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Time regularity in Gevrey classes of solutions to general nonlinear partial differential equations

By

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Abstract

The paper considers general nonlinear partial differential equations

(E) $$F\left(t, x, \{(\partial/\partial t)^j(\partial/\partial x)^\alpha u\}_{j\le m, |\alpha|\le L}\right) = 0$$

(with $1 \le m \le L$) in Gevrey classes, and gives a sufficient condition for the following assertion to be valid: if a solution $u(t,x)$ is in C^∞ class with respect to the time variable t and in the Gevrey class $\mathcal{E}^{\{\sigma\}}$ in the space variable x, then it is in the Gevrey class $\mathcal{E}^{\{s\}}$ also with respect to the time variable t for a suitable s. In [4] we have discussed this problem in a class of nonlinear partial differential equations; in this paper we will discuss the problem in the general case (E).

§ 1. Introduction

We denote by t the time variable in $\mathbb{R}_t$, and by $x = (x_1, \dots, x_n)$ the space variable in $\mathbb{R}^n_x$. We use the notations: $\mathbb{N} = \{0, 1, 2, \dots\}$, $\mathbb{N}^* = \{1, 2, \dots\}$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\partial_t = \partial/\partial t$, $\partial_x = (\partial_{x_1}, \dots, \partial_{x_n})$ with $\partial_{x_i} = \partial/\partial x_i$ $(i = 1, \dots, n)$ and $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$.

For $\sigma \ge 1$ and an open subset V of $\mathbb{R}^n_x$ we denote by $\mathcal{E}^{\{\sigma\}}(V)$ the set of all functions $f(x) \in C^\infty(V)$ satisfying the following: for any compact subset K of V there are $C > 0$ and $h > 0$ such that

$$\max_{x\in K} |\partial_x^\alpha f(x)| \le C h^{|\alpha|} |\alpha|!^\sigma, \quad \forall \alpha \in \mathbb{N}^n.$$

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A function in the class $\mathcal{E}^{\{\sigma\}}(V)$ is called *a function of the Gevrey class of order* σ.

If $\sigma = 1$, the class $\mathcal{E}^{\{1\}}(V)$ is nothing but the set of all analytic functions on V and usually it is denoted by $\mathcal{A}(V)$. If $1 < \sigma_1 < \sigma_2 < \infty$ we have

$$\mathcal{A}(V) \subset \mathcal{E}^{\{\sigma_1\}}(V) \subset \mathcal{E}^{\{\sigma_2\}}(V) \subset C^\infty(V).$$

Thus, functions in the class $\mathcal{E}^{\{\sigma_1\}}(V)$ are closer to analytic functions than those in $\mathcal{E}^{\{\sigma_2\}}(V)$; in this sense, we can say that functions in $\mathcal{E}^{\{\sigma_1\}}(V)$ are more regular than those in $\mathcal{E}^{\{\sigma_2\}}(V)$.

For an interval $[0,T] = \{t \in \mathbb{R}\,;\, 0 \le t \le T\}$ we denote by $C^\infty([0,T],\, \mathcal{E}^{\{\sigma\}}(V))$ the set of all infinitely differentiable functions $u(t,x)$ in $t \in [0,T]$ with values in $\mathcal{E}^{\{\sigma\}}(V)$ equipped with the usual locally convex topology (see [2]).

Similarly, for $s \ge 1$ and $\sigma \ge 1$ we denote by $\mathcal{E}^{\{s,\sigma\}}([0,T]\times V)$ the set of all functions $u(t,x) \in C^\infty([0,T] \times V)$ satisfying the following: for any compact subset K of V there are $C > 0$ and $h > 0$ such that

$$\max_{(t,x)\in[0,T]\times K} |\partial_t^k \partial_x^\alpha u(t,x)| \le C h^{k+|\alpha|} k!^s |\alpha|!^\sigma, \quad \forall (k,\alpha) \in \mathbb{N} \times \mathbb{N}^n.$$

Obviously, we have

$$\mathcal{E}^{\{s,\sigma\}}([0,T] \times V) \subset C^\infty([0,T], \mathcal{E}^{\{\sigma\}}(V)).$$

In the case $s = \sigma$ we write $\mathcal{E}^{\{\sigma\}}([0,T] \times V)$ instead of $\mathcal{E}^{\{\sigma,\sigma\}}([0,T] \times V)$.

Let Ω be an open subset of $\mathbb{R}_t \times \mathbb{R}_x^n \times \mathbb{R}_z^d$: the Gevrey class $\mathcal{E}^{\{s_1,\sigma,s_2\}}(\Omega)$ is defined in the same way. In this case, s_1 denotes the Gevrey order in t, σ denotes the Gevrey order in x, and s_2 denotes the Gevrey order in z.

In this paper, we will consider the following nonlinear partial differential equation

$$F\Big(t, x, \{\partial_t^j \partial_x^\alpha u\}_{j \le m, |\alpha| \le L}\Big) = 0 \tag{1.1}$$

where $1 \le m \le L$ are positive integers, and $F(t,x,\{z_{j,\alpha}\}_{j+|\alpha|\le m})$ is a suitable function in a Gevrey class (for the precise assumptions, see §2). And, we will consider the following problem on Gevrey regularity in time:

Problem 1.1. Let $u(t,x) \in C^\infty([0,T], \mathcal{E}^{\{\sigma\}}(V))$ be a solution of (1.1); can we have the result $u(t,x) \in \mathcal{E}^{\{s,\sigma\}}([0,T]\times V)$ for a suitable $s \ge 1$? If this is true, determine the precise bound of the index s of the time regularity.

In the previous paper [4], we have studied this problem for the equation

$$t^\gamma \partial_t^m u = F\Big(t, x, \{\partial_t^j \partial_x^\alpha u\}_{j<m, |\alpha| \le L}\Big)$$

(with $\gamma \geq 0$ and $1 \leq m \leq L$). The purpose of this paper is to discuss the problem 1.1 for general equation (1.1).

§ 2. Formulation and main theorem

Let $m \in \mathbb{N}^*$ be fixed, Λ_0 be a finite subset of $\{(j,\alpha) \in \mathbb{N} \times \mathbb{N}^n\,;\, j < m\}$, and set $\Lambda = \Lambda_0 \cup \{(m,0)\}$ and $d = \#\Lambda$. We denote by $z = \{z_{j,\alpha}\}_{(j,\alpha)\in\Lambda}$ the variable in $\mathbb{R}^d_z$. Let Ω be an open subset of $\mathbb{R}_t \times \mathbb{R}^n_x \times \mathbb{R}^d_z$, let $F(t,x,z)$ be a C^∞ function on Ω, and let us consider the following nonlinear partial differential equation:

$$F\big(t,x,Du\big) = 0 \quad \text{with } Du = \{\partial_t^j \partial_x^\alpha u\}_{(j,\alpha)\in\Lambda}. \tag{2.1}$$

Let $s_1 \geq 1$ and $\sigma \geq s_2 \geq 1$ be real numbers, V be an open subset of $\mathbb{R}^n_x$, and $T > 0$. Our basic assumptions are:

a_1) $m \geq 1$, $s_1 \geq 1$ and $\sigma \geq s_2 \geq 1$;
a_2) $\Lambda = \Lambda_0 \cup \{(m,0)\}$ and Λ_0 is a finite subset of $\{(j,\alpha) \in \mathbb{N} \times \mathbb{N}^n\,;\, j < m\}$,
a_3) $F(t,x,z) \in \mathcal{E}^{\{s_1,\sigma,s_2\}}(\Omega)$,
a_4) $u(t,x) \in C^\infty([0,T], \mathcal{E}^{\{\sigma\}}(V))$ is a solution of (2.1) on $[0,T] \times V$; this includes the property: $(t,x) \in [0,T] \times V \Longrightarrow (t,x,Du(t,x)) \in \Omega$.

Let us define

Definition 2.1. (1) Let $u(t)$ be a C^∞-function in a neighborhood of $t = t_0$; we define the order of zero of $u(t)$ at $t = t_0$ (which we denote by $\mathrm{ord}_{t=t_0}(u(t))$) by the following:

$$\mathrm{ord}_{t=t_0}(u(t)) = \min\{k \in \mathbb{N}\,;\, u^{(k)}(t_0) \neq 0\}$$

(if $u^{(k)}(t_0) = 0$ for all $k \in \mathbb{N}$ we set $\mathrm{ord}_{t=t_0}(u(t)) = \infty$).

