - 12(b^2+1)(bT+a)TR'(T)C(T)^2\\
&- 8ab(bT+a)R'(T)C(T)^2 - 4(bT+a)R(T)R'(T)C(T)C'(T)\\
&+ 4bR(T)R'(T)C(T)^2 - 4(bT+a)R(T)^2C'(T)^2\\
&- 8c_2(bT+a)R(T)^2C(T) + 4bR(T)^2C(T)C'(T)\big\}\\
&- 2(bT+a)K'(T)C(T)\big\{2abTR'(T)C(T) + 5R'(T)R(T)C(T)\\
&+ 6R(T)^2C'(T)\big\} - 4(bT+a)K''(T)R(T)C(T)^2\big(R(T) + abT\big)\\
&- 8(a^2+1)(b^2+1)(bT+a)R'(T)C(T)^3 + 8(bT+a)\big((a^2+1)c_2\\
&+ (b^2+1)c_0\big)R(T)R'(T)C(T)^2\\
&- 8(a^2+1)(b^2+1)(bT+a)R(T)C(T)^2C'(T),\\
B_4(T) &:= 2\big(3(b^2+1)T + 5ab\big)K(T)C(T) + 5R(T)K(T)C'(T)\\
&+ 2\big((b^2+1)T^2 + a^2 + 1\big)K'(T)C(T) - 8(a^2+1)(b^2+1)C(T)^2\\
&+ 4R(T)C(T)\big((a^2+1)c_2 + (b^2+1)c_0\big),\\
B_5(T) &:= 9C(T)^2K(T)^3 + K(T)^2C(T)^2\big\{6R'(T)C(T) + 4R(T)C'(T)\\
&+ 8b(bT+a)C(T) - 8(bT+a)^2C'(T)\big\}\\
&- 8(bT+a)^2K(T)K'(T)C(T)^3 + K(T)C(T)^2\big\{-8(bT+a)(bc_1\\
&- 2ac_2)R(T)C(T) - 8b(bT+a)R(T)C'(T)C(T) + 8b^2R(T)C(T)^2\\
&- 4b(bT+a)R(T)C(T)C'(T) - 48(bT+a)^2C(T)\big((a^2-1)c_2\\
&+ (b^2+1)c_0\big) + 24ac_1(bT+a)R'(T)C(T) - 48(bT+a)C(T)\big(2a(c_2
\end{aligned}
$$

$$
\begin{aligned}
&- c_0) - bc_1\big) + 12(bT+a)^2R(T)C'(T)^2 - 8b(bT+a)R'(T)C(T)^2 \\
&+ 8(bT+a)^2R'(T)C(T)C'(T) + 16c_2(bT+a)^2R(T)C(T) \\
&- 8b(bT+a)R(T)C(T)C'(T)\big\} \\
&+ 8(bT+a)^2K'(T)C(T)^3\big\{R'(T)C(T) + 3R(T)C'(T)\big\} \\
&+ 8(bT+a)^2R(T)C(T)^4K''(T) \\
&- 16b(bT+a)R(T)C(T)^4\big((a^2-1)c_2 + (b^2+1)c_0\big) \\
&- 32(bT+a)^2R(T)C(T)^3C'(T)\big((a^2-1)c_2 + (b^2+1)c_0\big) \\
&+ 16a(bT+a)R(T)R'(T)c_1C(T)^3C'(T) + 4abR(T)R'(T)c_1C(T)^4 \\
&+ 8a(b^2+1)(bT+a)R(T)c_1C(T)^4 - 8bR(T)C(T)^4\big(2a(c_2-c_0) \\
&- bc_1\big) - 32(bT+a)R(T)C(T)^3C'(T)\big(2a(c_2-c_0) - bc_1\big), \\
B_6(T) &:= -(b^2+1)(bT+a) + 2a, \\
B_7(T) &:= -(b^2+1)T(bT+a) + aR'(T) + 2(aT+b), \\
B_8(T) &:= -(a^2-3)(bT+a) + aTR'(T) - 4a - 4(bT+a)R(T), \\
B_9(T) &:= (a^2-1)(bT+a) + 2a + 4(bT+a)R(T), \\
B_{10}(T) &:= K(T)^2\big\{2bC(T) - 4(bT+a)C'(T)\big\} \\
&- 2(bT+a)C(T)K(T)K'(T) + K(T)\big\{aR'(T)c_1C(T) \\
&- 4(bT+a)C(T)\big((a^2-1)c_2 + (b^2+1)c_0\big) - 4a(c_2-c_0)C(T) \\
&+ 2bc_1C(T) - 2a(bT+a)^2c_2(2c_0+c_1T) - 2b(bT+a)^2c_0C'(T) \\
&+ 2ab(bT+a)c_1C(T) + a(bT+a)^2c_1C'(T) \\
&+ \frac{1}{2}(bT+a)R'(T)c_1(2c_0+c_1T) - 2(bT+a)(c_2-c_0)(2c_0+c_1T) \\
&+ (bT+a)c_1C'(T)\big\} + 8b(bT+a)^2\big((a^2-1)c_2 + (b^2+1)c_0\big)C(T)^2 \\
&+ 16(bT+a)^3C(T)C'(T)\big((a^2-1)c_2 + (b^2+1)c_0\big) \\
&- 8a(bT+a)^2R'(T)c_1C(T)C'(T) - 2ab(bT+a)R'(T)c_1C(T)^2 \\
&- 4a(b^2+1)(bT+a)^2c_1C(T)^2 + 4b(bT+a)C(T)^2\big(2a(c_2-c_0) \\
&- bc_1\big) + 16(bT+a)^2C(T)C'(T)\big(2a(c_2-c_0) - bc_1\big).
\end{aligned}
$$

Theorem 1.4. *Let $z = f(x,y)$ be a C^5-class function defined in a neighborhood of the origin satisfying* (0.2). *Assume that the origin is not an umbilical point of $M := \{z = f(x,y)\}$. Let $P(t)$ be the characteristic polynomial at the origin. For an integer ℓ $(1 \le \ell \le 10)$, we suppose that there exist ℓ non-zero real numbers $\{t_k\}_{k=1}^{\ell}$ satisfying $P(t_k) = 0,\ P'(t_k) \neq 0$ for each $k = 1,\dots,\ell$, and that M includes ℓ continuous families of circular arcs associated with $\{t_k\}_{k=1}^{\ell}$. Let $T_k(x,y)$ be the function T corresponding to non-zero simple root t_k; that is, $T_k(0,0) = t_k$ $(k = 1,\dots,\ell)$. Then f is a solution of* *the following system of fifth-order partial differential equations:*

$$
\left\{
\begin{array}{l}
Z(T_k(x,y)) = 0, \quad T_k(0,0) = t_k, \\
\displaystyle\sum_{j=0}^{5} \binom{5}{j} T_k(x,y)^j \partial_x^{5-j}\partial_y^j f(x,y) = \frac{24N(T_k(x,y))}{R(T_k(x,y))K(T_k(x,y))^3} \\
(1 \le k \le \ell),
\end{array}
\right.
\tag{1.11}
$$

where $N(T)$ *is defined at* (1.10)*. Further the converse statement also holds in the sense of* (*iii*) *of Theorem 1.2.*

Theorem 1.5. *Let* $M : z = f(x,y)$ *be a* $C^{5+\theta}$*-class surface germ at the origin satisfying condition* (0.2)*, where* θ $(0 < \theta < 1)$ *is an exponent for Hölder continuity. Assume that the origin is not an umbilical point of* M*. Let* $P(t)$ *be its characteristic polynomial at* $(0,0)$*. Suppose that* M *contains two continuous families of circles in the sense of* (*ii*) *of Theorem 1.2, where these families correspond to two distinct non-zero real simple roots* t_1, t_2 *of* $P(t) = 0$*, respectively. Then,* f *is an analytic function which is uniquely determined only by the partial derivatives at* $(0,0)$ *up to 8th-order. More precisely, such surface-germs are classified by at most* 21 *real parameters.*

§ 2. General cyclides and Theorem 0.2

Before giving a rough sketch of the proof of Theorem 0.2, we introduce some results on general cyclides.

