Time regularity in Gevrey classes of solutions to general nonlinear partial differential equations

By

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Abstract

The paper considers general nonlinear partial differential equations

$$F(t, x, \{\partial/\partial t\}^{j}\{\partial/\partial x\}^{\alpha}u\}_{j \leq m, |\alpha| \leq L} = 0$$

(with $1 \leq m \leq L$) in Gevrey classes, and gives a sufficient condition for the following assertion to be valid: if a solution $u(t, x)$ is in $C^\infty$ class with respect to the time variable $t$ and in the Gevrey class $\mathcal{E}^{(s)}$ in the space variable $x$, then it is in the Gevrey class $\mathcal{E}^{(s)}$ also with respect to the time variable $t$ for a suitable $s$. In [4] we have discussed this problem in a class of nonlinear partial differential equations; in this paper we will discuss the problem in the general case (E).

§1. Introduction

We denote by $t$ the time variable in $\mathbb{R}_t$, and by $x = (x_1, \ldots, x_n)$ the space variable in $\mathbb{R}_x^n$. We use the notations: $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{N}^* = \{1, 2, \ldots\}$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\partial_t = \partial/\partial t$, $\partial_x = (\partial_{x_1}, \ldots, \partial_{x_n})$ with $\partial_{x_i} = \partial/\partial x_i$ ($i = 1, \ldots, n$) and $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$.

For $\sigma \geq 1$ and an open subset $V$ of $\mathbb{R}_x^n$ we denote by $\mathcal{E}^{(\sigma)}(V)$ the set of all functions $f(x) \in C^\infty(V)$ satisfying the following: for any compact subset $K$ of $V$ there are $C > 0$ and $h > 0$ such that

$$\max_{x \in K} |\partial_x^\alpha f(x)| \leq Ch^{|\alpha|!|\alpha|!^\sigma}, \quad \forall \alpha \in \mathbb{N}^n.$$
A function in the class \( \mathcal{E}^{(\sigma)}(V) \) is called \textit{a function of the Gevrey class of order \( \sigma \)}.

If \( \sigma = 1 \), the class \( \mathcal{E}^{(1)}(V) \) is nothing but the set of all analytic functions on \( V \) and usually it is denoted by \( \mathcal{A}(V) \). If \( 1 < \sigma_1 < \sigma_2 < \infty \) we have

\[
\mathcal{A}(V) \subset \mathcal{E}^{(\sigma_1)}(V) \subset \mathcal{E}^{(\sigma_2)}(V) \subset C^\infty(V).
\]

Thus, functions in the class \( \mathcal{E}^{(\sigma_1)}(V) \) are closer to analytic functions than those in \( \mathcal{E}^{(\sigma_2)}(V) \); in this sense, we can say that functions in \( \mathcal{E}^{(\sigma_1)}(V) \) are more regular than those in \( \mathcal{E}^{(\sigma_2)}(V) \).

For an interval \([0, T] = \{ t \in \mathbb{R} ; 0 \leq t \leq T \}\) we denote by \( C^\infty([0, T], \mathcal{E}^{(\sigma)}(V)) \) the set of all infinitely differentiable functions \( u(t, x) \) in \( t \in [0, T] \) with values in \( \mathcal{E}^{(\sigma)}(V) \) equipped with the usual locally convex topology (see [2]).

Similarly, for \( s \geq 1 \) and \( \sigma \geq 1 \) we denote by \( \mathcal{E}^{(s, \sigma)}([0, T] \times V) \) the set of all functions \( u(t, x) \in C^\infty([0, T] \times V) \) satisfying the following: for any compact subset \( K \) of \( V \) there are \( C > 0 \) and \( h > 0 \) such that

\[
\max_{(t,x)\in[0,T]\times K} |\partial_t^k \partial_x^\alpha u(t, x)| \leq Ch^{k+|\alpha|} |\alpha|! |\sigma|!, \quad \forall (k, \alpha) \in \mathbb{N} \times \mathbb{N}^n.
\]

Obviously, we have

\[
\mathcal{E}^{(s, \sigma)}([0, T] \times V) \subset C^\infty([0, T], \mathcal{E}^{(\sigma)}(V)).
\]

In the case \( s = \sigma \) we write \( \mathcal{E}^{(\sigma)}([0, T] \times V) \) instead of \( \mathcal{E}^{(s, \sigma)}([0, T] \times V) \).

Let \( \Omega \) be an open subset of \( \mathbb{R}_t \times \mathbb{R}_x^n \times \mathbb{R}_z^d \): the Gevrey class \( \mathcal{E}^{(s_1, \sigma, s_2)}(\Omega) \) is defined in the same way. In this case, \( s_1 \) denotes the Gevrey order in \( t \), \( \sigma \) denotes the Gevrey order in \( x \), and \( s_2 \) denotes the Gevrey order in \( z \).

In this paper, we will consider the following nonlinear partial differential equation

\[
(1.1) \quad F\left(t, x, \{ \partial_t^j \partial_x^\alpha u \}_{j \leq m, |\alpha| \leq L} \right) = 0
\]

where \( 1 \leq m \leq L \) are positive integers, and \( F(t, x, \{ z_{j, \alpha} \}_{j+|\alpha| \leq m}) \) is a suitable function in a Gevrey class (for the precise assumptions, see §2). And, we will consider the following problem on Gevrey regularity in time:

**Problem 1.1.** Let \( u(t, x) \in C^\infty([0, T], \mathcal{E}^{(\sigma)}(V)) \) be a solution of (1.1); can we have the result \( u(t, x) \in \mathcal{E}^{(s, \sigma)}([0, T] \times V) \) for a suitable \( s \geq 1 \)? If this is true, determine the precise bound of the index \( s \) of the time regularity.

In the previous paper [4], we have studied this problem for the equation

\[
t^\gamma \partial_t^m u = F\left(t, x, \{ \partial_t^j \partial_x^\alpha u \}_{j \leq m, |\alpha| \leq L} \right)
\]
(with $\gamma \geq 0$ and $1 \leq m \leq L$). The purpose of this paper is to discuss the problem 1.1 for general equation (1.1).

§ 2. Formulation and main theorem

Let $m \in \mathbb{N}^*$ be fixed, $\Lambda_0$ be a finite subset of $\{(j, \alpha) \in \mathbb{N} \times \mathbb{N}_n; j < m\}$, and set

$\Lambda = \Lambda_0 \cup \{(m, 0)\}$ and $d = \#\Lambda$. We denote by $z = \{z_{j, \alpha}\}_{(j, \alpha) \in \Lambda}$ the variable in $\mathbb{R}^d$. Let $\Omega$ be an open subset of $\mathbb{R}_t \times \mathbb{R}_x^n \times \mathbb{R}_z^d$, let $F(t, x, z)$ be a $C^\infty$ function on $\Omega$, and let us consider the following nonlinear partial differential equation:

\[
 F(t, x, Du) = 0 \quad \text{with} \quad Du = \{\partial_t^j \partial_x^\alpha u\}_{(j, \alpha) \in \Lambda}.
\]

Let $s_1 \geq 1$ and $\sigma \geq s_2 \geq 1$ be real numbers, $V$ be an open subset of $\mathbb{R}_x^n$, and $T > 0$. Our basic assumptions are:

- $a_1) m \geq 1$, $s_1 \geq 1$ and $\sigma \geq s_2 \geq 1$;
- $a_2) \Lambda = \Lambda_0 \cup \{(m, 0)\}$ and $\Lambda_0$ is a finite subset of $\{(j, \alpha) \in \mathbb{N} \times \mathbb{N}_n; j < m\}$,
- $a_3) F(t, x, z) \in \mathcal{E}^{s_1, \sigma, s_2}(\Omega)$,
- $a_4) u(t, x) \in C^\infty([0, T], \mathcal{E}^\sigma(V))$ is a solution of (2.1) on $[0, T] \times V$; this includes the property: $(t, x) \in [0, T] \times V \implies (t, x, Du(t, x)) \in \Omega$.

Let us define

**Definition 2.1.**

1. Let $u(t)$ be a $C^\infty$-function in a neighborhood of $t = t_0$; we define the order of zero of $u(t)$ at $t = t_0$ (which we denote by $\text{ord}_{t=t_0}(u(t))$) by the following:

$$\text{ord}_{t=t_0}(u(t)) = \min\{k \in \mathbb{N}; u^{(k)}(t_0) \neq 0\}$$

(if $u^{(k)}(t_0) = 0$ for all $k \in \mathbb{N}$ we set $\text{ord}_{t=t_0}(u(t)) = \infty$).