(2) Let W be an open subset of $\mathbb{R}_t \times \mathbb{R}^n_x$, and let $f(t,x) \in C^\infty(W)$. We define a $\mathbb{N} \cup \{\infty\}$-valued function $q(t_0,x_0;f)$ on W in the following way. Take any $(t_0,x_0) \in W$; then $f(t,x_0)$ is a C^∞-function in a neighborhood of $t = t_0$ and so we can define the order $\mathrm{ord}_{t=t_0}(f(t,x_0))$ of zero of $f(t,x_0)$ at $t = t_0$. We set $q(t_0,x_0;f) = \mathrm{ord}_{t=t_0}(f(t,x_0))$.

Under the conditions a_1), a_2), a_3) and a_4) we set

$$\begin{aligned} k_{j,\alpha}(t_0,x_0) &= q(t_0,x_0;(\partial F/\partial z_{j,\alpha})(t,x,Du(t,x))) \\ &= \mathrm{ord}_{t=t_0}((\partial F/\partial z_{j,\alpha})(t,x_0,(Du)(t,x_0))) \end{aligned} \tag{2.2}$$

(which is the order of zero of $(\partial F/\partial z_{j,\alpha})(t, x_0, (Du)(t, x_0))$ at $t = t_0$) and suppose:

(M) For any $(t_0, x_0) \in [0, T] \times V$ there are $\gamma \in \mathbb{N}$ and a neighborhood V_0 of $x_0 \in V$ which satisfy the following properties:

1) $k_{m,0}(t_0, x) = \gamma$ for any $x \in V_0$;
2) $k_{j,0}(t_0, x) \geq \gamma - m + j$ for any $x \in V_0$,
3) $k_{j,\alpha}(t_0, x) \geq \gamma - m + j + 1$ for any $x \in V_0$, if $|\alpha| > 0$.

In the condition (M), the constant γ may depends on (t_0, x_0) and so we may write $\gamma = \gamma(t_0, x_0)$: this function $\gamma(t, x)$ is locally constant with respect to x. Thus, if (M) is satisfied, we can take any connected neighborhood of x_0 as V_0 in the condition (M).

For any fixed $(t_0, x_0) \in [0, T] \times V$, by using $\gamma = \gamma(t_0, x_0)$ and a connected neighborhood V_0 of x_0 we set

$$k_{j,\alpha}(t_0, V_0) = \min_{x \in V_0} k_{j,\alpha}(t_0, x) \quad (\text{for } (j, \alpha) \in \Lambda_0), \quad \text{and}$$

$$s_0(t_0, V_0) = 1 + \max\left[0, \max_{(j,\alpha) \in \Lambda_0, |\alpha| > 0} \left(\frac{j + \sigma|\alpha| - m}{\min\{k_{j,\alpha}(t_0, V_0) - \gamma + m - j, m - j\}} \right)\right].$$

Note that these $k_{j,\alpha}(t_0, V_0)$ and $s_0(t_0, V_0)$ depend on $(t_0, x_0) \in [0, T] \times V$ and V_0, and that by 2) and 3) we have the conditions: $k_{j,0}(t_0, V_0) \geq \gamma - m + j$, and $k_{j,\alpha}(t_0, V_0) \geq \gamma - m + j + 1$ if $|\alpha| > 0$. We also note that if $V_1 \subset V_0$ we have $k_{j,\alpha}(t_0, V_1) \geq k_{j,\alpha}(t_0, V_0)$ and so we have $s_0(t_0, V_1) \leq s_0(t_0, V_0)$.

By using these indices $s_0(t_0, V_0)$ (for $(t_0, x_0) \in [0, T] \times V$ and V_0) we define the index $s_0 \geq 1$ by the following:

$$s_0 = \sup_{(t_0, x_0) \in [0, T] \times V} \left(\inf_{V_0 \ni x_0} s_0(t_0, V_0) \right). \tag{2.3}$$

We note:

Lemma 2.2. *Under the above situation, for any $(t_0, x_0) \in [0, T] \times V$ and any sufficient small neighborhood V_0 of x_0 we have $s_0 \geq s(t_0, V_0)$.*

Proof. We set

$$K = \sup_{V_0 \ni x_0} k_{j,\alpha}(t_0, V_0).$$

If $K = \infty$, for any $N > 0$ we can take a V_0 such that $k_{j,\alpha}(t_0, V_0) > N$ and so we have $\min\{k_{j,\alpha}(t_0, V_0) - \gamma + m - j, m - j\} = m - j$ for any sufficiently small V_0. If $K < \infty$, by the condition that $k_{j,\alpha}(t_0, V_0)$ is a $\mathbb{N}$-valued function we see that $k_{j,\alpha}(t_0, V_0) = K$ holds for any sufficiently small V_0. Thus, by the definition of $s_0(t_0, V_0)$ we can conclude that $s_0(t_0, V_0)$ is independent of V_0 if V_0 is sufficiently small. This proves Lemma 2.2. □

The following result is the main theorem of this paper.

Theorem 2.3. *Suppose the conditions* $a_1)$, $a_2)$, $a_3)$, $a_4)$ *and* (M); *then we have* $u(t,x) \in \mathcal{E}^{\{s,\sigma\}}([0,T] \times V)$ *for any* $s \geq \max\{s_0, s_1, s_2\}$.

§ 3. Proof of Theorem 2.3

Take any $s \geq \max\{s_0, s_1, s_2\}$. To prove Theorem 2.3, it is enough to show the following assertion: *for any* $(t_0, x_0) \in [0,T] \times V$ *we can find a* $\delta > 0$ *and a small compact neighborhood* V_1 *of* x_0 *such that* $u(t,x) \in \mathcal{E}^{\{s,\sigma\}}([t_0, t_0+\delta] \times V_1)$ (*or* $u(t,x) \in \mathcal{E}^{\{s,\sigma\}}([t_0-\delta, t_0] \times V_1)$) *holds.* By changing the variable $t \longrightarrow t - t_0$, we have only to discuss the case $t_0 = 0$.

Take any $(0, x_0)$ and fix it. Let $\gamma = \gamma(0, x_0)$ and take a sufficiently small connected neighborhood V_0 of x_0: then by Lemma 2.2 we have $s_0 \geq s(t_0, V_0)$. For simplicity we set $k_{j,\alpha} = k_{j,\alpha}(0, V_0)$ (for $(j,\alpha) \in \Lambda_0$). Take a sufficiently small $T_0 > 0$, and we have $u(t,x) \in C^\infty([0,T_0], \mathcal{E}^{\{\sigma\}}(V_0))$. Since $k_{m,0}(0,x) = \gamma$ holds on V_0, we have

$$\left[t^{-\gamma} \frac{\partial F}{\partial z_{m,0}}(t,x,Du) \right]\bigg|_{t=0} \neq 0 \quad \text{on } V_0. \tag{3.1}$$

Moreover, we have $k_{j,0} \geq \gamma - m + j$, and $k_{j,\alpha} \geq \gamma - m + j + 1$ if $|\alpha| > 0$. By the condition $s \geq \max\{s_0, s_1, s_2\}$ we have $s \geq \max\{s_1, s_2\}$ and

$$s \geq 1 + \max\left[0, \max_{(j,\alpha)\in\Lambda_0, |\alpha|>0} \left(\frac{j + \sigma|\alpha| - m}{\min\{k_{j,\alpha} - \gamma + m - j, m - j\}} \right) \right]. \tag{3.2}$$

We will consider the equation only on $[0, T_0] \times V_0$.