Proposition 2.1. *Let* $z = f(x,y)$ *be a* C^4*-function defined in a neighborhood of the origin satisfying conditions* (0.2), (0.3)*. Assume that* $M = \{z = f(x,y)\}$ *coincides with a general cyclide as a surface germ at the origin. Then we have*

$$
e_1 = \frac{2d_0d_1 - d_1d_2 - d_0d_3}{c_0 - c_2}, \quad e_3 = \frac{-2d_2d_3 + d_1d_2 + d_0d_3}{c_0 - c_2}
\tag{2.1}
$$

at $(0,0)$*. Here* c_*, d_*, e_* *are the derivatives of* $f(x,y)$ *introduced in Definition 1.1. Conversely, if* f *satisfies conditions* (2.1)*, then there is a unique germ of a general cyclide* M' *such that the local defining function* $z = g(x,y)$ *of* M' *coincides with* $z = f(x,y)$ *up to the fourth-order derivatives at* $(0,0)$*.*

Example 2.2. Let M be a six-circle Blum cyclide

$$
(x^2 + y^2 + z^2)^2 - 2ax^2 - 2by^2 - 2cz^2 + d^2 = 0,
\tag{2.2}
$$

such that a, b, c, d are real numbers satisfying $a > c > d > 0$, $b < -d$. Then the characteristic polynomial at $(0, 0, \sigma_2\sqrt{c + \sigma_1\sqrt{c^2 - d^2}})$ with $\sigma_1 = \pm 1, \sigma_2 = \pm 1$ is given by

$$P(t) = \frac{(a-b)(b-c)(d-b)(d+b)}{4(c^2-d^2)^2(c+\sigma_1\ell)^2} t^2 \left(t^2 - \frac{a-c}{c-b}\right)\left(t^2 - \frac{a-d}{d-b}\right)\left(t^2 - \frac{a+d}{-b-d}\right).$$

Indeed, since $z^2 = c - x^2 - y^2 + \sigma_1\sqrt{c^2 - d^2 + 2(a-c)x^2 + 2(b-c)y^2}$ in a neighborhood of $(0, 0)$, we have

$$\begin{aligned} z = \sigma_2\sqrt{c+\sigma_1\ell}\Big(1 &+ \frac{(a-c-\sigma_1\ell)x^2 + (b-c-\sigma_1\ell)y^2}{2\sigma_1\ell(c+\sigma_1\ell)} - \frac{((a-c)x^2+(b-c)y^2)^2}{4\sigma_1\ell^3(c+\sigma_1\ell)} \\ &- \frac{\big((a-c-\sigma_1\ell)x^2+(b-c-\sigma_1\ell)y^2\big)^2}{8\ell^2(c+\sigma_1\ell)^2} + o((x^2+y^2)^2)\Big), \end{aligned}$$

where $\ell = \sqrt{c^2 - d^2}$. Therefore $f(x, y) := z(x, y) - \sigma_2\sqrt{c + \sigma_1\ell}$ satisfies conditions (0.2), (0.3), and so we have the following Taylor coefficients at the origin:

$$\begin{aligned} c_0 &= \frac{\sigma_2\,(\sigma_1(a-c)-\ell)}{2\ell\sqrt{c+\sigma_1\ell}}, \; c_2 = \frac{\sigma_2\,(\sigma_1(b-c)-\ell)}{2\ell\sqrt{c+\sigma_1\ell}}, \\ e_0 &= -\sigma_2\,\frac{2\sigma_1(a-c)^2(c+\sigma_1\ell)+\ell(a-c-\sigma_1\ell)^2}{8\ell^3\sqrt{c+\sigma_1\ell}^3}, \\ e_2 &= -\sigma_2\,\frac{2\sigma_1(a-c)(b-c)(c+\sigma_1\ell)+\ell(a-c-\sigma_1\ell)(b-c-\sigma_1\ell)}{4\ell^3\sqrt{c+\sigma_1\ell}^3}, \\ e_4 &= -\sigma_2\,\frac{2\sigma_1(b-c)^2(c+\sigma_1\ell)+\ell(b-c-\sigma_1\ell)^2}{8\ell^3\sqrt{c+\sigma_1\ell}^3}, \end{aligned}$$

and all the other coefficients up to the fourth order vanish at the origin. In particular we have the 6 non-zero real roots for $P(t) = 0$.

The following lemma and Proposition 2.4 are the original results in this paper. For the readers to check the calculations, we have a Mathematica source file Blumcircles-parameter in the following website for download:

http://agusta.ms.u-tokyo.ac.jp/microlocal/manycircles.html

Lemma 2.3. *Let $x_0, y_0, z_0, U_1, U_2, U_3, V_1, V_2, V_3, \lambda$ be real constants satisfying $U_3V_1 - U_1V_3 \neq 0$. Consider the following curve with parameter $t \in \mathbb{R} \cup \{\infty\}$ in $\mathbb{R}^3$:*

$$x = x_0 + \frac{2\lambda\big(U_1 + V_1 t\big)}{(U_1+V_1t)^2 + (U_2+V_2t)^2 + (U_3+V_3t)^2},$$
$$y = y_0 + \frac{2\lambda\big(U_2 + V_2 t\big)}{(U_1+V_1t)^2 + (U_2+V_2t)^2 + (U_3+V_3t)^2},$$

$$z = x_0 + \frac{2\lambda\big(U_3 + V_3 t\big)}{(U_1 + V_1 t)^2 + (U_2 + V_2 t)^2 + (U_3 + V_3 t)^2}.$$

Then, this is a circle contained in a plane

$$y - y_0 = \frac{U_3V_2 - U_2V_3}{U_3V_1 - U_1V_3}(x - x_0) + \frac{U_2V_1 - U_1V_2}{U_3V_1 - U_1V_3}(z - z_0)$$

with center

$$x = x_0 + \frac{\lambda\big(U_1(V_2^2 + V_3^2) - V_1(U_2V_2 + U_3V_3)\big)}{(U_2V_3 - U_3V_2)^2 + (U_3V_1 - U_1V_3)^2 + (U_1V_2 - U_2V_1)^2},$$
$$y = y_0 + \frac{\lambda\big(U_2(V_3^2 + V_1^2) - V_2(U_3V_3 + U_1V_1)\big)}{(U_2V_3 - U_3V_2)^2 + (U_3V_1 - U_1V_3)^2 + (U_1V_2 - U_2V_1)^2},$$
$$z = z_0 + \frac{\lambda\big(U_3(V_1^2 + V_2^2) - V_3(U_1V_1 + U_2V_2)\big)}{(U_2V_3 - U_3V_2)^2 + (U_3V_1 - U_1V_3)^2 + (U_1V_2 - U_2V_1)^2},$$

and radius

$$R = \frac{|\lambda|\sqrt{V_1^2 + V_2^2 + V_3^2}}{\sqrt{(U_2V_3 - U_3V_2)^2 + (U_3V_1 - U_1V_3)^2 + (U_1V_2 - U_2V_1)^2}}.$$