2. Let $W$ be an open subset of $\mathbb{R}_t \times \mathbb{R}_x^n$, and let $f(t, x) \in C^\infty(W)$. We define a $\mathbb{N} \cup \{\infty\}$-valued function $q(t_0, x_0; f)$ on $W$ in the following way. Take any $(t_0, x_0) \in W$; then $f(t, x_0)$ is a $C^\infty$-function in a neighborhood of $t = t_0$ and so we can define the order $\text{ord}_{t=t_0}(f(t, x_0))$ of zero of $f(t, x_0)$ at $t = t_0$. We set $q(t_0, x_0; f) = \text{ord}_{t=t_0}(f(t, x_0))$.

Under the conditions $a_1), a_2), a_3$ and $a_4$ we set

\[
 k_{j, \alpha}(t_0, x_0) = q(t_0, x_0; (\partial F/\partial z_{j, \alpha})(t, x, Du(t, x)))
 = \text{ord}_{t=t_0}((\partial F/\partial z_{j, \alpha})(t, x_0, (Du)(t, x_0)))
\]
(which is the order of zero of \( \frac{\partial F}{\partial z_{j,\alpha}}(t, x_0, (Du)(t, x_0)) \) at \( t = t_0 \) and suppose:

(M) For any \( (t_0, x_0) \in [0, T] \times V \) there are \( \gamma \in \mathbb{N} \) and a neighborhood \( V_0 \)
of \( x_0 \in V \) which satisfy the following properties:
1) \( k_{m,0}(t_0, x) = \gamma \) for any \( x \in V_0 \);
2) \( k_{j,0}(t_0, x) \geq \gamma - m + j \) for any \( x \in V_0 \),
3) \( k_{j, \alpha}(t_0, x) \geq \gamma - m + j + 1 \) for any \( x \in V_0 \), if \( |\alpha| > 0 \).

In the condition (M), the constant \( \gamma \) may depends on \( (t_0, x_0) \) and so we may write
\( \gamma = \gamma(t_0, x_0) \): this function \( \gamma(t, x) \) is locally constant with respect to \( x \). Thus, if (M) is satisfied, we can take any connected neighborhood of \( x_0 \) as \( V_0 \) in the condition (M).

For any fixed \( (t_0, x_0) \in [0, T] \times V \), by using \( \gamma = \gamma(t_0, x_0) \) and a connected neighborhood \( V_0 \) of \( x_0 \) we set
\[
k_{j, \alpha}(t_0, V_0) = \min_{x \in V_0} k_{j, \alpha}(t_0, x) \quad \text{(for } (j, \alpha) \in \Lambda_0 \text{)},
\]
and
\[
s_0(t_0, V_0) = 1 + \max \left[ 0, \max_{(j, \alpha) \in \Lambda_0, |\alpha| > 0} \left( \frac{j + \sigma|\alpha| - m}{\min \{k_{j, \alpha}(t_0, V_0) - \gamma + m - j, m - j\}} \right) \right].
\]

Note that these \( k_{j, \alpha}(t_0, V_0) \) and \( s_0(t_0, V_0) \) depend on \( (t_0, x_0) \in [0, T] \times V \) and \( V_0 \), and that by 2) and 3) we have the conditions: \( k_{j,0}(t_0, V_0) \geq \gamma - m + j \), \( k_{j,\alpha}(t_0, V_0) \geq \gamma - m + j + 1 \) if \( |\alpha| > 0 \). We also note that if \( V_1 \subset V_0 \) we have \( k_{j,\alpha}(t_0, V_1) \geq k_{j,\alpha}(t_0, V_0) \)
and so we have \( s_0(t_0, V_1) \leq s_0(t_0, V_0) \).

By using these indices \( s_0(t_0, V_0) \) (for \( (t_0, x_0) \in [0, T] \times V \) and \( V_0 \)) we define the index \( s_0 \geq 1 \) by the following:
\[
(2.3) \quad s_0 = \sup_{(t_0, x_0) \in [0, T] \times V} \left( \inf_{V_0 \ni x_0} s_0(t_0, V_0) \right).
\]

We note:

**Lemma 2.2.** Under the above situation, for any \( (t_0, x_0) \in [0, T] \times V \) and any sufficient small neighborhood \( V_0 \) of \( x_0 \) we have \( s_0 \geq s(t_0, V_0) \).

**Proof.** We set
\[
K = \sup_{V_0 \ni x_0} k_{j,\alpha}(t_0, V_0).
\]
If \( K = \infty \), for any \( N > 0 \) we can take a \( V_0 \) such that \( k_{j,\alpha}(t_0, V_0) > N \) and so we have \( \min \{k_{j,\alpha}(t_0, V_0) - \gamma + m - j, m - j\} = m - j \) for any sufficiently small \( V_0 \). If \( K < \infty \), by the condition that \( k_{j,\alpha}(t_0, V_0) \) is a \( \mathbb{N} \)-valued function we see that \( k_{j,\alpha}(t_0, V_0) = K \) holds for any sufficiently small \( V_0 \). Thus, by the definition of \( s_0(t_0, V_0) \) we can conclude that \( s_0(t_0, V_0) \) is independent of \( V_0 \) if \( V_0 \) is sufficiently small. This proves Lemma 2.2.

The following result is the main theorem of this paper.
Theorem 2.3. Suppose the conditions \( a_1, a_2, a_3, a_4 \) and (M); then we have \( u(t, x) \in \mathcal{E}^{\{s, \sigma\}}([0, T] \times V) \) for any \( s \geq \max\{s_0, s_1, s_2\} \).

§3. Proof of Theorem 2.3

Take any \( s \geq \max\{s_0, s_1, s_2\} \). To prove Theorem 2.3, it is enough to show the following assertion: for any \( (t_0, x_0) \in [0, T] \times V \) we can find a \( \delta > 0 \) and a small compact neighborhood \( V_1 \) of \( x_0 \) such that \( u(t, x) \in \mathcal{E}^{\{s, \sigma\}}([t_0, t_0 + \delta] \times V_1) \) (or \( u(t, x) \in \mathcal{E}^{\{s, \sigma\}}([t_0 - \delta, t_0] \times V_1) \)) holds. By changing the variable \( t \rightarrow t - t_0 \), we have only to discuss the case \( t_0 = 0 \).

Take any \((0, x_0)\) and fix it. Let \( \gamma = \gamma(0, x_0) \) and take a sufficiently small connected neighborhood \( V_0 \) of \( x_0 \); then by Lemma 2.2 we have \( s_0 \geq s(t_0, V_0) \). For simplicity we set \( k_{j, \alpha} = k_{j, \alpha}(0, V_0) \) (for \((j, \alpha)\in\Lambda_0\)). Take a sufficiently small \( T_0 > 0 \), and we have \( u(t, x) \in C^\infty([0, T_0], \mathcal{E}^{\{\sigma\}}(V_0)) \). Since \( k_{m,0}(0, x) = \gamma \) holds on \( V_0 \), we have

\[
(3.1) \quad \left[ t^{-\gamma} \frac{\partial F}{\partial z_{m,0}}(t, x, Du) \right]_{t=0} \neq 0 \quad \text{on} \ V_0.
\]

Moreover, we have \( k_{j,0} \geq \gamma - m + j \), and \( k_{j,\alpha} \geq \gamma - m + j + 1 \) if \(|\alpha| > 0\). By the condition \( s \geq \max\{s_0, s_1, s_2\} \) we have \( s \geq \max\{s_1, s_2\} \) and

\[
(3.2) \quad s \geq 1 + \max_{(j, \alpha) \in \Lambda_0, |\alpha| > 0} \left( \frac{j + \sigma|\alpha| - m}{\min\{k_{j, \alpha} - \gamma + m - j, m - j\}} \right).
\]

We will consider the equation only on \([0, T_0] \times V_0\).