Let us reduce our equation (2.1) to an equation discussed in [4]. First, we take an integer q with

$$q \geq \gamma + m + 1 \tag{3.3}$$

and set

$$u(t,x) = \varphi(t,x) + t^q w(t,x) \quad \text{with } \varphi(t,x) = \sum_{k=0}^{q-1} \frac{(\partial_t^k u)(0,x)}{k!} t^k;$$

then we have $\varphi(t,x) \in \mathcal{E}^{\{1,\sigma\}}(\mathbb{R} \times V_0)$ and $w(t,x) \in C^\infty([0,T_0], \mathcal{E}^{\{\sigma\}}(V_0))$. Since

$$Du = D\varphi + \{t^{q-j}[t\partial_t + q]_j \partial_x^\alpha w\}_{(j,\alpha)\in\Lambda}$$

(where $[\lambda]_0 = 1$ and $[\lambda]_p = \lambda(\lambda-1)\cdots(\lambda-p+1)$ for $p \geq 1$), and since $u(t,x)$ is a solution of (2.1) we have

$$F\Big(t, x, D\varphi + \{t^{q-j}[t\partial_t + q]_j \partial_x^\alpha w\}_{(j,\alpha)\in\Lambda}\Big) = 0 \tag{3.4}$$

which is regarded as an equation with respect to $w(t,x)$. To see the result $u(t,x) \in \mathcal{E}^{\{s,\sigma\}}([0,\delta] \times V_1)$, it is enough to show the condition: $w(t,x) \in \mathcal{E}^{\{s,\sigma\}}([0,\delta] \times V_1)$.

Let us do a further reduction. By the formula

$$[\lambda + q]_j = (\lambda + q)(\lambda + q - 1) \cdots (\lambda + q - j + 1) = \sum_{0 \le i \le j} c_{j,i} \lambda^i,$$

we define the constants $c_{j,i}$ $(0 \le i \le j \le m)$: we see that $c_{j,j} = 1$ holds. Set

$$Z(t,z) = \{Z_{j,\alpha}(t,z)\}_{(j,\alpha)\in\Lambda} \quad \text{with} \quad Z_{j,\alpha}(t,z) = t^{q-j} \sum_{0 \le i \le j} c_{j,i} z_{i,\alpha},$$

$$H(t,x,z) = \frac{1}{t^{q+\gamma-m}} \times F(t,x,D\varphi(t,x) + Z(t,z)),$$

$$\Theta^* w = \{(t\partial_t)^j \partial_x^\alpha w\}_{(j,\alpha)\in\Lambda} \quad \text{and} \quad z^* = (\Theta^* w)(0,x_0) \in \mathbb{R}^d.$$

By the definition we have $Z_{j,\alpha}(t,\Theta^* w) = t^{q-j}[t\partial_t + q]_j \partial_x^\alpha w$ for any $(j,\alpha) \in \Lambda$, and so we have $D\varphi(t,x) + Z(t,\Theta^* w) = Du(t,x)$. By the same argument as in [§3, [4]] we have

Lemma 3.1. *Under the above situation we have the following results:*

(1) *Set* $\Omega_0 = \{(t,x,z) \in \mathbb{R}_t \times V_0 \times \mathbb{R}_z^d;\ (t,x,D\varphi(t,x) + Z(t,z)) \in \Omega\}$*; then we have* $(0,x_0,z^*) \in \Omega_0$ *and* $H(t,x,z) \in \mathcal{E}^{\{s^*,\sigma,s_2\}}(\Omega_0)$ *for* $s^* = \max\{s_1,s_2\}$.

(2) $w(t,x) \in C^\infty([0,T_0],\mathcal{E}^{\{\sigma\}}(V_0))$ *is a solution of*

$$H(t,x,\Theta^* w) = 0, \quad \text{on } [0,T_0] \times V_0, \tag{3.5}$$

and we have $(t,x,\Theta^* w(t,x)) \in \Omega_0$ *for any* $(t,x) \in [0,T_0] \times V_0$.

(3) $H(0,x_0,z^*) = 0$ *and* $(\partial H/\partial z_{m,0})(0,x_0,z^*) \neq 0$. *In addition we have*

$$\frac{\partial H}{\partial z_{m,0}}(t,x,\Theta^* w(t,x))\Big|_{t=0} \neq 0 \quad \text{on } V_0. \tag{3.6}$$

(4) *We set*

$$q_{j,\alpha} = \min_{x\in V_0}\Big(\mathrm{ord}_{t=0}((\partial H/\partial z_{j,\alpha})(t,x,\Theta^* w(t,x)))\Big), \quad (j,\alpha) \in \Lambda_0. \tag{3.7}$$

Then we have $q_{j,\alpha} \ge 1$ *if* $|\alpha| > 0$. *Moreover we see that the condition* (3.2) *implies*

$$s \ge 1 + \max\left[0, \max_{(j,\alpha)\in\Lambda_0, |\alpha|>0}\left(\frac{j + \sigma|\alpha| - m}{\min\{q_{j,\alpha}, m-j\}}\right)\right]. \tag{3.8}$$

Proof. By the condition (M) and (3.3) we have

$$(\partial F/\partial z_{j,\alpha})(t,x,D\varphi(t,x)) = \begin{cases} O(t^{\gamma-m+j}) & \text{on } V_0, \quad \text{if } |\alpha| = 0, \\ O(t^{\gamma-m+j+1}) & \text{on } V_0, \text{if } |\alpha| > 0 \end{cases}$$

(where $f(t,x) = O(t^a)$ means that $f(t,x) = O(t^a)$ uniformly in x (as $t \longrightarrow +0$)). In particular, in the case $(j,\alpha) = (m,0)$ we have $(\partial F/\partial z_{m,0})(t,x,D\varphi(t,x)) = O(t^\gamma)$ and by (3.1) we have

$$\left[t^{-\gamma}\frac{\partial F}{\partial z_{m,0}}(t,x,D\varphi)\right]\Big|_{t=0} \neq 0 \quad \text{on } V_0.$$

Therefore, by (3.4) and Taylor's formula we have

$$\begin{aligned}
0 &= F(t,x,D\varphi(t,x)) + \sum_{(j,\alpha)\in\Lambda}\frac{\partial F}{\partial z_{j,\alpha}}(t,x,D\varphi)O(t^{q-j}) + \sum_{(j,\alpha),(i,\beta)\in\Lambda} O(t^{q-j})O(t^{q-i}) \\
&= F(t,x,D\varphi(t,x)) + \sum_{(j,\alpha)\in\Lambda} O(t^{\gamma-m+j}) \times O(t^{q-j}) + O(t^{2q-2m}) \\
&= F(t,x,D\varphi(t,x)) + O(t^{q+\gamma-m}) \quad \text{on } V_0 :
\end{aligned}$$

this shows that $F(t,x,D\varphi(t,x)) = O(t^{q+\gamma-m})$ holds on V_0. Hence, by the definition of $H(t,x,z)$ we have

$$\begin{aligned}
H(t,x,z) &= \frac{1}{t^{q+\gamma-m}} \times F(t,x,D\varphi(t,x) + Z(t,z)) \\
&= \frac{1}{t^{q+\gamma-m}}\left[F(t,x,D\varphi(t,x)) + \sum_{(j,\alpha)}\frac{\partial F}{\partial z_{j,\alpha}}(t,x,D\varphi)O(t^{q-j}) + O(t^{2q-2m})\right] \\
&= \frac{1}{t^{q+\gamma-m}}\left[O(t^{q+\gamma-m}) + \sum_{(j,\alpha)} O(t^{\gamma-m+j}) \times O(t^{q-j}) + O(t^{2q-2m})\right]
\end{aligned}$$

(where $f(t,x,z) = O(t^a)$ means that $f(t,x,z) = O(t^a)$ uniformly in (x,z) (as $t \longrightarrow +0$)): this proves that $H(t,x,z)$ is well-defined as a C^∞ function on Ω_0. By Proposition 5.1 (in Appendix) we have the condition: $H(t,x,z) \in \mathcal{E}^{\{s^*,\sigma,s_2\}}(\Omega_0)$ with $s^* = \max\{s_1,s_2\}$. This proves (1).