Proof. Suppose that this curve is a circle with center $(x_0 + W_1, y_0 + W_2, z_0 + W_3)$ included in $y - y_0 = \alpha(x - x_0) + \beta(z - z_0)$. Then, since

$$\sum_{j=1}^{3}\left(\frac{2\lambda(U_j + V_j t)}{(U_1 + V_1 t)^2 + (U_2 + V_2 t)^2 + (U_3 + V_3 t)^2} - W_j\right)^2$$
$$= \frac{4\lambda\big(\lambda - \sum_{j=1}^{3}(U_j + V_j t)W_j\big)}{(U_1 + V_1 t)^2 + (U_2 + V_2 t)^2 + (U_3 + V_3 t)^2} + \sum_{j=1}^{3} W_j^2,$$

we have the sufficient conditions:

$$V_2 = \alpha V_1 + \beta V_3, \quad U_2 = \alpha U_1 + \beta U_3,$$
$$W_2 = \alpha W_1 + \beta W_2, \quad \sum_{j=1}^{3} V_j W_j = 0, \quad \sum_{j=1}^{3} U_j W_j = \lambda.$$

Thus we obtain $\alpha = (U_3V_2 - U_2V_3)/(U_3V_1 - U_1V_3), \beta = (U_2V_1 - U_1V_2)/(U_3V_1 - U_1V_3)$, and W_1, W_2, W_3 as in the statement. Further we have radius $\sqrt{W_1^2 + W_2^2 + W_3^2} = R$. □

Proposition 2.4. *Let M be the six-circle Blum cyclide given at (2.2). Then for each non-zero real root of $P(t) = 0$, we have a continuous family of circles in a neighborhood of $(0, 0, \sigma_2\sqrt{c + \sigma_1\sqrt{c^2 - d^2}})$ with $\sigma_1, \sigma_2 = \pm 1$. In fact, we have the six non-zero real roots:*

$$\pm\sqrt{\frac{a - c}{c - b}}, \quad \pm\sqrt{\frac{a - d}{d - b}}, \quad \pm\sqrt{\frac{a + d}{-b - d}}.$$

(i) *For the characteristic roots* $\pm\sqrt{(a-c)/(c-b)}$ *with* $v_1 = \sqrt{(a-c)(c-b)(c^2-d^2)}$ *we have*

$$t_{1,\pm}(x,y) = T_{1,\pm}(x,y), \quad s_{1,\pm}(x,y) = 0,$$

and

$$T_{1,\pm}(x,y) := \frac{-2(a-c)(c-b)xy \mp v_1\sqrt{c^2-d^2+2(a-c)x^2+2(b-c)y^2}}{(b-c)\big(c^2-d^2+2(a-c)x^2\big)}.$$

(ii) *For the characteristic roots* $\pm\sqrt{(a-d)/(d-b)}$ *with* $v_2 = \sqrt{2(a-d)(d-b)(c-d)}$ *we have*

$$\begin{aligned} t_{2,\pm}(x,y) := {} & \Big\{2(b-c)(a-d)xyz \mp \sigma_1\sigma_2 v_2\big(-by^2-cz^2 \\ & + (x^2+y^2+z^2)(y^2+z^2)\big)\Big\}\Big/\Big\{(d-b)z\big((d-c)(d+x^2+y^2 \\ & + z^2) - 2(a-c)x^2\big) \mp \sigma_1\sigma_2 v_2 xy(x^2+y^2+z^2-b)\Big\}, \\ s_{2,\pm}(x,y) := {} & \Big\{(c-d)y\big((b-d)(d+x^2+y^2+z^2)+2(a-b)x^2\big) \\ & \pm \sigma_1\sigma_2 v_2 xz(x^2+y^2+z^2-c)\Big\}\Big/\Big\{(d-b)z\big((d-c)(d+x^2+y^2 \\ & + z^2) - 2(a-c)x^2\big) \mp \sigma_1\sigma_2 v_2 xy(x^2+y^2+z^2-b)\Big\}. \end{aligned}$$

Further, $T_{2,\pm}(x,y) = P_{2,\pm}(x,y)/Q_2(x,y)$ *with*

$$\begin{aligned} P_{2,\pm}(x,y) := {} & xy\Big\{-(c^2-d^2)\big(3ab-2ac-2bc-(a+b-4c)d-d^2\big) \\ & + 2(a-c)\big(-2ab+ac+2bc+(a-3c)d+d^2\big)x^2 \\ & + 2(b-c)\big(-2ab+2ac+bc+(b-3c)d+d^2\big)y^2 \\ & + \sigma_1\sqrt{c^2-d^2+2(a-c)x^2+2(b-c)y^2}\,\big(2c^2(a+b)-3abc \\ & + (ab+ac+bc-4c^2)d \\ & - (a+b-c)d^2+d^3+2(a-c)(b-d)x^2+2(b-c)(a-d)y^2\big)\Big\} \\ & \pm \sigma_1 v_2\sqrt{c-x^2-y^2+\sigma_1\sqrt{c^2-d^2+2(a-c)x^2+2(b-c)y^2}} \\ & \times\Big\{\sigma_1\sqrt{c^2-d^2+2(a-c)x^2+2(b-c)y^2}\,\big(c^2-d^2+(a-c)x^2 \\ & + (b-c)y^2\big) + c\big(c^2-d^2+2(a-c)x^2+2(b-c)y^2\big)\Big\}, \\ Q_2(x,y) := {} & -(b-d)(c-d)^2(2c+d)(c+d) \\ & + (b-d)(d-c)\big(6ac-7c^2+4(a-c)d+d^2\big)x^2 \\ & - (b-c)(b-d)(c-d)(3c+d)y^2 - 2(a-c)(b-d)(2a-3c+d)x^4 \\ & + 2(b-c)\big(-2ab+ac+2bc+(a-3c)d+d^2\big)x^2y^2 \end{aligned}$$

$$
\begin{aligned}
&- \sigma_1 \sqrt{c^2 - d^2 + 2(a-c)x^2 + 2(b-c)y^2} \\
&\times \bigl\{(b-d)(c-d)(2c-d)(c+d) \\
&+ (b-d)\bigl(4ac - 5c^2 - 2(a-c)d + d^2\bigr)x^2 + (b-c)(b-d)(c-d)y^2 \\
&- 2(a-c)(b-d)x^4 - 2(b-c)(a-d)x^2y^2\bigr\}.
\end{aligned}
$$

(iii) *For the characteristic roots* $\pm\sqrt{(a+d)/(-d-b)}$, *setting* $v_3 = \sqrt{2(a+d)(-b-d)(c+d)}$ *we have a similar expression to* (ii) (only replace d in (ii) by $-d$). We omit the detailed forms of $T_{3,\pm}(x,y)$.