Let us reduce our equation (2.1) to an equation discussed in [4]. First, we take an integer \( q \) with

\[
(3.3) \quad q \geq \gamma + m + 1
\]

and set

\[
u(t, x) = \varphi(t, x) + t^q w(t, x) \quad \text{with} \quad \varphi(t, x) = \sum_{k=0}^{q-1} \frac{(\partial_t^k u)(0, x)}{k!} t^k;
\]

then we have \( \varphi(t, x) \in \mathcal{E}^{\{1, \sigma\}}(\mathbb{R} \times V_0) \) and \( w(t, x) \in C^\infty([0, T_0], \mathcal{E}^{\{\sigma\}}(V_0)) \). Since

\[
Du = D\varphi + \{t^{p-j}[t\partial_t + q]_j \partial_x^\alpha w\}_{(j, \alpha) \in \Lambda}
\]

(where \([\lambda]_0 = 1 \) and \([\lambda]_p = \lambda(\lambda - 1) \cdots (\lambda - p + 1) \) for \( p \geq 1 \)), and since \( u(t, x) \) is a solution of (2.1) we have

\[
(3.4) \quad F(t, x, D\varphi + \{t^{p-j}[t\partial_t + q]_j \partial_x^\alpha w\}_{(j, \alpha) \in \Lambda}) = 0
\]
which is regarded as an equation with respect to $w(t, x)$. To see the result $u(t, x) \in \mathcal{E}^{(s, \sigma)}([0, \delta] \times V_1)$, it is enough to show the condition: $w(t, x) \in \mathcal{E}^{\{s, \sigma\}}([0, \delta] \times V_1)$.

Let us do a further reduction. By the formula
\[
[\lambda + q]_j = (\lambda + q)(\lambda + q - 1) \cdots (\lambda + q - j + 1) = \sum_{0 \leq i \leq j} c_{j, i} \lambda^i,
\]
we define the constants $c_{j, i}$ ($0 \leq i \leq j \leq m$): we see that $c_{j, j} = 1$ holds. Set
\[
Z(t, z) = \{Z_{j, \alpha}(t, z)\}_{(j, \alpha) \in \Lambda} \quad \text{with} \quad Z_{j, \alpha}(t, z) = t^{q-j} \sum_{0 \leq i \leq j} c_{j, i} z_{i, \alpha},
\]
\[
H(t, x, z) = \frac{1}{t^{q+\gamma-m}} \times F(t, x, D\varphi(t, x) + Z(t, z)),
\]
\[
\Theta^*w = \{(i\partial_t)^j \partial_x^\alpha w\}_{(j, \alpha) \in \Lambda} \quad \text{and} \quad z^* = (\Theta^*w)(0, x_0) \in \mathbb{R}^d.
\]
By the definition we have $Z_{j, \alpha}(t, \Theta^*w) = t^{q-j}[t\partial_t + q]_j \partial_x^\alpha w$ for any $(j, \alpha) \in \Lambda$, and so we have $D\varphi(t, x) + Z(t, \Theta^*w) = Du(t, x)$. By the same argument as in [§3, [4]] we have

**Lemma 3.1.** Under the above situation we have the following results:

1. Set $\Omega_0 = \{(t, x, z) \in \mathbb{R}_t \times V_0 \times \mathbb{R}^d_0; (t, x, D\varphi(t, x) + Z(t, z)) \in \Omega\}$; then we have $(0, x_0, z^*) \in \Omega_0$ and $H(t, x, z) \in \mathcal{E}^{(s^*, \sigma, s_2)}(\Omega_0)$ for $s^* = \max\{s_1, s_2\}$.
2. $w(t, x) \in C^\infty([0, T_0], \mathcal{E}^{\{\sigma\}}(V_0))$ is a solution of
\[
(3.5) \quad H(t, x, \Theta^*w) = 0, \quad \text{on } [0, T_0] \times V_0,
\]
and we have $(t, x, \Theta^*w(t, x)) \in \Omega_0$ for any $(t, x) \in [0, T_0] \times V_0$.
3. $H(0, x_0, z^*) = 0$ and $(\partial H/\partial z_{m,0})(0, x_0, z^*) \neq 0$. In addition we have
\[
(3.6) \quad \frac{\partial H}{\partial z_{m,0}}(t, x, \Theta^*w(t, x)) \bigg|_{t=0} \neq 0 \quad \text{on } V_0.
\]
4. We set
\[
(3.7) \quad q_{j, \alpha} = \min_{x \in V_0} \left(\operatorname{ord}_{t=0}(\partial H/\partial z_{j, \alpha})(t, x, \Theta^*w(t, x))\right), \quad (j, \alpha) \in \Lambda_0.
\]
Then we have $q_{j, \alpha} \geq 1$ if $|\alpha| > 0$. Moreover we see that the condition (3.2) implies
\[
(3.8) \quad s \geq 1 + \max \left[0, \max_{(j, \alpha) \in \Lambda_0, |\alpha| > 0} \left(\frac{j + \sigma|\alpha| - m}{\min\{q_{j, \alpha}, m-j\}}\right)\right].
\]

**Proof.** By the condition (M) and (3.3) we have
\[
(\partial F/\partial z_{j, \alpha})(t, x, D\varphi(t, x)) = \begin{cases} O(t^{\gamma-m+j}) & \text{on } V_0, \quad \text{if } |\alpha| = 0, \\ O(t^{\gamma-m+j+1}) & \text{on } V_0, \quad \text{if } |\alpha| > 0 \end{cases}
\]
(where $f(t, x) = O(t^n)$ means that $f(t, x) = O(t^n)$ uniformly in $x$ (as $t \rightarrow +0$)). In particular, in the case $(j, \alpha) = (m, 0)$ we have $(\partial F/\partial z_{m,0})(t, x, D\varphi(t, x)) = O(t^\gamma)$ and by (3.1) we have
\[\left. t^{-\gamma} \frac{\partial F}{\partial z_{m,0}}(t, x, D\varphi) \right|_{t=0} \neq 0 \quad \text{on } V_0.\]
Therefore, by (3.4) and Taylor’s formula we have
\[0 = F(t, x, D\varphi(t, x)) + \sum_{(j, \alpha) \in \Lambda} \frac{\partial F}{\partial z_{j,\alpha}}(t, x, D\varphi)O(t^{q-j}) + \sum_{(j, \alpha), (i, \beta) \in \Lambda} O(t^{q-j})O(t^{q-i})\]
\[= F(t, x, D\varphi(t, x)) + \sum_{(j, \alpha) \in \Lambda} O(t^{\gamma-m+j}) \times O(t^{q-j}) + O(t^{2q-2m})\]
\[= F(t, x, D\varphi(t, x)) + O(t^{q+\gamma-m}) \quad \text{on } V_0:\]
this shows that $F(t, x, D\varphi(t, x)) = O(t^{q+\gamma-m})$ holds on $V_0$. Hence, by the definition of $H(t, x, z)$ we have
\[H(t, x, z) = \frac{1}{t^{q+\gamma-m}} \times F(t, x, D\varphi(t, x) + Z(t, z))\]
\[= \frac{1}{t^{q+\gamma-m}} \left[ F(t, x, D\varphi(t, x)) + \sum_{(j, \alpha)} \frac{\partial F}{\partial z_{j,\alpha}}(t, x, D\varphi)O(t^{q-j}) + O(t^{2q-2m}) \right]\]
\[= \frac{1}{t^{q+\gamma-m}} \left[ O(t^{q+\gamma-m}) + \sum_{(j, \alpha)} O(t^{\gamma-m+j}) \times O(t^{q-j}) + O(t^{2q-2m}) \right]\]
(where $f(t, x, z) = O(t^n)$ means that $f(t, x, z) = O(t^n)$ uniformly in $(x, z)$ (as $t \rightarrow +0$)). This proves that $H(t, x, z)$ is well-defined as a $C^\infty$ function on $\Omega_0$. By Proposition 5.1 (in Appendix) we have the condition: $H(t, x, z) \in \mathcal{E}^{\{s^*, \sigma, s_2\}}(\Omega_0)$ with $s^* = \max\{s_1, s_2\}$. This proves (1).

(2) is clear from the definition of $H(t, x, z)$ and $\Omega_0$. Since
\[\frac{\partial H}{\partial z_{m,0}}(t, x, \Theta^*w(t, x)) = \frac{1}{t^\gamma} \frac{\partial F}{\partial z_{m,0}}(t, x, Du(t, x))\]
holds, by (3.1) we have the result (3). Since
\[\frac{\partial H}{\partial z_{j,\alpha}}(t, x, \Theta^*w(t, x)) = \frac{1}{t^{q+\gamma-m}} \sum_{l \geq j} \frac{\partial F}{\partial z_{l,\alpha}}(t, x, Du(t, x))t^{q-l}c_{l,j}\]
\[= \sum_{l \geq j} O(t^{k_{l,\alpha}+\gamma+m-l}) \quad \text{on } V_0,\]
we have the result (4).