(2) is clear from the definition of $H(t,x,z)$ and Ω_0. Since

$$\frac{\partial H}{\partial z_{m,0}}(t,x,\Theta^* w(t,x)) = \frac{1}{t^\gamma}\frac{\partial F}{\partial z_{m,0}}(t,x,Du(t,x))$$

holds, by (3.1) we have the result (3). Since

$$\begin{aligned}
\frac{\partial H}{\partial z_{j,\alpha}}(t,x,\Theta^* w(t,x)) &= \frac{1}{t^{q+\gamma-m}}\sum_{l\geq j}\frac{\partial F}{\partial z_{l,\alpha}}(t,x,Du(t,x))t^{q-l}c_{l,j} \\
&= \sum_{l\geq j} O(t^{k_{l,\alpha}-\gamma+m-l}) \quad \text{on } V_0,
\end{aligned}$$

we have the result (4). □

Now, let us apply the implicit function theorem [Theorem 5.2 (with $\sigma_1 = \sigma$ and $\sigma_2 = s_2$) in Appendix] to the functional equation

(3.9) $$H(t,x,z) = 0 \quad \text{in a neighborhood of } (0,x_0,z^*).$$

We write $z^* = (z'^*, z^*_{m,0})$ and $z = (z', z_{m,0})$. Since

$$H(0, x_0, z^*) = 0 \quad \text{and} \quad (\partial H/\partial z_{m,0})(0, x_0, z^*) \neq 0$$

hold, we can find an open neighborhood Ω_1 of $(0, x_0, z'^*) \in \mathbb{R}_t \times V_0 \times \mathbb{R}^{d-1}$ and a function $G(t, x, z')$ on Ω_1 which satisfy the following properties:

1) $G(t, x, z') \in \mathcal{E}^{\{s^*,\sigma,s_2\}}(\Omega_1)$;

2) $z^*_{m,0} = G(0, x_0, z'^*)$;

3) the functional relation (3.9) is equivalent to $z_{m,0} = G(t, x, z')$;

4) if we take $\delta > 0$ and an open neighborhood $V_1(\subset V_0)$ of x_0 sufficiently small, we have $(t, x, \Theta w(t, x)) \in \Omega_1$ for any $(t, x) \in [0, \delta] \times V_1$ and the function $w(t, x) \in C^\infty([0, \delta], \mathcal{E}^{\{\sigma\}}(V_1))$ is a solution of the equation

$$(3.10) \qquad (t\partial_t)^m w = G\big(t, x, \Theta w\big) \quad \text{with } \Theta w = \{(t\partial_t)^j \partial^\alpha w\}_{(j,\alpha)\in\Lambda_0}.$$

Moreover, we have

Lemma 3.2. *We set*

$$q_{j,\alpha}(G) = \min_{x\in V_1}\Big(\mathrm{ord}_{t=0}((\partial G/\partial z_{j,\alpha})(t, x, \Theta w(t, x)))\Big), \quad (j, \alpha) \in \Lambda_0.$$

Then we have $q_{j,\alpha}(G) \geq 1$ *if* $|\alpha| > 0$, *and that the condition* (3.2) *implies*

$$(3.11) \qquad s \geq 1 + \max\left[0, \max_{(j,\alpha)\in\Lambda_0,|\alpha|>0}\left(\frac{j + \sigma|\alpha| - m}{\min\{q_{j,\alpha}(G), m - j\}}\right)\right].$$

We denote by $s_0(G)$ *the right-hand side of* (3.11).

Proof. By the implicit function theorem we have

$$\frac{\partial G}{\partial z_{j,\alpha}}(t, x, \Theta w(t, x)) = -\frac{(\partial H/\partial z_{j,\alpha})(t, x, \Theta^* w(t, x))}{(\partial H/\partial z_{m,0})(t, x, \Theta^* w(t, x))}.$$

Since (3.6) holds, we have

$$\begin{aligned} &\mathrm{ord}_{t=0}\big((\partial G/\partial z_{j,\alpha})(t, x, \Theta w(t, x))\big) \\ &\qquad = \mathrm{ord}_{t=0}\big((\partial H/\partial z_{j,\alpha})(t, x, \Theta^* w(t, x))\big), \quad \forall x \in V_1 \end{aligned}$$

and by the condition $V_1 \subset V_0$ we have $q_{j,\alpha}(G) \geq q_{j,\alpha}$ for any $(j, \alpha) \in \Lambda_0$, where $q_{j,\alpha}$ $((j, \alpha) \in \Lambda_0)$ are the ones in (3.7). Thus, by (3.8) we have (3.11). □

Thus, we have seen that $w(t, x)$ is a solution of (3.10). Since (3.10) is just an equation discussed in [4], by [Theorems 5.0.1 and 6.1 in [4]] we have the following result which proves Theorem 2.3.

Theorem 3.3. *Under the above situation, we have* $w(t,x) \in \mathcal{E}^{\{s,\sigma\}}([0,\delta] \times V_1)$ *for any* $s \geq \max\{s_0(G), s^*\}$ *with* $s^* = \max\{s_1, s_2\}$.

§ 4. Application

In this section, we will consider the equation (2.1) under the assumption

$$F(t,x,z) \in \mathcal{E}^{\{\sigma\}}(\Omega) \tag{4.1}$$

for some $\sigma \geq 1$, and we will consider the following problem:

Problem 4.1. Let $u(t,x) \in C^\infty([0,T], \mathcal{E}^{\{\sigma\}}(V))$ be a solution of (2.1); can we have the result $u(t,x) \in \mathcal{E}^{\{\sigma\}}([0,T] \times V)$?

By using Theorem 2.3, let us give a sufficient condition for this problem to be affirmative. We see: by Theorem 2.3 (with $s_1 = s_2 = \sigma$) we have the result $u(t,x) \in \mathcal{E}^{\{\sigma\}}([0,T] \times V)$ if the condition $\sigma \geq s_0$ holds, that is, if for any $(t_0, x_0) \in [0,T] \times V$ and a sufficiently small neighborhood V_0 of x_0 we have

$$\sigma \geq 1 + \max\left[0, \max_{(j,\alpha)\in\Lambda_0, |\alpha|>0} \left(\frac{j + \sigma|\alpha| - m}{\min\{k_{j,\alpha}(t_0,V_0) - \gamma + m - j, m - j\}} \right)\right] \tag{4.2}$$

which is equivalent to

$$\begin{aligned} & m - j - \min\{k_{j,\alpha}(t_0,V_0) - \gamma + m - j, m - j\} \\ & \quad \geq \sigma(|\alpha| - \min\{k_{j,\alpha}(t_0,V_0) - \gamma + m - j, m - j\}) \\ & \text{for any } (j,\alpha) \in \Lambda_0 \text{ with } |\alpha| > 0. \end{aligned} \tag{4.3}$$

If $j + |\alpha| > m$ holds for some $(j,\alpha) \in \Lambda_0$, we have

$$1 + \frac{j + \sigma|\alpha| - m}{\min\{k_{j,\alpha}(t_0,V_0) - \gamma + m - j, m - j\}} \geq 1 + \frac{j + \sigma|\alpha| - m}{m - j} = \frac{\sigma|\alpha|}{m - j} > \sigma.$$

This shows that if $\Lambda_0 \not\subset \{(j,\alpha) \in \mathbb{N} \times \mathbb{N}^n \,;\, j + |\alpha| \leq m\}$ the condition (4.2) is not satisfied.

Let us consider the case:

$$\Lambda_0 \subset \{(j,\alpha) \in \mathbb{N} \times \mathbb{N}^n \,;\, j + |\alpha| \leq m\}. \tag{4.4}$$

If $|\alpha| \leq \min\{k_{j,\alpha}(t_0,V_0) - \gamma + m - j, m - j\}$ holds, the condition (4.3) is clear from the fact that the right-hand side is nonpositive and the left-hand side is nonnegative. If $|\alpha| > \min\{k_{j,\alpha}(t_0,V_0) - \gamma + m - j, m - j\}$ holds, by the condition $|\alpha| \leq m - j$ we have

$$\begin{aligned} m - j \geq |\alpha| & > \min\{k_{j,\alpha}(t_0,V_0) - \gamma + m - j, m - j\} \\ & = k_{j,\alpha}(t_0,V_0) - \gamma + m - j \end{aligned}$$

and so the inequality (4.3) is equivalent to

$$\sigma \leq \frac{m-j-(k_{j,\alpha}(t_0,V_0)-\gamma+m-j)}{|\alpha|-(k_{j,\alpha}(t_0,V_0)-\gamma+m-j)} = 1+\frac{m-j-|\alpha|}{|\alpha|-(k_{j,\alpha}(t_0,V_0)-\gamma+m-j)}$$