Proof. Take a point

$$(x_0, y_0, z_0) \in S := \{(x^2+y^2+z^2)^2 - 2ax^2 - 2by^2 - 2cz^2 + d^2 = 0\}.$$

Following Blum's argument [1], we consider the inversion with center (x_0, y_0, z_0):

$$(x,y,z) = \left(x_0 + \frac{2x'}{x'^2+y'^2+z'^2},\ y_0 + \frac{2y'}{x'^2+y'^2+z'^2},\ z_0 + \frac{2z'}{x'^2+y'^2+z'^2} \right).$$

Putting

$$r_0 = \sqrt{x_0^2 + y_0^2 + z_0^2}, \quad A = (r_0^2 - a)/2, \quad B = (r_0^2 - b)/2, \quad C = (r_0^2 - c)/2,$$

we rewrite the equation by (x', y', z'):

$$(Ax_0x' + By_0y' + Cz_0z')r'^2 + Ax'^2 + By'^2 + Cz'^2 + \bigl(x_0x' + y_0y' + z_0z' + 1\bigr)^2 = 0.$$

Then by taking the intersection with a plane $Ax_0x' + By_0y' + Cz_0z' + k = 0$ for some $k \in \mathbb{R}$ we reduce the equation to the following system:

$$
\begin{cases}
(A-k)x'^2 + (B-k)y'^2 + (C-k)z'^2 + \bigl(x_0x' + y_0y' + z_0z' + 1\bigr)^2 = 0, \\
Ax_0x' + By_0y' + Cz_0z' + k = 0.
\end{cases}
\tag{2.3}
$$

To get the condition for k, we substitute z' for $z' = -(Ax_0x' + By_0y' + k)/(Cz_0)$ in the first equation. Hence we have a quadratic equation in x', y':

$$0 = A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = A'x'^2 + (B'y' + D')x' + C'y'^2 + E'y' + F',$$

where

$$
\begin{aligned}
(C^2z_0^2)A' &= C^2z_0^2(A-k) + A^2x_0^2(C-k) + z_0^2x_0^2(C-A)^2 =: A'', \\
(C^2z_0^2)B' &= 2AB(C-k)x_0y_0 + 2(C-A)(C-B)x_0y_0z_0^2 =: B'', \\
(C^2z_0^2)C' &= C^2z_0^2(B-k) + B^2y_0^2(C-k) + z_0^2y_0^2(C-B)^2 =: C'',
\end{aligned}
$$

$$
\begin{aligned}
(C^2 z_0^2) D' &= 2(C-k)kAx_0 + 2(C-A)(C-k)x_0 z_0^2 =: D'', \\
(C^2 z_0^2) E' &= 2(C-k)kBy_0 + 2(C-B)(C-k)y_0 z_0^2 =: E'', \\
(C^2 z_0^2) F' &= (C-k)k^2 + (C-k)^2 z_0^2 =: F''.
\end{aligned}
$$

Then the condition for k is that the equation for x', y' splits into a product of two first order equations in x', y'. Therefore the discriminant

$$
\begin{aligned}
&(B''y' + D'')^2 - 4A''(C''y'^2 + E''y' + F'') \\
&= (B''^2 - 4A''C'')y'^2 + 2(B''D'' - 2A''E'')y' + D''^2 - 4A''F''
\end{aligned}
$$

in x' should be the square of some first order polynomial in y'. Hence we get the following condition for k:

$$
\begin{cases}
(B''D'' - 2A''E'')^2 - (B''^2 - 4A''C'')(D''^2 - 4A''F'') = 0, \\
B''^2 - 4A''C'' \geq 0.
\end{cases}
$$

Since the first equation is equivalent to

$$
4A''\big(- B''D''E'' + A''E''^2 + B''^2F'' + C''(D''^2 - 4A''F'')\big) = 0,
$$

we obtain a sufficient condition:

$$
\begin{cases}
-B''D''E'' + A''E''^2 + B''^2F'' + C''(D''^2 - 4A''F'') = 0, \\
B''^2 - 4A''C'' \geq 0.
\end{cases}
$$

Finally we get an equation for k:

$$
\begin{aligned}
0 &= -B''D''E'' + A''E''^2 + B''^2F'' + C''(D''^2 - 4A''F'') \\
&= 4C^4 z_0^4 (k-A)(k-B)(k-C)\big(k^2 - k(x_0^2 + y_0^2 + z_0^2) + Ax_0^2 + By_0^2 + Cz_0^2\big) \\
&= 4C^4 z_0^4 (k-A)(k-B)(k-C)\left(k - \frac{r_0^2 - d}{2}\right)\left(k - \frac{r_0^2 + d}{2}\right),
\end{aligned}
$$

because

$$
\begin{aligned}
&r_0^4 - d^2 - 4(Ax_0^2 + By_0^2 + Cz_0^2) = r_0^4 - d^2 - 2(r_0^2 - a)x_0^2 - 2(r_0^2 - b)y_0^2 - 2(r_0^2 - c)z_0^2 \\
&= -r_0^4 - d^2 + 2ax_0^2 + 2by_0^2 + 2cz_0^2 = 0.
\end{aligned}
$$

Thus k should be either one of the following:

$$
(r_0^2 - a)/2, \quad (r_0^2 - b)/2, \quad (r_0^2 - c)/2, \quad (r_0^2 - d)/2, \quad (r_0^2 + d)/2.
$$

On the other hand, putting $G(k) := B''^2 - 4A''C''$, we have the inequalities for $\forall k$:

$$
0 \leq G((r_0^2 - a)/2) = -4(A-B)(A-C)C^2 x_0^2 z_0^2 \big(A^2 + By_0^2 + Cz_0^2 - A(y_0^2 + z_0^2)\big)
$$

$$
\begin{aligned}
&\equiv (a-b)(c-a)(a+d)(a-d),\\
0 \le G((r_0^2-b)/2) &= -4(A-B)(-B+C)C^2y_0^2z_0^2\big(B^2+Ax_0^2+Cz_0^2-B(x_0^2+z_0^2)\big)\\
&\equiv (a-b)(b-c)(b+d)(b-d),\\
0 \le G((r_0^2-c)/2) &= 4(A-C)(-B+C)C^2z_0^4\big(C^2+Ax_0^2+By_0^2-C(x_0^2+y_0^2)\big)\\
&\equiv (a-c)(b-c)(d-c)(c+d),\\
0 \le G((r_0^2-d)/2) &= C^2z_0^2(d-a)(d-b)(d-c)(x_0^2+y_0^2+z_0^2+d)^2/8\\
&\equiv (d-a)(d-b)(d-c),\\
0 \le G((r_0^2+d)/2) &= -C^2z_0^2(d+a)(d+b)(d+c)(x_0^2+y_0^2+z_0^2-d)^2/8\\
&\equiv -(d+a)(d+b)(d+c).
\end{aligned}
$$

Hence for our case $a > c > d > 0, -b > d$, only $k = (r_0^2-c)/2, (r_0^2-d)/2, (r_0^2+d)/2$ give the positive $G(k)$'s.