Now, let us apply the implicit function theorem [Theorem 5.2 (with $\sigma_1 = \sigma$ and $\sigma_2 = s_2$) in Appendix] to the functional equation
\[(3.9) \quad H(t, x, z) = 0 \quad \text{in a neighborhood of } (0, x_0, z^*).\]
We write $z^* = (z', z^*_{m,0})$ and $z = (z', z_{m,0})$. Since

$$H(0, x_0, z^*) = 0 \quad \text{and} \quad (\partial H/\partial z_{m,0})(0, x_0, z^*) \neq 0$$

hold, we can find an open neighborhood $\Omega_1$ of $(0, x_0, z^*) \in \mathbb{R}_t \times V_0 \times \mathbb{R}^{d-1}$ and a function $G(t, x, z')$ on $\Omega_1$ which satisfy the following properties:

1) $G(t, x, z') \in \mathcal{E}^{s^*, \sigma, s_2} (\Omega_1)$;
2) $z^*_{m,0} = G(0, x_0, z^*)$;
3) the functional relation (3.9) is equivalent to $z_{m,0} = G(t, x, z')$;
4) if we take $\delta > 0$ and an open neighborhood $V_1(\subset V_0)$ of $x_0$ sufficiently small, we have $(t, x, \Theta w(t, x)) \in \Omega_1$ for any $(t, x) \in [0, \delta] \times V_1$ and the function $w(t, x) \in C^\infty([0, \delta], \mathcal{E}^{\sigma}(V_1))$ is a solution of the equation

$$(3.10) \quad (t\partial_t)^m w = G(t, x, \Theta w) \quad \text{with} \quad \Theta w = \{(t\partial_t)^j \partial^{\alpha} w\}_{(j, \alpha) \in \Lambda_0}. $$

Moreover, we have

**Lemma 3.2.** We set

$$q_{j, \alpha}(G) = \min_{x \in V_1} \left( \mathrm{ord}_{t=0}((\partial G/\partial z_{j,\alpha})(t, x, \Theta w(t, x))) \right), \quad (j, \alpha) \in \Lambda_0.$$ 

Then we have $q_{j, \alpha}(G) \geq 1$ if $|\alpha| > 0$, and that the condition (3.2) implies

$$(3.11) \quad s \geq 1 + \max \left[ 0, \max_{(j, \alpha) \in \Lambda_0, |\alpha| > 0} \left( \frac{j + \sigma|\alpha| - m}{\min\{q_{j,\alpha}(G), m-j\}} \right) \right].$$

We denote by $s_0(G)$ the right-hand side of (3.11).

**Proof.** By the implicit function theorem we have

$$\frac{\partial G}{\partial z_{j,\alpha}}(t, x, \Theta w(t, x)) = -\frac{(\partial H/\partial z_{j,\alpha})(t, x, \Theta^* w(t, x))}{(\partial H/\partial z_{m,0})(t, x, \Theta^* w(t, x))}.$$ 

Since (3.6) holds, we have

$$\mathrm{ord}_{t=0}((\partial G/\partial z_{j,\alpha})(t, x, \Theta w(t, x))) = \mathrm{ord}_{t=0}((\partial H/\partial z_{j,\alpha})(t, x, \Theta^* w(t, x))), \quad \forall x \in V_1$$

and by the condition $V_1 \subset V_0$ we have $q_{j,\alpha}(G) \geq q_{j,\alpha}$ for any $(j, \alpha) \in \Lambda_0$, where $q_{j,\alpha}$ ($(j, \alpha) \in \Lambda_0$) are the ones in (3.7). Thus, by (3.8) we have (3.11).

Thus, we have seen that $w(t, x)$ is a solution of (3.10). Since (3.10) is just an equation discussed in [4], by [Theorems 5.0.1 and 6.1 in [4]] we have the following result which proves Theorem 2.3.
Theorem 3.3. Under the above situation, we have \( w(t, x) \in \mathcal{E}^{\{s, \sigma\}}([0, \delta] \times V_1) \) for any \( s \geq \max\{s_0(G), s^*\} \) with \( s^* = \max\{s_1, s_2\} \).

§ 4. Application

In this section, we will consider the equation (2.1) under the assumption

\[(4.1) \quad F(t, x, z) \in \mathcal{E}^{\{\sigma\}}(\Omega)\]

for some \( \sigma \geq 1 \), and we will consider the following problem:

Problem 4.1. Let \( u(t, x) \in C^\infty([0, T], \mathcal{E}^{\{\sigma\}}(V)) \) be a solution of (2.1); can we have the result \( u(t, x) \in \mathcal{E}^{\{\sigma\}}([0, T] \times V) \)?

By using Theorem 2.3, let us give a sufficient condition for this problem to be affirmative. We see: by Theorem 2.3 (with \( s_1 = s_2 = \sigma \)) we have the result \( u(t, x) \in \mathcal{E}^{\{\sigma\}}([0, T] \times V) \) if the condition \( \sigma \geq s_0 \) holds, that is, if for any \((t_0, x_0) \in [0, T] \times V\) and a sufficiently small neighborhood \( V_0 \) of \( x_0 \) we have

\[(4.2) \quad \sigma \geq 1 + \max \left[ 0, \max_{(j, \alpha) \in \Lambda_0, |\alpha| > 0} \left( \frac{j + \sigma|\alpha| - m}{\min\{k_{j,\alpha}(t_0, V_0) - \gamma + m - j, m - j\}} \right) \right] \]

which is equivalent to

\[(4.3) \quad m - j - \min\{k_{j,\alpha}(t_0, V_0) - \gamma + m - j, m - j\} \geq \sigma(|\alpha| - \min\{k_{j,\alpha}(t_0, V_0) - \gamma + m - j, m - j\}) \]

for any \((j, \alpha) \in \Lambda_0\) with \(|\alpha| > 0\).

If \( j + |\alpha| > m \) holds for some \((j, \alpha) \in \Lambda_0\), we have

\[1 + \frac{j + \sigma|\alpha| - m}{\min\{k_{j,\alpha}(t_0, V_0) - \gamma + m - j, m - j\}} \geq 1 + \frac{j + \sigma|\alpha| - m}{m - j} = \frac{\sigma|\alpha|}{m - j} > \sigma.\]

This shows that if \( \Lambda_0 \not\subset \{(j, \alpha) \in \mathbb{N} \times \mathbb{N}^n; j + |\alpha| \leq m\} \) the condition (4.2) is not satisfied.

Let us consider the case:

\[(4.4) \quad \Lambda_0 \subset \{(j, \alpha) \in \mathbb{N} \times \mathbb{N}^n; j + |\alpha| \leq m\}.\]

If \(|\alpha| \leq \min\{k_{j,\alpha}(t_0, V_0) - \gamma + m - j, m - j\}\) holds, the condition (4.3) is clear from the fact that the right-hand side is nonpositive and the left-hand side is nonnegative. If \(|\alpha| > \min\{k_{j,\alpha}(t_0, V_0) - \gamma + m - j, m - j\}\) holds, by the condition \(|\alpha| \leq m - j\) we have

\[m - j \geq |\alpha| > \min\{k_{j,\alpha}(t_0, V_0) - \gamma + m - j, m - j\} = k_{j,\alpha}(t_0, V_0) - \gamma + m - j\]
and so the inequality (4.3) is equivalent to
\[
\sigma \leq \frac{m - j - (k_{j, \alpha}(t_0, V_0) - \gamma + m - j)}{|\alpha| - (k_{j, \alpha}(t_0, V_0) - \gamma + m - j)} = 1 + \frac{m - j - |\alpha|}{|\alpha| - (k_{j, \alpha}(t_0, V_0) - \gamma + m - j)}
\]
Therefore, if we set \( \Delta = \{(j, \alpha) \in \Lambda_0; k_{j, \alpha}(t_0, V_0) - \gamma + m - j < |\alpha|\} \), our condition (4.2) is equivalent to
\[
(4.5) \quad 1 \leq \sigma \leq 1 + \min \left[ \infty, \min_{(j, \alpha) \in \Delta} \left( \frac{m - j - |\alpha|}{|\alpha| - (k_{j, \alpha}(t_0, V_0) - \gamma + m - j)} \right) \right] .
\]
Thus, summing up we have the following result. Set
\[
(4.6) \quad \Delta(t_0, V_0) = \{(j, \alpha); k_{j, \alpha}(t_0, V_0) - \gamma + m - j < |\alpha|\},
\]
\[
(4.7) \quad \sigma_0(t_0, V_0) = 1 + \min \left[ \infty, \min_{(j, \alpha) \in \Delta(t_0, V_0)} \left( \frac{m - j - |\alpha|}{|\alpha| - (k_{j, \alpha}(t_0, V_0) - \gamma + m - j)} \right) \right]
\]
where \( \gamma = \gamma(t_0, x_0) \). We have:

**Theorem 4.2.** Let \( \sigma \geq 1 \). Suppose the conditions (4.1), (4.4), (a4), (M), and
\[
(4.8) \quad \sigma \leq \inf_{(t_0, x_0) \in [0, T] \times V} \left( \sup_{V_0 \ni x_0} \sigma_0(t_0, V_0) \right)
\]
then we have \( u(t, x) \in \mathcal{E}^{\{\sigma\}}([0, T] \times V) \).