Therefore, if we set $\Delta = \{(j,\alpha) \in \Lambda_0\,;\, k_{j,\alpha}(t_0,V_0)-\gamma+m-j < |\alpha|\}$, our condition (4.2) is equivalent to

$$1 \leq \sigma \leq 1+\min\left[\infty, \min_{(j,\alpha)\in\Delta}\left(\frac{m-j-|\alpha|}{|\alpha|-(k_{j,\alpha}(t_0,V_0)-\gamma+m-j)}\right)\right]. \tag{4.5}$$

Thus, summinig up we have the following result. Set

$$\Delta(t_0,V_0) = \{(j,\alpha)\,;\, k_{j,\alpha}(t_0,V_0)-\gamma+m-j < |\alpha|\}, \tag{4.6}$$

$$\sigma_0(t_0,V_0) = 1+\min\left[\infty, \min_{(j,\alpha)\in\Delta(t_0,V_0)}\left(\frac{m-j-|\alpha|}{|\alpha|-(k_{j,\alpha}(t_0,V_0)-\gamma+m-j)}\right)\right] \tag{4.7}$$

where $\gamma=\gamma(t_0,x_0)$. We have:

Theorem 4.2. *Let $\sigma \geq 1$. Suppose the conditions* (4.1), (4.4), a_4), (M), *and*

$$\sigma \leq \inf_{(t_0,x_0)\in[0,T]\times V}\left(\sup_{V_0\ni x_0}\sigma_0(t_0,V_0)\right): \tag{4.8}$$

then we have $u(t,x) \in \mathcal{E}^{\{\sigma\}}([0,T]\times V)$.

§ 5. Appendix

In §3, we have used two results: one is a result on the composition of Gevrey functions, and the other is the implicit function theorem in Gevrey classes. We present here precise formulations and their proofs.

First, let us show a result on the composition of Gevrey functions. We write $t \in \mathbb{R}_t$, $x=(x_1,\ldots,x_n) \in \mathbb{R}^n_x$ and $z=(z_1,\ldots,z_d)\in\mathbb{R}^d_z$. We have

Proposition 5.1. *Let $s\geq 1$, $s_1 \geq 1$, $s_2\geq 1$, $\sigma \geq 1$, let Ω and W be open subsets of $\mathbb{R}_t\times\mathbb{R}^n_x\times\mathbb{R}^d_z$. If the conditions*

1) $F(t,x,z)\in\mathcal{E}^{\{s_1,\sigma,s_2\}}(\Omega)$,
2) $u_i(t,x,z)\in\mathcal{E}^{\{s,\sigma,s_2\}}(W)$ $(i=1,\ldots,d)$,
3) $W\ni(t,x,z)\Longrightarrow(t,x,u(t,x,z))\in\Omega$, *where* $u=(u_1,\ldots,u_d)$,
4) $\sigma\geq s_2$ *and* $s\geq\max\{s_1,s_2\}$

hold, we have $F(t,x,u(t,x,z))\in\mathcal{E}^{\{s,\sigma,s_2\}}(W)$.

Proof. Take any compact subset Z of W; then the image L of Z by the mapping $(t,x,z) \longrightarrow (t,x,u(t,x,z))$ is also a compact subset of Ω. We take constants $A_{p,q,\nu} \geq 0$ and $B_{i,k,\beta,\gamma} \geq 0$ so that

- $\displaystyle \max_{(t,x,z)\in L} \Big| \frac{F^{(p,q,\nu)}(t,x,z)}{p!q!\nu!} \Big| \leq A_{p,q,\nu} p!^{s_1-1} |q|!^{\sigma-1} |\nu|!^{s_2-1},$
- $\displaystyle \max_{(t,x,z)\in Z} \Big| \frac{u_i^{(k,\beta,\gamma)}(t,x,z)}{k!\beta!\gamma!} \Big| \leq B_{i,k,\beta,\gamma} (k-1)!^{s-1} |\beta|!^{\sigma-1} |\gamma|!^{s_2-1}, \text{ if } k \geq 1,$
- $\displaystyle \max_{(t,x,z)\in Z} \Big| \frac{u_i^{(0,\beta,\gamma)}(t,x,z)}{\beta!\gamma!} \Big| \leq B_{i,0,\beta,\gamma} (|\beta|-1)!^{\sigma-1} |\gamma|!^{s_2-1}, \text{ if } |\beta| \geq 1,$
- $\displaystyle \max_{(t,x,z)\in Z} \Big| \frac{u_i^{(0,0,\gamma)}(t,x,z)}{\gamma!} \Big| \leq B_{i,0,0,\gamma} (|\gamma|-1)!^{s_2-1}, \text{ if } |\gamma| \geq 1,$

where $p \in \mathbb{N}$, $q \in \mathbb{N}^n$, $\nu \in \mathbb{N}^d$, $k \in \mathbb{N}$, $\beta \in \mathbb{N}^n$, and $\gamma \in \mathbb{N}^d$. We set also

$$
\begin{aligned}
G(t,x,z) &= \sum_{p+|q|+|\nu|\geq 0} A_{p,q,\nu} t^p x^q z^\nu, \\
w_i(t,x,z) &= \sum_{k+|\beta|+|\gamma|\geq 1} B_{i,k,\beta,\gamma} t^k x^\beta z^\gamma, \quad i = 1,\dots,d.
\end{aligned}
$$

Then, $G(t,x,z) = G(t,x,z_1,\dots,z_d)$ and $w_i(t,x,z)$ $(i=1,\dots,d)$ are convergent in a neighborhoof of $(t,x,z) = (0,0,0)$, and so the function

$$
\begin{aligned}
H(t,x,z) &= G(t,x,w_1(t,x,z),\dots,w_d(t,x,z)) \\
&= \sum_{p+|q|+|\nu|\geq 0} A_{p,q,\nu} t^p x^q \Big(\sum_{k_1+|\beta_1|+|\gamma_1|\geq 1} B_{1,k_1,\beta_1,\gamma_1} t^{k_1} x^{\beta_1} z^{\gamma_1} \Big)^{\nu_1} \times \\
&\qquad\qquad \times \cdots \times \Big(\sum_{k_d+|\beta_d|+|\gamma_d|\geq 1} B_{d,k_d,\beta_d,\gamma_d} t^{k_d} x^{\beta_d} z^{\gamma_d} \Big)^{\nu_d}
\end{aligned}
$$

(with $\nu = (\nu_1,\dots,\nu_d)$) is also convergent in a neighborhood of $(t,x,z) = (0,0,0)$. If we set

$$
H(t,x,z) = \sum_{(m,\alpha,\mu)\in\mathbb{N}\times\mathbb{N}^n\times\mathbb{N}^d} C_{m,\alpha,\mu} t^m x^\alpha z^\mu
$$

we have $C_{0,0,0} = A_{0,0,0}$ and for $m + |\alpha| + |\mu| \geq 1$

$$
C_{m,\alpha,\mu} = \sum_{\substack{1\leq p+|q|+|\nu| \\ \leq m+|\alpha|+|\mu|}} A_{p,q,\nu} \sum_{\substack{|k^*|=m-p \\ s(\beta(\nu))=\alpha-q \\ s(\gamma(\nu))=\mu \\ k_i(j)+|\beta_i(j)|+|\gamma_i(j)|\geq 1}} \prod_{i=1}^d \prod_{j=1}^{\nu_i} \Big(B_{i,k_i(j),\beta_i(j),\gamma_i(j)} \Big), \tag{5.1}
$$

where $|k^*| = \sum_{i=1}^{d}(k_i(1) + \cdots + k_i(\nu_i))$ and

$$s(\beta(\nu)) = \sum_{i=1}^{d}\sum_{j=1}^{\nu_i}\beta_i(j) \in \mathbb{N}^n, \quad s(\gamma(\nu)) = \sum_{i=1}^{d}\sum_{j=1}^{\nu_i}\gamma_i(j) \in \mathbb{N}^d.$$