Case (i) $k = (r_0^2-c)/2$: Setting $v_1 = \pm\sqrt{(a-c)(c-b)(c^2-d^2)}$, we solve equations (2.3). Then we have

$$y' = \frac{\big(2(a-c)(c-b)x_0y_0 - v_1(r_0^2-c)\big)x'}{(c-b)\big(c^2-d^2+2(a-c)x_0^2\big)},$$

$$z' =$$

$$\frac{(c-b)\big(d^2-c^2+2(c-a)x_0^2\big)+\big\{(b-c)x_0\big((a-c)(r_0^2-2z_0^2)+ac-d^2\big)+v_1y_0(r_0^2-b)\big\}x'}{(c-b)z_0\big(c^2-d^2+2(a-c)x_0^2\big)}.$$

Therefore in the original coordinates x, y, z we have the following:

$$
\begin{aligned}
x &= x_0 + \frac{2B_1^2z_0^2x'}{B_1^2z_0^2x'^2+B_2^2z_0^2x'^2+(B_1+B_3x')^2},\\
y &= y_0 + \frac{-2B_1B_2z_0^2x'}{B_1^2z_0^2x'^2+B_2^2z_0^2x'^2+(B_1+B_3x')^2},\\
z &= z_0 + \frac{-2B_1z_0(B_1+B_3x')}{B_1^2z_0^2x'^2+B_2^2z_0^2x'^2+(B_1+B_3x')^2},
\end{aligned}
\tag{2.4}
$$

where

$$B_1 := (b-c)\big(c^2-d^2+2(a-c)x_0^2\big), \tag{2.5}$$

$$B_2 := 2(a-c)(-b+c)x_0y_0 - v_1(-c+x_0^2+y_0^2+z_0^2), \tag{2.6}$$

$$B_3 := v_1y_0(-b+x_0^2+y_0^2+z_0^2)+(b-c)x_0(ac-d^2+(a-c)(x_0^2+y_0^2-z_0^2)). \tag{2.7}$$

Then, setting $\lambda = z_0B_1, U_1 = U_2 = 0, U_3 = -B_1, V_1 = z_0B_1, V_2 = -z_0B_2, V_3 = -B_3$, we can apply Lemma 2.3. Therefore, concerning the plane $y - y_0 = t(x-x_0) + s(z-z_0)$ including the circle we have

$$t = -\frac{B_2}{B_1} = \frac{2(a-c)(b-c)x_0y_0 - v_1(c-x_0^2-y_0^2-z_0^2)}{(b-c)\big(c^2-d^2+2(a-c)x_0^2\big)}, \quad s = 0. \tag{2.8}$$

Further, in a neighborhood of $(0, 0, \sigma_2\sqrt{c + \sigma_1\sqrt{c^2 - d^2}})$ $(\sigma_1, \sigma_2 = \pm 1)$ we obtain

$$(2.9)\ \ T(x, y) = t(x, y) = \frac{2(a-c)(b-c)xy + \sigma_1 v_1\sqrt{c^2 - d^2 + 2(a-c)x^2 + 2(b-c)y^2}}{(b-c)\big(c^2 - d^2 + 2(a-c)x^2\big)},$$

which corresponds to the characteristic root

$$T(0, 0) = -\mathrm{sign}(v_1)\,\sigma_1\sqrt{(a-c)/(c-b)}.$$

The results corresponding to the characteristic roots are as in the statement. The center of the circle is

$$x = x_0 - \frac{B_1B_3}{B_1^2 + B_2^2}, \quad y = y_0 + \frac{B_2B_3}{B_1^2 + B_2^2}, \quad z = 0. \tag{2.10}$$

and the radius is

$$\sqrt{\frac{B_3^2 + z_0^2(B_1^2 + B_2^2)}{B_1^2 + B_2^2}}. \tag{2.11}$$

Case (ii) $k = (r_0^2 - d)/2$: Setting $v_2 = \pm\sqrt{2(-b+d)(c-d)(a-d)}$, we solve (2.3). Hence we get

$$\begin{aligned}
y' = {} & \frac{1}{2\big\{(d-c)y_0^2\big(b^2 - d^2 + 2(a-b)x_0^2\big) + (b-d)z_0^2\big(d^2 - c^2 + 2(c-a)x_0^2\big)\big\}} \\
& \times \Big\{2(c-d)y_0\big(bd - d^2 + 2(a-b)x_0^2 + (b-d)r_0^2\big) + 2x_0y_0\Big((ab - d^2)(c-d) \\
& + (a-b)(c-d)(x_0^2 - y_0^2) + \big((a+b)(c+d) - 2(ab+cd)\big)z_0^2\Big)x' \\
& - v_2z_0(r_0^2 - c)\big((d + r_0^2)x' + 2x_0\big)\Big\}, \\
z' = {} & \frac{1}{2\big\{(d-c)y_0^2\big(b^2 - d^2 + 2(a-b)x_0^2\big) + (b-d)z_0^2\big(d^2 - c^2 + 2(c-a)x_0^2\big)\big\}} \\
& \times \Big\{ -2z_0\big\{(b-d)\big(d^2 - cd - (2a - c - d)x_0^2 + (d-c)(y_0^2 + z_0^2)\big) \\
& - \Big((ac - d^2)(b-d) + (b-d)(a-c)(x_0^2 - z_0^2) + (bc - 2bd + dc - 2ac \\
& + ab + ad)y_0^2\Big)x_0x'\big\} + v_2y_0(-b + r_0^2)\big(2x_0 + (d + r_0^2)x'\big)\Big\}.
\end{aligned}$$

Therefore in the original coordinates we have the following:

$$x = x_0 + \frac{8C_1^2x'}{4C_1^2x'^2 + (2C_2 + C_3x')^2 + (2C_4 + C_5x')^2}, \tag{2.12}$$

$$y = y_0 + \frac{4C_1(2C_2 + C_3x')}{4C_1^2x'^2 + (2C_2 + C_3x')^2 + (2C_4 + C_5x')^2}, \tag{2.13}$$

$$z = z_0 + \frac{4C_1(2C_4 + C_5x')}{4C_1^2x'^2 + (2C_2 + C_3x')^2 + (2C_4 + C_5x')^2}, \tag{2.14}$$

where

$$\begin{aligned}
C_1 &:= -(c-d)\big(b^2-d^2+2(a-b)x_0^2\big)y_0^2-(b-d)\big(c^2-d^2+2(a-c)x_0^2\big)z_0^2,\\
C_2 &:= -v_2x_0z_0(r_0^2-c)+(c-d)y_0\big(bd-d^2+2(a-b)x_0^2+(b-d)r_0^2\big),\\
C_3 &:= v_2z_0(c-r_0^2)(d+r_0^2)+2x_0y_0\big\{(c-d)(ab-d^2)+(a-b)(c-d)x_0^2\\
&\qquad -(a-b)(c-d)y_0^2+\big((a+b)(c+d)-2(ab+cd)\big)z_0^2\big\},\\
C_4 &:= v_2x_0y_0(r_0^2-b)+(-b+d)z_0\big(d^2-cd-(2a-c-d)x_0^2+(d-c)(y_0^2+z_0^2)\big),\\
C_5 &:= v_2y_0(-b+r_0^2)(d+r_0^2)+2x_0z_0\big\{(b-d)(ac-d^2)\\
&\qquad +\big(b(c-2d)+cd+a(b-2c+d)\big)y_0^2+(a-c)(b-d)(x_0^2-z_0^2)\big\}.
\end{aligned}$$