§ 5. Appendix

In §3, we have used two results: one is a result on the composition of Gevrey functions, and the other is the implicit function theorem in Gevrey classes. We present here precise formulations and their proofs.

First, let us show a result on the composition of Gevrey functions. We write \( t \in \mathbb{R}_t \), \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n_\times \) and \( z = (z_1, \ldots, z_d) \in \mathbb{R}^d_\times \). We have

**Proposition 5.1.** Let \( s \geq 1 \), \( s_1 \geq 1 \), \( s_2 \geq 1 \), \( \sigma \geq 1 \), let \( \Omega \) and \( W \) be open subsets of \( \mathbb{R}_t \times \mathbb{R}^n_\times \times \mathbb{R}^d_\times \). If the conditions
1) \( F(t, x, z) \in \mathcal{E}^{\{s_1, \sigma, s_2\}}(\Omega) \),
2) \( u_i(t, x, z) \in \mathcal{E}^{\{s, \sigma, s_2\}}(W) \) (\( i = 1, \ldots, d \)),
3) \( W \ni (t, x, z) \Rightarrow (t, x, u(t, x, z)) \in \Omega \), where \( u = (u_1, \ldots, u_d) \),
4) \( \sigma \geq s_2 \) and \( s \geq \max\{s_1, s_2\} \)
hold, we have \( F(t, x, u(t, x, z)) \in \mathcal{E}^{\{s, \sigma, s_2\}}(W) \).
Proof. Take any compact subset \( Z \) of \( W \); then the image \( L \) of \( Z \) by the mapping \((t, x, z) \rightarrow (t, x, u(t, x, z))\) is also a compact subset of \( \Omega \). We take constants \( A_{p,q,\nu} \geq 0 \) and \( B_{i,k,\beta,\gamma} \geq 0 \) so that

\[
\begin{align*}
\bullet \quad & \max_{(t,x,z) \in L} \left| \frac{F^{(p,q,\nu)}(t,x,z)}{p!q!\nu!} \right| \leq A_{p,q,\nu} p^{s_1-1} |q|^{\sigma-1} |\nu|^{s_2-1}, \\
\bullet \quad & \max_{(t,x,z) \in Z} \left| \frac{u_i^{(k,\beta,\gamma)}(t,x,z)}{k!\beta!\gamma!} \right| \leq B_{i,k,\beta,\gamma} (k-1)!^{s-1} |\beta|^{\sigma-1} |\gamma|^{s_2-1}, \text{ if } k \geq 1, \\
\bullet \quad & \max_{(t,x,z) \in Z} \left| \frac{u_i^{(0,\beta,\gamma)}(t,x,z)}{\gamma!} \right| \leq B_{i,0,\beta,\gamma} (|\beta|-1)!^{\sigma-1} |\gamma|^{s_2-1}, \text{ if } |\beta| \geq 1, \\
\bullet \quad & \max_{(t,x,z) \in Z} \left| \frac{u_i^{(0,0,\gamma)}(t,x,z)}{\gamma!} \right| \leq B_{i,0,0,\gamma} (|\gamma|-1)!^{s_2-1}, \text{ if } |\gamma| \geq 1,
\end{align*}
\]

where \( p \in \mathbb{N}, q \in \mathbb{N}^n, \nu \in \mathbb{N}^d, k \in \mathbb{N}, \beta \in \mathbb{N}^n \), and \( \gamma \in \mathbb{N}^d \). We set also

\[
G(t,x,z) = \sum_{p+|q|+|\nu| \geq 0} A_{p,q,\nu} t^p x^q z^\nu,
\]

\[
w_i(t,x,z) = \sum_{k+|\beta|+|\gamma| \geq 1} B_{i,k,\beta,\gamma} t^k x^\beta z^\gamma, \quad i = 1, \ldots, d.
\]

Then, \( G(t,x,z) = G(t,x,z_1, \ldots, z_d) \) and \( w_i(t,x,z) \) \((i = 1, \ldots, d)\) are convergent in a neighborhood of \((t,x,z) = (0,0,0)\), and so the function

\[
H(t,x,z) = G(t,x, w_1(t,x,z), \ldots, w_d(t,x,z))
\]

\[
= \sum_{p+|q|+|\nu| \geq 0} A_{p,q,\nu} t^p x^q \left( \sum_{k_1+|\beta_1|+|\gamma_1| \geq 1} B_{1,k_1,\beta_1,\gamma_1} t^{k_1} x^{\beta_1} z^{\gamma_1} \right)^{\nu_1} \times \\
\times \cdots \times \left( \sum_{k_d+|\beta_d|+|\gamma_d| \geq 1} B_{d,k_d,\beta_d,\gamma_d} t^{k_d} x^{\beta_d} z^{\gamma_d} \right)^{\nu_d}
\]

(with \( \nu = (\nu_1, \ldots, \nu_d) \)) is also convergent in a neighborhood of \((t,x,z) = (0,0,0)\). If we set

\[
H(t,x,z) = \sum_{(m,\alpha,\mu) \in \mathbb{N} \times \mathbb{N}^n \times \mathbb{N}^d} C_{m,\alpha,\mu} t^m x^\alpha z^\mu
\]

we have \( C_{0,0,0} = A_{0,0,0} \) and for \( m + |\alpha| + |\mu| \geq 1 \)

\[
(5.1) \quad C_{m,\alpha,\mu} = \sum_{1 \leq p+|q|+|\nu| \leq m+|\alpha|+|\mu|} A_{p,q,\nu} \prod_{i=1}^d \prod_{j=1}^{\nu_i} \left( B_{i,k_i(j),\beta_i(j),\gamma_i(j)} \right),
\]
where $|k^*| = \sum_{i=1}^{d} (k_i(1) + \cdots + k_i(\nu_i))$ and

$$s(\beta(\nu)) = \sum_{i=1}^{d} \sum_{j=1}^{\nu_i} \beta_i(j) \in \mathbb{N}^d, \quad s(\gamma(\nu)) = \sum_{i=1}^{d} \sum_{j=1}^{\nu_i} \gamma_i(j) \in \mathbb{N}^d.$$ 

Since $H(t, x, z)$ is a holomorphic function in a neighborhood of $(t, x, z) = (0, 0, 0)$, by Cauchy’s inequality we have $C_{m, \alpha, \mu} \leq M \eta^{m+|\alpha|+|\mu|}$ for some $M > 0$ and $\eta > 0$. Therefore, to see the condition $h(t, x, z) = F(t, x, u(t, x, z)) \in \mathcal{E}^{s, \sigma, s_2}(Z)$ it is sufficient to prove the following inequalities:

$$\frac{1}{m!s-1|\alpha|!\sigma-1|\mu|!s_2-1} \max_{(t,x,z) \in Z} \left| \frac{1}{m!\alpha!\mu!} h^{(m,\alpha,\mu)}(t, x, z) \right| \leq 3^{(m+|\alpha|+|\mu|)(s_2-1)} C_{m, \alpha, \mu}$$

for any $m + |\alpha| + |\mu| \geq 1$.