Since $H(t,x,z)$ is a holomorphic function in a neighborhood of $(t,x,z) = (0,0,0)$, by Cauchy's inequality we have $C_{m,\alpha,\mu} \leq M\eta^{m+|\alpha|+|\mu|}$ $(m+|\alpha|+|\mu| = 0,1,2,\ldots)$ for some $M > 0$ and $\eta > 0$. Therefore, to see the condition $h(t,x,z) = F(t,x,u(t,x,z)) \in \mathcal{E}^{\{s,\sigma,s_2\}}(Z)$ it is sufficient to prove the following inequalities:

$$\begin{aligned}(5.2)\qquad &\frac{1}{m!^{s-1}|\alpha|!^{\sigma-1}|\mu|!^{s_2-1}} \max_{(t,x,z)\in Z}\left|\frac{1}{m!\alpha!\mu!}h^{(m,\alpha,\mu)}(t,x,z)\right| \\ &\quad \leq 3^{(m+|\alpha|+|\mu|)(s_2-1)}C_{m,\alpha,\mu} \qquad \text{for any } m+|\alpha|+|\mu| \geq 1.\end{aligned}$$

Now, let us recall that by Faá di Bruno's formula (see [1]) or [[4], Lemma 4.3] we have

$$\begin{aligned}\frac{1}{m!\alpha!\mu!}h^{(m,\alpha,\mu)} = &\sum_{1\leq p+|q|+|\nu|\leq m+|\alpha|+|\mu|}\frac{1}{p!q!\nu!}F^{(p,q,\nu)}\times \\ &\times \sum_{\substack{|k^*|=m-p \\ s(\beta(\nu))=\alpha-q \\ s(\gamma(\nu))=\mu \\ k_i(j)+|\beta_i(j)|+|\gamma_i(j)|\geq 1}} \prod_{i=1}^{d}\prod_{j=1}^{\nu_i}\left(\frac{1}{k_i(j)!\beta_i(j)!\gamma_i(j)!}u_i^{(k_i(j),\beta_i(j),,\gamma_i(j))}\right).\end{aligned}$$

Thereofre, by using estimates $(1/p!q!\nu!)|F^{(p,q,\nu)}| \leq A_{p,q,\nu}p!^{s_1-1}|q|!^{\sigma-1}|\nu|!^{s_2-1}$ on L and $(1/k!\beta!\gamma!)|u_i^{(k,\beta,\gamma)}| \leq B_{i,k,\beta,\gamma}(k-\delta_1)!^{s-1}(|\beta|-\delta_2)!^{\sigma-1}(|\gamma|-\delta_3)^{s_2-1}$ (where $(\delta_1,\delta_2,\delta_3) = (1,0,0)$ if $k \geq 1$, $(\delta_1,\delta_2,\delta_3) = (0,1,0)$ if $k = 0$ and $|\beta| \geq 1$, and $(\delta_1,\delta_2,\delta_3) = (0,0,1)$ if $k = 0$, $\beta = 0$ and $|\gamma| \geq 1$) on Z, we have

$$\begin{aligned}(5.3)\qquad &\frac{1}{m!^{s-1}|\alpha|!^{\sigma-1}|\mu|!^{s_2-1}}\left|\frac{1}{m!\alpha!\mu!}h^{(m,\alpha,\mu)}\right| \\ &\leq \frac{1}{m!^{s-1}|\alpha|!^{\sigma-1}|\mu|!^{s_2-1}}\sum_{1\leq p+|q|+|\nu|\leq m+|\alpha|+|\mu|}A_{p,q,\nu}p!^{s_1-1}|q|!^{\sigma-1}|\nu|!^{s_2-1}\times \\ &\quad\times \sum_{\substack{|k^*|=m-p \\ s(\beta(\nu))=\alpha-q \\ s(\gamma(\nu))=\mu \\ k_i(j)+|\beta_i(j)|+|\gamma_i(j)|\geq 1}} \prod_{i=1}^{d}\prod_{j=1}^{\nu_i}\Bigl(B_{i,k_i(j),\beta_i(j),\gamma_i(j)}(k_i(j)-\delta_1)!^{s-1}\times \\ &\qquad\qquad \times (|\beta_i(j)|-\delta_2)!^{\sigma-1}(|\gamma_i(j)|-\delta_3)!^{s_2-1}\Bigr)\end{aligned}$$

$$\leq \sum_{1\leq p+|q|+|\nu|\leq m+|\alpha|+|\mu|} A_{p,q,\nu} \frac{p!^{s_1-1}|q|!^{\sigma-1}|\nu|!^{s_2-1}}{m!^{s-1}|\alpha|!^{\sigma-1}|\mu|!^{s_2-1}} \times$$

$$\times \sum_{\substack{|k^*|=m-p \\ s(\beta(\nu))=\alpha-q \\ s(\gamma(\nu))=\mu \\ k_i(j)+|\beta_i(j)|+|\gamma_i(j)|\geq 1}} (|k^*|-n_1)!^{s-1}(|\beta(\nu)^*|-n_2)!^{\sigma-1}(|\gamma(\nu)^*|-n_3)!^{s_2-1}$$

$$\times \prod_{i=1}^{d}\prod_{j=1}^{\nu_i}\Big(B_{i,k_i(j),\beta_i(j),\gamma_i(j)}\Big),$$

where $|\beta(\nu)^*| = \sum_{i=1}^{d}\sum_{j=1}^{\nu_i}|\beta_i(j)|$, $|\gamma(\nu)^*| = \sum_{i=1}^{d}\sum_{j=1}^{\nu_i}|\gamma_i(j)|$, $n_1 = \#\{(i,j)\,;k_i(j)\geq 1\}$, $n_2 = \#\{(i,j)\,;k_i(j) = 0, |\beta_i(j)|\geq 1\}$, and $n_3 = \#\{(i,j)\,;k_i(j) = 0, \beta_i(j) = 0, |\gamma_i(j)|\geq 1\}$. Since $n_1+n_2+n_3 = |\nu|$, $s\geq s_2$ and $\sigma\geq s_2$ hold, we have

$$|\nu|!^{s_2-1} \leq (3^{|\nu|}n_1!n_2!n_3!)^{s_2-1} \leq 3^{|\nu|(s_2-1)}n_1!^{s-1}n_2!^{\sigma-1}n_3!^{s_2-1},$$

and so

$$\begin{aligned}
&\frac{p!^{s_1-1}|q|!^{\sigma-1}|\nu|!^{s_2-1}}{m!^{s-1}|\alpha|!^{\sigma-1}|\mu|!^{s_2-1}}(|k^*|-n_1)!^{s-1}(|\beta(\nu)^*|-n_2)!^{\sigma-1}(|\gamma(\nu)^*|-n_3)!^{s_2-1} \\
&\leq \frac{p!^{s-1}|q|!^{\sigma-1}\times 3^{|\nu|(s_2-1)}n_1!^{s-1}n_2!^{\sigma-1}n_3!^{s_2-1}}{m!^{s-1}|\alpha|!^{\sigma-1}|\mu|!^{s_2-1}}\times \\
&\qquad\qquad \times (m-p-n_1)!^{s-1}(|\alpha|-|q|-n_2)!^{\sigma-1}(|\mu|-n_3)!^{s_2-1} \\
&= 3^{|\nu|(s_2-1)}\frac{p!^{s-1}n_1!^{s-1}(m-p-n_1)!^{s-1}}{m!^{s-1}} \\
&\qquad \times \frac{|q|!^{\sigma-1}n_2!^{\sigma-1}(|\alpha|-|q|-n_2)!^{\sigma-1}}{|\alpha|!^{\sigma-1}}\frac{n_3!^{s_2-1}(|\mu|-n_3)!^{s_2-1}}{|\mu|!^{s_2-1}} \\
&\leq 3^{|\nu|(s_2-1)} \leq 3^{(m+|\alpha|+|\mu|)(s_2-1)}.
\end{aligned}$$

Thus, by applying this to (5.3) we obtain

$$\begin{aligned}
&\frac{1}{m!^{s-1}|\alpha|!^{\sigma-1}|\mu|!^{s_2-1}}\Big|\frac{1}{m!\alpha!\mu!}h^{(m,\alpha,\mu)}\Big| \\
&\leq 3^{(m+|\alpha|+|\mu|)(s_2-1)}\sum_{\substack{1\leq p+|q|+|\nu| \\ \leq m+|\alpha|+|\mu|}} A_{p,q,\nu}\sum_{\substack{|k^*|=m-p \\ s(\beta(\nu))=\alpha-q \\ s(\gamma(\nu))=\mu \\ k_i(j)+|\beta_i(j)|+|\gamma_i(j)|\geq 1}}\prod_{i=1}^{d}\prod_{j=1}^{\nu_i}\Big(B_{i,k_i(j),\beta_i(j),\gamma_i(j)}\Big) \\
&= 3^{(m+|\alpha|+|\mu|)(s_2-1)}C_{m,\alpha,\mu}
\end{aligned}$$

on Z. This proves (5.2). □

Some versions of the implicit function theorem in ultra-differentiable classes are given in Komatsu [3] and Yamanaka [5]. For the self-containedness, we will give here an implicit function theorem which is used in §3.