In the same way as in case (i), putting $\lambda=2C_1, U_1=0, U_2=2C_2, U_3=2C_4, V_1=2C_1, V_2=C_3, V_3=C_5$, we get the plane

$$y-y_0=\frac{C_3C_4-C_2C_5}{2C_1C_4}\,(x-x_0)+\frac{C_2}{C_4}\,(z-z_0).$$

Therefore we obtain $t=(C_3C_4-C_2C_5)/(2C_1C_4), s=C_2/C_4$. Since the numerator of t has the factor $2C_1$, we have

$$\begin{aligned}
t &= \big\{2(b-c)(a-d)x_0y_0z_0+v_2\big(-by_0^2-cz_0^2+r_0^2(y_0^2+z_0^2)\big)\big\}/C_4\\
&= \frac{2(b-c)(a-d)x_0y_0z_0+v_2\big(-by_0^2-cz_0^2+r_0^2(y_0^2+z_0^2)\big)}{v_2x_0y_0(r_0^2-b)+(-b+d)z_0\big(d^2-cd-(2a-c-d)x_0^2+(d-c)(y_0^2+z_0^2)\big)},\\
s &= \frac{-v_2x_0z_0(r_0^2-c)+(c-d)y_0\big(bd-d^2+2(a-b)x_0^2+(b-d)r_0^2\big)}{v_2x_0y_0(r_0^2-b)+(-b+d)z_0\big(d^2-cd-(2a-c-d)x_0^2+(d-c)(y_0^2+z_0^2)\big)}.
\end{aligned}$$

As for $T(x_0,y_0)=(t+sz_x)/(1-sz_y)|_{(x_0,y_0)}$, we can calculate z_x, z_y by the implicit function theorem applied to $(x^2+y^2+z^2)^2-2ax^2-2by^2-2cz^2+d^2=0$. Hence we have

$$z_x=\frac{x(a-x^2-y^2-z^2)}{z(x^2+y^2+z^2-c)},\quad z_y=\frac{y(b-x^2-y^2-z^2)}{z(x^2+y^2+z^2-c)}.$$

Therefore in a neighborhood of $\big(0,0,\sigma_2\sqrt{c+\sigma_1\sqrt{c^2-d^2}}\big)(\sigma_1,\sigma_2=\pm1)$ we get $T(x_0,y_0)=P(x_0,y_0)/Q(x_0,y_0)$, where

$$\begin{aligned}
Q(x_0,y_0) &= (d-b)(c-d)^2(c+d)(2c+d)+(b-d)(d-c)\big(6ac-7c^2+4(a-c)d+d^2\big)x_0^2\\
&\quad -(b-c)(b-d)(c-d)(3c+d)y_0^2-2(a-c)(b-d)(2a-3c+d)x_0^4\\
&\quad +2(b-c)\big(-2ab+ac+2bc+(a-3c)d+d^2\big)x_0^2y_0^2\\
&\quad +\Big\{-(b-d)(c-d)(c+d)(2c-d)-(b-d)\big(4ac-5c^2-2(a-c)d+d^2\big)x_0^2\\
&\quad -(b-c)(b-d)(c-d)y_0^2+2(a-c)(b-d)x_0^4+2(b-c)(a-d)x_0^2y_0^2\Big\}
\end{aligned}$$

$$
\begin{aligned}
&\times \sigma_1\sqrt{c^2-d^2+2(a-c)x_0^2+2(b-c)y_0^2},\\
P(x_0,y_0) = {}& x_0y_0\Big\{(c^2-d^2)\big(-3ab+2ac+2bc+(a+b-4c)d+d^2\big)+2(a-c)\big(-2ab\\
&+ac+2bc+(a-3c)d+d^2\big)x_0^2+2(b-c)\big(-2ab+2ac+bc+(b-3c)d+d^2\big)y_0^2\\
&+\sigma_1\sqrt{c^2-d^2+2(a-c)x_0^2+2(b-c)y_0^2}\Big(-3abc+2ac^2+2bc^2+(ab+ac\\
&+bc-4c^2)d-(a+b-c)d^2+d^3+2(a-c)(b-d)x_0^2+2(b-c)(a-d)y_0^2\Big)\Big\}\\
&+\sigma_2 v_2\sqrt{c-x_0^2-y_0^2+\sigma_1\sqrt{c^2-d^2+2(a-c)x_0^2+2(b-c)y_0^2}}\,\Big\{-c^3+cd^2\\
&-2(a-c)cx_0^2-2(b-c)cy_0^2+\sigma_1\sqrt{c^2-d^2+2(a-c)x_0^2+2(b-c)y_0^2}\\
&\times\Big(-c^2+d^2+(-a+c)x_0^2+(-b+c)y_0^2\Big)\Big\}.
\end{aligned}
$$

In particular, for any signature of d, the characteristic root $T(0,0)$ is given by

$$T(0,0) = -\sigma_1\sigma_2\sigma_3\,\mathrm{sign}(v_2)\sqrt{(a-d)/(d-b)},$$

where $\sigma_3 = \mathrm{sign}(\sqrt{c+d}+\sigma_1\sqrt{c-d})$ (hence $\sigma_3=-1$ for $d<0, \sigma_1=-1$, and $\sigma_3=1$ for other cases). The results corresponding to the characteristic roots are as in the statement. The center of the circle is

$$x = x_0 - \frac{2C_1^2(C_2C_3+C_4C_5)}{4C_1^2(C_2^2+C_4^2)+(C_3C_4-C_2C_5)^2}, \tag{2.15}$$

$$y = y_0 + \frac{C_1\big(C_2(4C_1^2+C_5^2)-C_3C_4C_5\big)}{4C_1^2(C_2^2+C_4^2)+(C_3C_4-C_2C_5)^2}, \tag{2.16}$$

$$z = z_0 + \frac{C_1\big(C_4(4C_1^2+C_3^2)-C_2C_3C_5\big)}{4C_1^2(C_2^2+C_4^2)+(C_3C_4-C_2C_5)^2}. \tag{2.17}$$

Further the radius is

$$|C_1|\sqrt{\frac{4C_1^2+C_3^2+C_5^2}{4C_1^2(C_2^2+C_4^2)+(C_3C_4-C_2C_5)^2}}. \tag{2.18}$$

Case (iii) $k=(r_0^2+d)/2$: Setting $v_2 = \pm\sqrt{2(-b-d)(c+d)(a+d)}$, we solve (2.3). Easily to see, the results are obtained by exchanging d by $-d$ in the results in case (ii). $\square$

The following is the key lemma for the calculation in the proof of Theorem 0.2:

Lemma 2.5. *Let $Q(t)=\sum_{k=0}^n q_kt^k$ be a polynomial in t of degree $n(>0)$ with coefficients $q_0,\dots,q_n\in\mathbb{C}$ $(q_n\neq 0)$. Suppose that $Q(t)=0$ has n separate roots $t_1,\dots,t_n\in\mathbb{C}$. Let $A(t)$ be a polynomial in t with complex coefficients, and $s_1,\dots,s_m$ be $m(>0)$*

different complex numbers such that $\{s_p; p = 1, ..., m\} \cap \{t_j; j = 1, ..., n\} = \emptyset$. *Then the solution* $(g_0, ..., g_{n-1}) \in \mathbb{C}^n$ *to the system*

$$\sum_{k=0}^{n-1} (t_j)^k g_k = \left. \frac{A(t)}{\prod_{p=1}^m (t - s_p)} \right|_{t=t_j} \qquad (j = 1, ..., n) \tag{2.19}$$

is given by

$$g_k = \alpha_k + \sum_{p=1}^m \beta_p \gamma_{k,p} \tag{2.20}$$

for $k = 0, ..., n-1$. *Here* β_p *is the residue of* $A(t)/(\prod_{r=1}^m (t - s_p))$ *at* $t = s_p$ *given by*

$$\beta_p := \frac{A(s_p)}{\prod_{r \neq p} (s_p - s_r)},$$

and α_k*'s are the coefficients of the remainder* $\sum_{k=0}^{n-1} \alpha_k t^k$ *of the division* $B(t)/Q(t)$ *for the polynomial*

$$B(t) := \frac{A(t)}{\prod_{p=1}^m (t - s_p)} - \sum_{p=1}^m \frac{\beta_p}{t - s_p}.$$

Further, the coefficient $\gamma_{k,p}$ *at* (2.20) *is given by*

$$\gamma_{k,p} := - \sum_{\ell=0}^{n-k-1} q_{\ell+k+1}(s_p)^\ell \Big/ Q(s_p) \quad \left(\textit{hence } 1/(t_j - s_p) = \sum_{k=0}^{n-1} \gamma_{k,p}(t_j)^k, \ \forall j = 1, ..., n\right).$$

A rough sketch of the proof of Theorem 0.2. By Theorem 1.4 we have the following system of 6 differential equations for $f(x, y)$:

$$\begin{cases} Z(T_k(x, y)) = 0, \quad T_k(0, 0) = t_k, \\ \displaystyle\sum_{j=0}^5 T_k(x, y)^j \left(\binom{5}{j} \partial_x^{5-j} \partial_y^j f(x, y) \right) = \frac{24N(T_k(x, y))}{R(T_k(x, y))K(T_k(x, y))^3} \\ (1 \leq k \leq 6). \end{cases} \tag{2.21}$$

Since

$$\det \left(T_j(0, 0)^{k-1}; j, k = 1, ..., 6 \right) = \prod_{j>k} (t_j - t_k) \neq 0,$$

the coefficient matrix $\left(T_j(x, y)^{k-1}; j, k = 1, ..., 6\right)$ is invertible in a neighborhood of $x = y = 0$. Therefore we can rewrite (2.21) as follows:

$$\begin{cases} Z(T_{j+1}(x, y)) = 0, \quad T_{j+1}(0, 0) = t_{j+1}, \\ \partial_x^{5-j} \partial_y^j f(x, y) = G_j(\nabla f, \nabla^2 f, \nabla^3 f, \nabla^4 f, T_1, ..., T_6), \\ (0 \leq j \leq 5), \end{cases} \tag{2.22}$$

where G_j $(j = 0, ..., 5)$ are analytic functions of $\nabla f, \nabla^2 f, \nabla^3 f, \nabla^4 f, T_1, ..., T_6$. In particular, we know that f is a C^6-class function in a neighborhood of the origin. Hence f satisfies all the assumptions in Theorem 1.5, and so we obtain the analyticity of $f(x, y)$ at $x = y = 0$. On the other hand, it is easy to see that all the Taylor coefficients at the origin of $f(x, y)$ are determined successively by the Taylor coefficients c_0, c_2, d_*, e_* at the origin. Thus the proof of the former part of Theorem 0.2 is completed. Now we suppose the additional conditions

$$d_0 = d_1 = d_2 = d_3 = 0$$

at $(0, 0)$. Since $a = b = c_1 = d_0 = d_1 = d_2 = d_3 = 0$ at $(0, 0)$, by the explicit forms of $N(T), B_{10}(T)$ in Proposition 1.3 we get $N(T; 0, 0) = 0$. Hence by equations (2.21) we obtain

$$\partial_x^{5-j} \partial_y^j f(0, 0) = 0 \quad (0 \le j \le 5).$$

To get some necessary conditions on the values $c_0, c_2, e_0, ..., e_4$ at the origin for a solution $f(x, y)$, we find sixth-order derivatives of f at the origin by using the differential equations (2.21). Precisely speaking, we find the following coefficients g_j, h_j $(j = 0, 1, ..., 5)$

$$\frac{\partial_x^{5-j} \partial_y^j f(x, y)}{(5-j)!j!} = g_j x + h_j y + O(x^2 + y^2)$$

as $x, y \to 0$. Since all the fifth order derivatives of f vanish at the origin, concerning a, b, c_*, d_*, e_* we have

$$\begin{aligned}
&a \equiv 2c_0^* x, \quad b \equiv 2c_2^* y, \quad c_0 \equiv c_0^*, \quad c_1 \equiv 0, \quad c_2 \equiv c_2^*, \\
&d_0 \equiv 4e_0^* x + e_1^* y, \quad d_1 \equiv 3e_1^* x + 2e_2^* y, \quad d_2 \equiv 2e_2^* x + 3e_3^* y, \quad d_3 \equiv e_3^* x + 4e_4^* y, \\
&e_0 \equiv e_0^*, \quad e_1 \equiv e_1^*, \quad e_2 \equiv e_2^*, \quad e_3 \equiv e_3^*, \quad e_4 \equiv e_4^*.
\end{aligned}$$

Here, c_j^*, e_j^* mean their values at the origin, and "$A \equiv 0$" means "$A = O(x^2 + y^2)$ as $x, y \to 0$". Concerning $T(x, y)$, noting $K(T) \equiv 2T(c_0^* + c_2^* T^2) - 2c_2^* T(T^2 + 1) \equiv 2T(c_0^* - c_2^*)$, we have

$$Z(T) \equiv 4(c_0^* - c_2^*)^2 T^2 \Big((T^2 + 1) \sum_{j=0}^{4} e_j^* T^j - (c_2^* T^2 + c_0^*)^3 \Big).$$

Therefore, since $Z'(T) \neq 0$, we obtain $T_j(x, y) \equiv t_j$ $(j = 1, ..., 6)$. From now on, c_*, e_* mean their values at $x = y = 0$. By Proposition 1.3 we have the following:

$$N(T) \equiv 20(c_0 - c_2)^2 T^2 \big(A_1(T) x + A_2(T) y \big),$$

where

$$A_1(T) = \Big\{ -(T^2 + 1)^2 E'(T) + 2TC(T)^2 \big(2C(T) + c_0 - c_2 \big) \Big\}$$

$$\times\big(4E(T)-TE'(T)\big)+4c_0C(T)^2\Big\{(T^2+1)E'(T)-4c_2TC(T)^2\Big\},$$
$$A_2(T)=E'(T)\Big\{-(T^2+1)^2E'(T)+2TC(T)^2\big(2C(T)+c_0-c_2\big)\Big\}$$
$$+4c_2TC(T)^2\Big\{(T^2+1)E'(T)-4c_2TC(T)^2\Big\}.$$