Now, let us recall that by Faá di Bruno’s formula (see [1]) or [[4], Lemma 4.3] we have

$$\frac{1}{m!\alpha!\mu!} h^{(m,\alpha,\mu)} = \sum_{1 \leq p+q+\nu \leq m+|\alpha|+|\mu|} \frac{1}{p!q!\nu!} F^{(p,q,\nu)} \times \sum_{|k^*|=m-p} \prod_{i=1}^{d} \prod_{j=1}^{\nu_i} \left( \frac{1}{k_i(j)!(\beta_i(j)!(\gamma_i(j)!} u_i^{(k_i(j),\beta_i(j),\gamma_i(j))}\right),$$

Therefore, by using estimates $(1/p!q!\nu!)|F^{(p,q,\nu)}| \leq A_{p,q,\nu} p!\nu!$ on $L$ and $(1/k!\beta!\gamma!) u_i^{(k_i,\beta_i,\gamma_i)} \leq B_{i,k,\beta,\gamma} \delta_1^{-1}(|\beta| - \delta_2)^s - 1 (|\gamma| - \delta_3)^{s_2 - 1}$ (where $(\delta_1, \delta_2, \delta_3) = (1, 0, 0)$ if $k \geq 1$, $(\delta_1, \delta_2, \delta_3) = (0, 1, 0)$ if $k = 0$ and $|\beta| \geq 1$, and $(\delta_1, \delta_2, \delta_3) = (0, 0, 1)$ if $k = 0, \beta = 0$ and $|\gamma| \geq 1$) on $Z$, we have

$$\frac{1}{m!s-1|\alpha|!\sigma-1|\mu|!s_2-1} \frac{1}{m!\alpha!\mu!} h^{(m,\alpha,\mu)} \leq \sum_{1 \leq p+q+\nu \leq m+|\alpha|+|\mu|} A_{p,q,\nu} p!\nu! \times \sum_{|k^*|=m-p} \prod_{i=1}^{d} \prod_{j=1}^{\nu_i} \left( B_{i,k_i(j),\beta_i(j),\gamma_i(j)} (k_i(j) - \delta_1)^s - 1 \times (|\beta_i(j)| - \delta_2)^{\sigma - 1} (|\gamma_i(j)| - \delta_3)^{s_2 - 1}\right)$$
\[
\leq \sum_{1 \leq p+|q|+|\nu| \leq m+|\alpha|+|\mu|} A_{p,q,\nu} \frac{p!^{s_{1}-1} |q|!^{\sigma-1} |\nu|!^{s_{2}-1}}{m!^{s-1} |\alpha|!^{\sigma-1} |\mu|!^{s_{2}-1}} \times \\
\sum_{k^{*}: |k^{*}| = m-p, s(\beta(\nu)) = \alpha-q, s(\gamma(\nu)) = \mu} \prod_{i=1}^{d} \prod_{j=1}^{\nu_{i}} \left( B_{i,k_{i}(j),\beta_{i}(j),\gamma_{i}(j)} \right),
\]
where \(|\beta(\nu)^{*}| = \sum_{i=1}^{d} \sum_{j=1}^{\nu_{i}} |\beta_{i}(j)|, |\gamma(\nu)^{*}| = \sum_{i=1}^{d} \sum_{j=1}^{\nu_{i}} |\gamma_{i}(j)|, n_{1} = \#\{(i,j); k_{i}(j) \geq 1\}, n_{2} = \#\{(i,j); k_{i}(j) = 0, |\beta_{i}(j)| \geq 1\}, \text{ and } n_{3} = \#\{(i,j); k_{i}(j) = 0, \beta_{i}(j) = 0, |\gamma_{i}(j)| \geq 1\}. \text{ Since } n_{1} + n_{2} + n_{3} = |\nu|, s \geq s_{2} \text{ and } \sigma \geq s_{2} \text{ hold, we have}
\[
|\nu|!^{s_{2}-1} \leq (3|\nu|n_{1}!n_{2}!n_{3}!)^{s_{2}-1} \leq 3^{|\nu|(s_{2}-1)} n_{1}!^{s-1} n_{2}!^{\sigma-1} n_{3}!^{s_{2}-1},
\]
and so
\[
\frac{p!^{s_{1}-1} |q|!^{\sigma-1} |\nu|!^{s_{2}-1}}{m!^{s-1} |\alpha|!^{\sigma-1} |\mu|!^{s_{2}-1}} \times \\
\times (|k^{*}| - n_{1})!^{s-1}(|\beta(\nu)^{*}| - n_{2})!^{\sigma-1}(|\gamma(\nu)^{*}| - n_{3})!^{s_{2}-1}
\]
\[
\leq \frac{p!^{s_{1}-1} |q|!^{\sigma-1} |\nu|!^{s_{2}-1}}{m!^{s-1} |\alpha|!^{\sigma-1} |\mu|!^{s_{2}-1}} \times \\
\times (m-p-n_{1})!^{s-1}(|\alpha| - |q| - n_{2})!^{\sigma-1}(|\mu| - n_{3})!^{s_{2}-1}
\]
\[
= 3^{|\nu|(s_{2}-1)} \frac{p!^{s_{1}-1} n_{1}!^{s-1} (m-p-n_{1})!^{s-1}}{m!^{s-1}} \\
\times \frac{|q|!^{\sigma-1} n_{2}!^{\sigma-1} (|\alpha| - |q| - n_{2})!^{\sigma-1} n_{3}!^{s_{2}-1} (|\mu| - n_{3})!^{s_{2}-1}}{|\mu|!^{s_{2}-1}}
\]
\[
\leq 3^{|\nu|(s_{2}-1)} \leq 3^{(m+|\alpha|+|\mu|)(s_{2}-1)}.
\]
Thus, by applying this to (5.3) we obtain
\[
\frac{1}{m!^{s-1} |\alpha|!^{\sigma-1} |\mu|!^{s_{2}-1}} \left| \frac{1}{m!^{s-1} |\alpha|!^{\sigma-1} |\mu|!^{s_{2}-1}} h^{(m,\alpha,\mu)} \right|
\]
\[
\leq 3^{(m+|\alpha|+|\mu|)(s_{2}-1)} \sum_{1 \leq p+|q|+|\nu| \leq m+|\alpha|+|\mu|} A_{p,q,\nu} \sum_{k^{*}: |k^{*}| = m-p, s(\beta(\nu)) = \alpha-q, s(\gamma(\nu)) = \mu} \prod_{i=1}^{d} \prod_{j=1}^{\nu_{i}} \left( B_{i,k_{i}(j),\beta_{i}(j),\gamma_{i}(j)} \right)
\]
\[
= 3^{(m+|\alpha|+|\mu|)(s_{2}-1)} C_{m,\alpha,\mu}
\]
on \mathbb{Z}. \text{ This proves (5.2).}\]

Some versions of the implicit function theorem in ultra-differentiable classes are given in Komatsu [3] and Yamanaka [5]. For the self-containedness, we will give here an implicit function theorem which is used in §3.
As before, we write $t \in \mathbb{R}_t$, $x = (x_1, \ldots, x_n) \in \mathbb{R}_x^n$, $z = (z_1, \ldots, z_d) \in \mathbb{R}_z^d$ and $w \in \mathbb{R}_w$. Let $\Omega$ be an open neighborhood of $(0, 0, 0, 0) \in \mathbb{R}_t \times \mathbb{R}_x^n \times \mathbb{R}_z^d \times \mathbb{R}_w$, let $F(t, x, z, w)$ be a $C^\infty$-function on $\Omega$, let $s_1 \geq 1$, $s_2 \geq 1$, $\sigma_1 \geq 1$ and $\sigma_2 \geq 1$, and suppose: $\sigma_i \geq s_2$ for $i = 1, 2$. We have

**Theorem 5.2.** Suppose the following conditions: $F(t, x, z, w) \in \mathcal{E}^{\{s_1, \sigma_1, \sigma_2, s_2\}}(\Omega)$, $F(0, 0, 0, 0) = 0$ and $(\partial F/\partial w)(0, 0, 0, 0) \neq 0$. Then, there are an open neighborhood $W$ of $(0, 0, 0) \in \mathbb{R}_t \times \mathbb{R}_x^n \times \mathbb{R}_z^d$ and a function $\phi(t, x, z) \in C^\infty(W)$ which satisfy $\phi(0, 0, 0) = 0$ and the following properties:

(5.4) \[ W \ni (t, x, z) \implies (t, x, z, \phi(t, x, z)) \in \Omega, \]

(5.5) \[ F(t, x, z, \phi(t, x, z)) = 0 \quad \text{on } W. \]

Moreover, we have $\phi(t, x, z) \in \mathcal{E}^{\{s, \sigma_1, \sigma_2\}}(W)$ for any $s \geq \max\{s_1, s_2\}$.

**Proof.** The former half of the result is nothing but the result of the implicit function theorem in the $C^\infty$ class, and so we know that there are an open neighborhood $W$ of $(0, 0, 0) \in \mathbb{R}_t \times \mathbb{R}_x^n \times \mathbb{R}_z^d$ and a function $\phi(t, x, z) \in C^\infty(W)$ which satisfy $\phi(0, 0, 0) = 0$, (5.4) and (5.5).