As before, we write $t \in \mathbb{R}_t$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n_x$, $z = (z_1, \dots, z_d) \in \mathbb{R}^d_z$ and $w \in \mathbb{R}_w$. Let Ω be an open neighborhood of $(0,0,0,0) \in \mathbb{R}_t \times \mathbb{R}^n_x \times \mathbb{R}^d_z \times \mathbb{R}_w$, let $F(t,x,z,w)$ be a C^∞-function on Ω, let $s_1 \geq 1$, $s_2 \geq 1$, $\sigma_1 \geq 1$ and $\sigma_2 \geq 1$, and suppose: $\sigma_i \geq s_2$ for $i = 1, 2$. We have

Theorem 5.2. *Suppose the following conditions: $F(t,x,z,w) \in \mathcal{E}^{\{s_1,\sigma_1,\sigma_2,s_2\}}(\Omega)$, $F(0,0,0,0) = 0$ and $(\partial F/\partial w)(0,0,0,0) \neq 0$. Then, there are an open neighborhood W of $(0,0,0) \in \mathbb{R}_t \times \mathbb{R}^n_x \times \mathbb{R}^d_z$ and a function $\phi(t,x,z) \in C^\infty(W)$ which satisfy $\phi(0,0,0) = 0$ and the following properties:*

$$W \ni (t,x,z) \Longrightarrow (t,x,z,\phi(t,x,z)) \in \Omega, \tag{5.4}$$

$$F(t,x,z,\phi(t,x,z)) = 0 \quad \text{on } W. \tag{5.5}$$

Moreover, we have $\phi(t,x,z) \in \mathcal{E}^{\{s,\sigma_1,\sigma_2\}}(W)$ for any $s \geq \max\{s_1, s_2\}$.

Proof. The former half of the result is nothing but the result of the implicit function theorem in the C^∞ class, and so we know that there are an open neighborhood W of $(0,0,0) \in \mathbb{R}_t \times \mathbb{R}^n_x \times \mathbb{R}^d_z$ and a function $\phi(t,x,z) \in C^\infty(W)$ which satisfy $\phi(0,0,0) = 0$, (5.4) and (5.5).

Let us show that $\phi(t,x,z) \in \mathcal{E}^{\{s,\sigma_1,\sigma_2\}}(W)$ holds for any $s \geq \max\{s_1, s_2\}$. Since the problem is set in a local sense, without loss of generality we may suppose that $(\partial F/\partial w)(t,x,z,w) \neq 0$ holds on Ω. Then, by (5.5) we have the equality

$$\begin{aligned} \phi_t(t,x,z) &= -\frac{F_t(t,x,z,\phi)}{F_w(t,x,z,\phi)}, \\ \phi_{x_i}(t,x,z) &= -\frac{F_{x_i}(t,x,z,\phi)}{F_w(t,x,z,\phi)}, \quad i = 1, \dots, n, \\ \phi_{z_j}(t,x,z) &= -\frac{F_{z_j}(t,x,z,\phi)}{F_w(t,x,z,\phi)}, \quad j = 1, \dots, d, \end{aligned}$$

where $\phi_t = \partial\phi/\partial t$, $F_t = \partial F/\partial t$, $\phi_{x_i} = \partial\phi/\partial x_i$, $F_{x_i} = \partial F/\partial x_i$ and so on. Therefore, if we set $G(t,x,z) = -(F_t/F_w)(t,x,z,w)$, $H_i(t,x,z) = -(F_{x_i}/F_w)(t,x,z,w)$ $(i = 1, \dots, n)$ and $K_j(t,x,z,w) = -(F_{z_j}/F_w)(t,x,z,w)$ $(j = 1, \dots, d)$ we have the system of equations

$$\begin{cases} \phi_t = G(t,x,z,\phi), \\ \phi_{x_i} = H_i(t,x,z,\phi), \quad i = 1, \dots, n, \\ \phi_{z_j} = K_j(t,x,\phi), \quad j = 1, \dots, d. \end{cases} \tag{5.6}$$

Since $F_w(t,x,z,w) \neq 0$ on Ω, by the condition $F(t,x,z,w) \in \mathcal{E}^{\{s_1,\sigma_1,\sigma_2,s_2\}}(\Omega)$ we have

$$\begin{aligned} &\bullet\, G(t,x,z,w) \in \mathcal{E}^{\{s_1,\sigma_1,\sigma_2,s_2\}}(\Omega), \\ &\bullet\, H_i(t,x,z,w) \in \mathcal{E}^{\{s_1,\sigma_1,\sigma_2,s_2\}}(\Omega), \quad i = 1, \dots, n, \\ &\bullet\, K_j(t,x,z,w) \in \mathcal{E}^{\{s_1,\sigma_1,\sigma_2,s_2\}}(\Omega), \quad j = 1, \dots, d. \end{aligned} \tag{5.7}$$

Thus, to complete the proof of Theorem 5.2 it is enough to show

Proposition 5.3. *Let $s_1 \geq 1$, $s_2 \geq 1$, $\sigma_1 \geq 1$ and $\sigma_2 \geq 1$. Suppose the conditions* (5.7) *and $\sigma_i \geq s_2$ for $i = 1, 2$. If $\phi(t,x,z) \in C^\infty(W)$ is a solution of* (5.6), *we have $\phi(t,x,z) \in \mathcal{E}^{\{s,\sigma_1,\sigma_2\}}(W)$ for any $s \geq \max\{s_1, s_2\}$.*

Proof of Proposition 5.3. Let Z be a compact subset of W, and let L be the image of Z by the mapping: $(t,x,z) \longrightarrow (t,x,z,\phi(t,x,z)) \in \Omega$. We take $A_{p,q_1,q_2,k} \geq 0$ ($p \in \mathbb{N}$, $q_1 \in \mathbb{N}^n$, $q_2 \in \mathbb{N}^d$ and $k \in \mathbb{N}$), $B^{(i)}_{q_1,q_2,k} \geq 0$ ($i = 1,\dots,n$, $q_1 \in \mathbb{N}^n$, $q_2 \in \mathbb{N}^d$ and $k \in \mathbb{N}$) and $C^{(j)}_{q_2,k} \geq 0$ ($j = 1,\dots,d$, $q_2 \in \mathbb{N}^d$ and $k \in \mathbb{N}$) so that

- $\displaystyle \max_{(t,x,z,w)\in L} \left| \frac{G^{(p,q_1,q_2,k)}(t,x,z,w)}{p!q_1!q_2!k!} \right| \leq A_{p,q_1,q_2,k} p!^{s_1-1} |q_1|!^{\sigma_1-1} |q_2|!^{\sigma_2-1} k!^{s_2-1},$
- $\displaystyle \max_{(t,x,z,w)\in L} \left| \frac{H_i^{(0,q_1,q_2,k)}(t,x,z,w)}{q_1!q_2!k!} \right| \leq B^{(i)}_{q_1,q_2,k} |q_1|!^{\sigma_1-1} |q_2|!^{\sigma_2-1} k!^{s_2-1},$
- $\displaystyle \max_{(t,x,z,w)\in L} \left| \frac{K_j^{(0,0,q_2,k)}(t,x,z,w)}{q_2!k!} \right| \leq C^{(j)}_{q_2,k} |q_2|!^{\sigma_2-1} k!^{s_2-1}.$

Let us consider the following functional equation:

$$\begin{aligned} (5.8)\quad Y = t &\sum_{p+|q_1|+|q_2|+k\geq 0} A_{p,q_1,q_2,k} t^p x^{q_1} z^{q_2} (3^{s_2-1}Y)^k \\ &+ \sum_{i=1}^n x_i \sum_{|q_1|+|q_2|+k\geq 0} B^{(i)}_{q_1,q_2,k} x^{q_1} z^{q_2} (2^{s_2-1}Y)^k + \sum_{j=1}^d z_j \sum_{|q_2|+k\geq 0} C^{(j)}_{q_2,k} z^{q_2} Y^k. \end{aligned}$$