Since we have the equations

$$\sum_{k=0}^{5}(t_j)^k(g_kx+h_ky)\equiv\frac{N(t_j)}{5R(t_j)K(t_j)^3}\equiv\frac{N(t_j)}{40(c_0-c_2)^3t_j^3(t_j^2+1)}$$

for $j=1,2,...,6$, we get the following equations for g_j,h_j:

$$\sum_{k=0}^{5}(t_j)^kg_k=\frac{A_1(t_j)}{2(c_0-c_2)t_j(t_j^2+1)},\quad \sum_{k=0}^{5}(t_j)^kh_k=\frac{A_2(t_j)}{2(c_0-c_2)t_j(t_j^2+1)}$$

for $j=1,2,...,6$. On the other hand, since $\{t_j\}$ are the non-zero real roots of the characteristic polynomial $P(t)$, we have $Q(t_j)=0$ for $j=1,...,6$ with $Q(t)=\sum_{j=0}^{6}q_jt^j:=(t^2+1)E(t)-C(t)^3$. Therefore we can find $(g_0,...,g_5),(h_0,...,h_5)$ by using Lemma 2.5 and some Mathematica program. Since $g_j=\partial_x^{6-j}\partial_y^jf(0,0)/(j!(5-j)!),h_j=\partial_x^{5-j}\partial_y^{j+1}f(0,0)/(j!(5-j)!)$, we have 5 compatibility conditions:

$$f_1:=5h_0-g_1=0\quad f_2:=2h_1-g_2=0,\quad f_3:=h_2-g_3=0,$$
$$f_4:=h_3-2g_4=0,\quad f_5:=h_4-5g_5=0.$$

We put $f_j^*=2(c_0-c_2)(c_0^3-e_0)(c_2^3-e_4)f_j$. Then we obtain the following expressions:

$$f_2^*=(c_0^3-e_0)(3c_0^2c_2-e_0-e_2)e_3^2-24(c_0^3-e_0)(c_2^3-e_4)e_1e_3$$
$$-(9c_0^3-6c_0^2c_2-7e_0+2e_2)(c_2^3-e_4)e_1^2,$$
$$f_4^*=-24(c_0^3-e_0)(c_2^3-e_4)e_1e_3+(c_0^3-e_0)(6c_0c_2^2-9c_2^3-2e_2+7e_4)e_3^2$$
$$+(c_2^3-e_4)(3c_0c_2^2-e_2-e_4)e_1^2.$$

In particular we have

$$0=f_2^*-f_4^*=(c_0^3-e_0)(3c_0^2c_2-6c_0c_2^2+9c_2^3-e_0+e_2-7e_4)e_3^2$$
$$-(c_2^3-e_4)(9c_0^3-6c_0^2c_2+3c_0c_2^2-7e_0+e_2-e_4)e_1^2.$$

Hence if $(c_0^3-e_0)(3c_0^2c_2-6c_0c_2^2+9c_2^3-e_0+e_2-7e_4)\neq0$, we get $e_3=te_1$ with

$$t=\pm\sqrt{\frac{(c_2^3-e_4)(9c_0^3-6c_0^2c_2+3c_0c_2^2-7e_0+e_2-e_4)}{(c_0^3-e_0)(3c_0^2c_2-6c_0c_2^2+9c_2^3-e_0+e_2-7e_4)}},$$

and so from $f_2^*=0$ we have

$$\Big((c_0^3-e_0)(3c_0^2c_2-e_0-e_2)t^2-24(c_0^3-e_0)(c_2^3-e_4)t$$

$$- (9c_0^3 - 6c_0^2c_2 - 7e_0 + 2e_2)(c_2^3 - e_4)\Big)e_1^2 = 0.$$

Therefore we conclude that, for generic c_0, c_2, e_0, e_2, e_4, we have $e_1 = e_3 = 0$. On the other hand, under $c_1 = d_* = e_1 = e_3 = 0$ we can apply Proposition 2.1. Hence there exists a unique germ M' of a general cyclide at the origin with the same data a, b, c_*, d_*, e_* at the origin. Since the characteristic polynomial of M' at the origin coincides with $P(t)$, it has the six distinct non-zero characteristic roots $t_1, ..., t_6$ with $C(t_j; 0, 0) \neq 0$ $(\forall j)$. Indeed $C(t_j; 0, 0) \neq 0$ follows from the assumptions on M at $(0, 0)$ and (i) of Theorem 1.2. Therefore by Lemma 2.6 we know that M' includes 6 continuous families of circular arcs corresponding to characteristic roots $t_1, ..., t_6$, and it is conformally equivalent to a general cyclide of type (0.4). Hence by the former part of Theorem 0.2 we conclude that M' coincides with our surface germ $z = f(x, y)$. This completes the proof except for the proof of Lemma 2.6, which is given independently of Theorem 0.2.

Lemma 2.6. *Let $M = \{z = f(x, y)\}$ be a C^4-class surface germ at $(0, 0, 0)$ with the following Taylor expansion at $(0, 0)$:*

$$f(x, y) = c_0x^2 + c_2y^2 + e_0x^4 + e_2x^2y^2 + e_4y^4 + o((x^2 + y^2)^2),$$

where c_0, c_2, e_0, e_2, e_4 are real coefficients with $c_0 - c_2 \neq 0$. We suppose that M is a general cyclide as a germ at the origin, and that the characteristic polynomial $P(t)$ at $(0, 0)$ of M has 6 distinct non-zero real roots $t_1, ..., t_6$ with $C(t_j; 0, 0) \neq 0$ $(\forall j)$. Then by some conformal transformation Φ, M is transformed into a germ at $\tau = (0, 0, *)$ of the following 6-circle Blum cyclide:

$$(x^2 + y^2 + z^2)^2 - 2a_1x^2 - 2a_2y^2 - 2a_3z^2 + a_4^2 = 0, \tag{2.23}$$

where $a_1 > a_3 > a_4 > 0$, $-a_2 > a_4$. In particular this surface (2.23) has the same characteristic roots $\{t_k\}_{k=1}^6$ at τ, and for every $j = 1, ..., 6$ the continuous family of circular arcs corresponding to t_j is transformed by Φ^{-1} into the continuous family of circular arcs on M corresponding to t_j.

Remark. Takeuchi [6] proved that a general cyclide can be transformed into (0.4) by a conformal transformation. The arguments there are geometrically very interesting, but they are not germ-fixing arguments. Indeed, it is not so easy to construct a similar conformal transformation fixing the reference point.

In October 2011, we found in the internet arXiv (110.2338v1) with title:

A surface containing a line and a circle through each point is a quadric by Fedor Nilov and Mikhail Skopenkov concerning surfaces including several circular arcs. They found a surface which is not a cyclide, but includes 2 families of circles:

$$(x^2 + y^2 + z^2)^2 - 4y^2z^2 - 4x^2 = 0 \quad \Longleftrightarrow \quad x = \sqrt{1 - y^2} + \sqrt{1 - z^2},$$

where y =Const., z =Const. becomes circles. Further they proved the following:

Theorem 2.7. *Let Φ be a smooth closed surface in $\mathbb{R}^3$ homeomorphic to either a sphere or a torus. If through each point of the surface one can draw at least* **4 distinct circles fully contained in the surface** *(and continuously depending on the point) then the surface is* **a cyclide**.

They extended Takeuchi's idea on intersection numbers of fundamental groups and used a classical theorem on the relationship between cospherical circles and cyclides. So the proof relies on the global information of the surface. On the other hand their counter example is not a closed surface, but a surface with singularities. At the same time, they gave a conjecture (also see [5]):

3 distinct continuous families of circles $\Longrightarrow$ cyclides.

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