Let us show that $\phi(t, x, z) \in \mathcal{E}^{\{s, \sigma_1, \sigma_2\}}(W)$ holds for any $s \geq \max\{s_1, s_2\}$. Since the problem is set in a local sense, without loss of generality we may suppose that $(\partial F/\partial w)(t, x, z, w) \neq 0$ holds on $\Omega$. Then, by (5.5) we have the equality

\[
\begin{align*}
\phi_t(t, x, z) &= -\frac{F_t(t, x, z, \phi)}{F_w(t, x, z, \phi)}, \\
\phi_{x_i}(t, x, z) &= -\frac{F_{x_i}(t, x, z, \phi)}{F_w(t, x, z, \phi)}, \quad i = 1, \ldots, n, \\
\phi_{z_j}(t, x, z) &= -\frac{F_{z_j}(t, x, z, \phi)}{F_w(t, x, z, \phi)}, \quad j = 1, \ldots, d,
\end{align*}
\]

where $\phi_t = \partial \phi/\partial t$, $F_t = \partial F/\partial t$, $\phi_{x_i} = \partial \phi/\partial x_i$, $F_{x_i} = \partial F/\partial x_i$ and so on. Therefore, if we set $G(t, x, z) = -(F_t/F_w)(t, x, z, w)$, $H_i(t, x, z) = -(F_{x_i}/F_w)(t, x, z, w)$ ($i = 1, \ldots, n$) and $K_j(t, x, z, w) = -(F_{z_j}/F_w)(t, x, z, w)$ ($j = 1, \ldots, d$) we have the system of equations

\[
\begin{align*}
\phi_t &= G(t, x, z, \phi), \\
\phi_{x_i} &= H_i(t, x, z, \phi), \quad i = 1, \ldots, n, \\
\phi_{z_j} &= K_j(t, x, \phi), \quad j = 1, \ldots, d.
\end{align*}
\]

Since $F_w(t, x, z, w) \neq 0$ on $\Omega$, by the condition $F(t, x, z, w) \in \mathcal{E}^{\{s_1, \sigma_1, \sigma_2, s_2\}}(\Omega)$ we have

- $G(t, x, z, w) \in \mathcal{E}^{\{s_1, \sigma_1, \sigma_2, s_2\}}(\Omega)$,
- $H_i(t, x, z, w) \in \mathcal{E}^{\{s_1, \sigma_1, \sigma_2, s_2\}}(\Omega)$, \quad $i = 1, \ldots, n$,
- $K_j(t, x, z, w) \in \mathcal{E}^{\{s_1, \sigma_1, \sigma_2, s_2\}}(\Omega)$, \quad $j = 1, \ldots, d$. 

(5.7)
Thus, to complete the proof of Theorem 5.2 it is enough to show

**Proposition 5.3.** Let \( s_1 \geq 1, s_2 \geq 1, \sigma_1 \geq 1 \) and \( \sigma_2 \geq 1 \). Suppose the conditions (5.7) and \( \sigma_i \geq s_2 \) for \( i = 1, 2 \). If \( \phi(t, x, z) \in C^\infty(W) \) is a solution of (5.6), we have \( \phi(t, x, z) \in \mathcal{E}^{(s,\sigma_1,\sigma_2)}(W) \) for any \( s \geq \max\{s_1, s_2\} \).

*Proof of Proposition 5.3.* Let \( Z \) be a compact subset of \( W \), and let \( L \) be the image of \( Z \) by the mapping: \( (t, x, z) \mapsto (t, x, z, \phi(t, x, z)) \in \Omega \). We take \( A_{p,q_1,q_2,k} \geq 0 \) \( (p \in \mathbb{N}, q_1 \in \mathbb{N}^n, q_2 \in \mathbb{N}^d \) and \( k \in \mathbb{N} \)), \( B_{q_1,q_2,k}^{(i)} \geq 0 \) \( (i = 1, \ldots, n, q_1 \in \mathbb{N}^n, q_2 \in \mathbb{N}^d \) and \( k \in \mathbb{N} \)) and \( C_{q_2,k}^{(j)} \geq 0 \) \( (j = 1, \ldots, d, q_2 \in \mathbb{N}^d \) and \( k \in \mathbb{N} \)) so that

- \( \max_{(t,x,z,w) \in L} \left| \frac{G^{(p,q_1,q_2,k)}(t,x,z,w)}{p!q_1!q_2!k!} \right| \leq A_{p,q_1,q_2,k} p!^{s_1-1} |q_1|!^{\sigma_1-1} |q_2|!^{\sigma_2-1} k!^{s_2-1} \),
- \( \max_{(t,x,z,w) \in L} \left| \frac{H_{i}^{(0,q_1,q_2,k)}(t,x,z,w)}{q_1!q_2!k!} \right| \leq B_{q_1,q_2,k}^{(i)} |q_1|!^{\sigma_1-1} |q_2|!^{\sigma_2-1} k!^{s_2-1} \),
- \( \max_{(t,x,z,w) \in L} \left| \frac{K_{j}^{(0,0,q_2,k)}(t,x,z,w)}{q_2!k!} \right| \leq C_{q_2,k}^{(j)} |q_2|!^{\sigma_2-1} k!^{s_2-1} \).

Let us consider the following functional equation:

\[
(5.8) \quad Y(t, x, z) = t \sum_{p+|q_1|+|q_2|+k \geq 0} A_{p,q_1,q_2,k} t^p x^{q_1} z^{q_2} (3^{s_2-1} Y)^k \\
+ \sum_{i=1}^n x_i \sum_{|q_1|+|q_2|+k \geq 0} B_{q_1,q_2,k}^{(i)} x^{q_1} z^{q_2} (2^{s_2-1} Y)^k + \sum_{j=1}^d z_j \sum_{|q_2|+k \geq 0} C_{q_2,k}^{(j)} z^{q_2} Y^k.
\]

Since this is an analytic functional equation, by the implicit function theorem in holomorphic category we see that (5.8) has a unique holomorphic solution \( Y(t, x, z) \) in a neighborhood of \( (0,0,0) \in \mathbb{C}_t^n \times \mathbb{C}_x^n \times \mathbb{C}_z^d \) satisfying \( Y(0,0,0) = 0 \). Let

\[
Y(t, x, z) = \sum_{m+|\alpha_1|+|\alpha_2| \geq 1} Y_{m,\alpha_1,\alpha_2} t^m x^{\alpha_1} z^{\alpha_2},
\]

be the Taylor expansion of \( Y(t, x, z) \). We see that the coefficients \( Y_{m,\alpha_1,\alpha_2} \) \( (m + |\alpha_1| + |\alpha_2| \geq 1) \) are determined by the following recurrent formulas:

\[
Y_{1,0,0} = A_{0,0,0,0}, \quad Y_{0,e_i,0} = B_{0,0,0}^{(i)} (i = 1, \ldots, n), \quad Y_{0,0,e_j} = C_{0,0}^{(j)} (j = 1, \ldots, d)
\]

(where \( e_1 = (1,0,\ldots,0), \ldots, e_n = (0,\ldots,0,1) \in \mathbb{N}^n \) and \( e_1 = (1,0,\ldots,0), \ldots, e_d = \ldots \)).
(0, \ldots, 0, 1) \in \mathbb{N}^d$, and for $M = m + |\alpha_1| + |\alpha_2| \geq 2$ we have

\begin{align*}
Y_{m, \alpha_1, \alpha_2} &= \sum_{0 \leq p + |q_1| + |q_2| + k \leq M - 1} \sum_{|\mu^*| = m - p - 1} A_{p, q_1, q_2, k} \prod_{l=1}^{k} 3^{s_2(l)} Y_{\mu(l), \beta_1(l), \beta_2(l)} \\
&\quad + \sum_{i=1}^{n} \sum_{0 \leq |q_1| + |q_2| + k \leq M - 1} B_{q_1, q_2, k}^{(i)} \prod_{l=1}^{k} 2^{s_2(l)} Y_{\mu(l), \beta_1(l), \beta_2(l)} \\
&\quad + \sum_{j=1}^{d} \sum_{0 \leq |q_2| + k \leq M - 1} C_{q_2, k}^{(j)} \prod_{l=1}^{k} Y_{\mu(l), \beta_1(l), \beta_2(l)}
\end{align*}

where $|\mu^*| = \mu(1) + \cdots + \mu(k)$, $s(\beta_1^*) = \sum_{l=1}^{k} \beta_1(l)$ and $s(\beta_2^*) = \sum_{l=1}^{k} \beta_2(l)$. We have

**Lemma 5.4.** Take any $s \geq \max\{s_1, s_2\}$. Then, we have:

\begin{enumerate}
\item $\max_{(t,x,z) \in Z} \left| \frac{\phi^{(m, \alpha_1, \alpha_2)}(t,x,z)}{m! \alpha_1! \alpha_2!} \right| \leq Y_{m, \alpha_1, \alpha_2} (m - 1)!^{s - 1} |\alpha_1|!^{s_1 - 1} |\alpha_2|!^{s_2 - 1}, \text{ if } m \geq 1,$
\item $\max_{(t,x,z) \in Z} \left| \frac{\phi^{(0, \alpha_1, \alpha_2)}(t,x,z)}{\alpha_1! \alpha_2!} \right| \leq Y_{0, \alpha_1, \alpha_2} (|\alpha_1| - 1)!^{s_1 - 1} |\alpha_2|!^{s_2 - 1}, \text{ if } |\alpha_1| \geq 1,$
\item $\max_{(t,x,z) \in Z} \left| \frac{\phi^{(0, 0, \alpha_2)}(t,x,z)}{\alpha_2!} \right| \leq Y_{0, 0, \alpha_2} (|\alpha_2| - 1)!^{s_2 - 1}, \text{ if } |\alpha_2| \geq 1.$
\end{enumerate}

**Proof of Lemma 5.4.** We will prove this by induction on $M = m + |\alpha_1| + |\alpha_2|$. By (5.6) we have $|\phi_i| = |G(t, x, z, \phi)| \leq A_{0,0,0,0} = Y_{1,0,0,0}$, $|\phi_{x_1}| = |H_i(t, x, z, \phi)| \leq B_{0,0,0}^{(i)} = Y_{0,0,0,0}$ ($i = 1, \ldots, n$) and $|\phi_{z_j}| = |K_j(t, x, z, \phi)| \leq C_{0,0}^{(j)} = Y_{0,0,0,\epsilon_j}$ ($j = 1, \ldots, d$). This proves the case $M = 1$.

Suppose that $M = m + |\alpha_1| + |\alpha_2| \geq 2$. If $m \geq 1$, by (5.6) we have

\begin{align*}
\frac{\phi^{(m, \alpha_1, \alpha_2)}(t, x, z)}{m! \alpha_1! \alpha_2!} &= \frac{\phi^{(m-1, \alpha_1, \alpha_2)}(t, x, z)}{m! \alpha_1! \alpha_2!} \frac{1}{m! \alpha_1! \alpha_2!} \partial_t^{m-1} \partial_{x_1}^{\alpha_1} \partial_{z_2}^{\alpha_2} G(t, x, z, \phi) \\
&= \frac{1}{m} \sum_{0 \leq p + |q_1| + |q_2| + k \leq M - 1} \frac{G^{(p, q_1, q_2, k)}}{p! q_1! q_2! k!} \prod_{l=1}^{k} \frac{\phi^{(\mu(l), \beta_1(l), \beta_2(l))}}{\mu(l)! \beta_1(l)! \beta_2(l)!}
\end{align*}
and therefore
\[(5.10)\]
\[
\frac{|\phi^{(m,\alpha_{1},\alpha_{2})}(t, x, z)|}{m!\alpha_{1}!\alpha_{2}!} \leq \frac{1}{m} \sum_{0 \leq p + |q_{1}| + |q_{2}| + k \leq M-1} A_{p,q_{1},q_{2},k} p!^{s_{1}-1} |q_{1}|!^{\sigma_{1}-1} |q_{2}|!^{\sigma_{2}-1} k!^{s_{2}-1} \times
\]
\[
\sum_{|\mu^{*}|=m-p-1} \prod_{l=1}^{k} Y_{\mu(l),\beta_{1}(l),\beta_{2}(l)} (\mu(l) - \delta_{0})!^{s_{2}-1} (|\beta_{1}(l)| - \delta_{1})!^{\sigma_{1}-1} (|\beta_{2}(l)| - \delta_{2})!^{\sigma_{2}-1}
\]
where \((\delta_{0}, \delta_{1}, \delta_{2}) = (1, 0, 0)\) if \(\mu(l) \geq 1\), \((\delta_{0}, \delta_{1}, \delta_{2}) = (0, 1, 0)\) if \(\mu(l) = 0\) and \(|\beta_{1}(l)| \geq 1\), and \((\delta_{0}, \delta_{1}, \delta_{2}) = (0, 0, 1)\) if \(\mu(l) = 0\), \(\beta_{1}(l) = 0\) and \(|\beta_{2}(l)| \geq 1\). If we set \(n_{0} = \#\{l; \mu(l) \geq 1\}\), \(n_{1} = \#\{l; \mu(l) = 0, |\beta_{1}(l)| \geq 1\}\) and \(n_{2} = \#\{l; \mu(l) = 0, \beta_{1}(l) = 0, |\beta_{2}(l)| \geq 1\}\), then we have \(n_{0} + n_{1} + n_{2} = k\) and
\[
k^{s_{2}-1} \leq (3^{k} n_{0}! n_{1}! n_{2}!)^{s_{2}-1} \leq 3^{k(s_{2}-1)} n_{0}!^{s_{1}-1} n_{1}!^{\sigma_{1}-1} n_{2}!^{\sigma_{2}-1}.
\]
By applying this to (5.10) and by using \(s \geq s_{1}\) we have
\[
\frac{|\phi^{(m,\alpha_{1},\alpha_{2})}(t, x, z)|}{m!\alpha_{1}!\alpha_{2}!} \leq \frac{1}{m} \sum_{0 \leq p + |q_{1}| + |q_{2}| + k \leq M-1} A_{p,q_{1},q_{2},k} p!^{s_{1}-1} |q_{1}|!^{\sigma_{1}-1} |q_{2}|!^{\sigma_{2}-1} \times
\]
\[
\times \sum_{|\mu^{*}|=m-p-1} \prod_{l=1}^{k} 3^{s_{2}-1} Y_{\mu(l),\beta_{1}(l),\beta_{2}(l)} (\mu(l) - \delta_{0})!^{s_{2}-1} (|\beta_{1}(l)| - \delta_{1})!^{\sigma_{1}-1} (|\beta_{2}(l)| - \delta_{2})!^{\sigma_{2}-1}
\]
\[
\leq (m - 1)!^{s_{1}-1} |\alpha_{1}|!^{\sigma_{1}-1} |\alpha_{2}|!^{\sigma_{2}-1} \times
\]
\[
\times \sum_{|\mu^{*}|=m-p-1} \prod_{l=1}^{k} 3^{s_{2}-1} Y_{\mu(l),\beta_{1}(l),\beta_{2}(l)} (\mu(l) - \delta_{0})!^{s_{2}-1} (|\beta_{1}(l)| - \delta_{1})!^{\sigma_{1}-1} (|\beta_{2}(l)| - \delta_{2})!^{\sigma_{2}-1}
\]
\[
\leq (m - 1)!^{s_{1}-1} |\alpha_{1}|!^{\sigma_{1}-1} |\alpha_{2}|!^{\sigma_{2}-1} \times Y_{m,\alpha_{1},\alpha_{2}}.
\]
This proves (1).

By using \(\phi_{x_{i}} = H_{i}(t, x, z, \phi)\) \((i = 1, \ldots, n)\) and \(\phi_{z_{j}} = K_{j}(t, x, z, \phi)\) \((j = 1, \ldots, d)\), we can prove (2) and (3) in the same way. \(\square\)
Completion of the proof of Proposition 5.3. Since $Y(t, x, z)$ is a holomorphic function in a neighborhood of $(0, 0, 0) \in \mathbb{C}_t \times \mathbb{C}^n_x \times \mathbb{C}^d_z$, by Cauchy’s inequality we can take $C > 0$ and $h > 0$ so that

$$Y_{m, \alpha_1, \alpha_2} \leq Ch^{m+|\alpha_1|+|\alpha_2|}$$

holds for all $m \in \mathbb{N}$, $\alpha_1 \in \mathbb{N}^n$ and $\alpha_2 \in \mathbb{N}^d$. By combining this with Lemma 5.4 we have the result: $\phi(t, x, z) \in \mathcal{E}^{s, \sigma_1, \sigma_2}(Z)$. This proves Proposition 5.3. □

This completes the proof of Theorem 5.2. □

References