Since this is an analytic functional equation, by the implicit function theorem in holomorphic category we see that (5.8) has a unique holomorphic solution $Y(t,x,z)$ in a neighborhood of $(0,0,0) \in \mathbb{C}^n_t \times \mathbb{C}^n_x \times \mathbb{C}^d_z$ satisfying $Y(0,0,0) = 0$. Let

$$Y(t,x,z) = \sum_{m+|\alpha_1|+|\alpha_2|\geq 1} Y_{m,\alpha_1,\alpha_2} t^m x^{\alpha_1} z^{\alpha_2},$$

be the Taylor expansion of $Y(t,x,z)$. We see that the coefficients Y_{m,α_1,α_2} ($m + |\alpha_1| + |\alpha_2| \geq 1$) are determined by the following recurrent formulas:

$$Y_{1,0,0} = A_{0,0,0,0},\ \ Y_{0,e_i,0} = B^{(i)}_{0,0,0}\ \ (i = 1,\dots,n),\ \ Y_{0,0,\epsilon_j} = C^{(j)}_{0,0}\ \ (j = 1,\dots,d)$$

(where $e_1 = (1,0,\dots,0),\dots,e_n = (0,\dots,0,1) \in \mathbb{N}^n$ and $\epsilon_1 = (1,0,\dots,0),\dots,\epsilon_d =$

$(0,\dots,0,1)\in\mathbb{N}^d$), and for $M=m+|\alpha_1|+|\alpha_2|\geq 2$ we have

$$(5.9)\qquad Y_{m,\alpha_1,\alpha_2}=\sum_{0\leq p+|q_1|+|q_2|+k\leq M-1}A_{p,q_1,q_2,k}\sum_{\substack{|\mu^*|=m-p-1\\ s(\beta_1^*)=\alpha_1-q_1\\ s(\beta_2^*)=\alpha_2-q_2}}\prod_{l=1}^{k}3^{s_2-1}Y_{\mu(l),\beta_1(l),\beta_2(l)}$$
$$+\sum_{i=1}^{n}\sum_{0\leq|q_1|+|q_2|+k\leq M-1}B^{(i)}_{q_1,q_2,k}\sum_{\substack{|\mu^*|=m\\ s(\beta_1^*)=\alpha_1-q_1-e_i\\ s(\beta_2^*)=\alpha_2-q_2}}\prod_{l=1}^{k}2^{s_2-1}Y_{\mu(l),\beta_1(l),\beta_2(l)}$$
$$+\sum_{j=1}^{d}\sum_{0\leq|q_2|+k\leq M-1}C^{(j)}_{q_2,k}\sum_{\substack{|\mu^*|=m\\ s(\beta_1^*)=\alpha_1\\ s(\beta_2^*)=\alpha_2-q_2-\epsilon_j}}\prod_{l=1}^{k}Y_{\mu(l),\beta_1(l),\beta_2(l)},$$

where $|\mu^*|=\mu(1)+\cdots+\mu(k)$, $s(\beta_1^*)=\sum_{l=1}^k\beta_1(l)$ and $s(\beta_2^*)=\sum_{l=1}^k\beta_2(l)$. We have

Lemma 5.4. *Take any $s\geq\max\{s_1,s_2\}$. Then, we have:*

(1) $\displaystyle\max_{(t,x,z)\in Z}\left|\frac{\phi^{(m,\alpha_1,\alpha_2)}(t,x,z)}{m!\alpha_1!\alpha_2!}\right|\leq Y_{m,\alpha_1,\alpha_2}(m-1)!^{s-1}|\alpha_1|!^{\sigma_1-1}|\alpha_2|!^{\sigma_2-1}$, *if* $m\geq 1$,

(2) $\displaystyle\max_{(t,x,z)\in Z}\left|\frac{\phi^{(0,\alpha_1,\alpha_2)}(t,x,z)}{\alpha_1!\alpha_2!}\right|\leq Y_{0,\alpha_1,\alpha_2}(|\alpha_1|-1)!^{\sigma_1-1}|\alpha_2|!^{\sigma_2-1}$, *if* $|\alpha_1|\geq 1$,

(3) $\displaystyle\max_{(t,x,z)\in Z}\left|\frac{\phi^{(0,0,\alpha_2)}(t,x,z)}{\alpha_2!}\right|\leq Y_{0,0,\alpha_2}(|\alpha_2|-1)!^{\sigma_2-1}$, *if* $|\alpha_2|\geq 1$.

Proof of Lemma 5.4. We will prove this by induction on $M=m+|\alpha_1|+|\alpha_2|$. By (5.6) we have $|\phi_t|=|G(t,x,z,\phi)|\leq A_{0,0,0,0}=Y_{1,0,0}$, $|\phi_{x_i}|=|H_i(t,x,z,\phi)|\leq B^{(i)}_{0,0,0}=Y_{0,e_i,0}$ $(i=1,\dots,n)$ and $|\phi_{z_j}|=|K_j(t,x,z,\phi)|\leq C^{(j)}_{0,0}=Y_{0,0,\epsilon_j}$ $(j=1,\dots,d)$. This proves the case $M=1$.

Suppose that $M=m+|\alpha_1|+|\alpha_2|\geq 2$. If $m\geq 1$, by (5.6) we have

$$\frac{\phi^{(m,\alpha_1,\alpha_2)}(t,x,z)}{m!\alpha_1!\alpha_2!}$$
$$=\frac{\phi_t^{(m-1,\alpha_1,\alpha_2)}(t,x,z)}{m!\alpha_1!\alpha_2!}=\frac{1}{m!\alpha_1!\alpha_2!}\partial_t^{m-1}\partial_x^{\alpha_1}\partial_z^{\alpha_2}G(t,x,z,\phi)$$
$$=\frac{1}{m}\sum_{0\leq p+|q_1|+|q_2|+k\leq M-1}\frac{G^{(p,q_1,q_2,k)}}{p!q_1!q_2!k!}\sum_{\substack{|\mu^*|=m-p-1\\ s(\beta_1^*)=\alpha_1-q_1\\ s(\beta_2^*)=\alpha_2-q_2}}\prod_{l=1}^{k}\frac{\phi^{(\mu(l),\beta_1(l),\beta_2(l))}}{\mu(l)!\beta_1(l)!\beta_2(l)!}$$

and therefore

(5.10)

$$
\begin{aligned}
&\Big|\frac{\phi^{(m,\alpha_1,\alpha_2)}(t,x,z)}{m!\alpha_1!\alpha_2!}\Big| \\
&\le \frac{1}{m}\sum_{0\le p+|q_1|+|q_2|+k\le M-1} A_{p,q_1,q_2,k}p!^{s_1-1}|q_1|!^{\sigma_1-1}|q_2|!^{\sigma_2-1}k!^{s_2-1}\times \\
&\times \sum_{\substack{|\mu^*|=m-p-1\\ s(\beta_1^*)=\alpha_1-q_1\\ s(\beta_2^*)=\alpha_2-q_2}} \prod_{l=1}^{k} Y_{\mu(l),\beta_1(l),\beta_2(l)}(\mu(l)-\delta_0)!^{s-1}(|\beta_1(l)|-\delta_1)!^{\sigma_1-1}(|\beta_2(l)|-\delta_2)!^{\sigma_2-1}
\end{aligned}
$$

where $(\delta_0,\delta_1,\delta_2)=(1,0,0)$ if $\mu(l)\ge 1$, $(\delta_0,\delta_1,\delta_2)=(0,1,0)$ if $\mu(l)=0$ and $|\beta_1(l)|\ge 1$, and $(\delta_0,\delta_1,\delta_2)=(0,0,1)$ if $\mu(l)=0$, $\beta_1(l)=0$ and $|\beta_2(l)|\ge 1$. If we set $n_0=\#\{l\,;\,\mu(l)\ge 1\}$, $n_1=\#\{l\,;\,\mu(l)=0,|\beta_1(l)|\ge 1\}$ and $n_2=\#\{l\,;\,\mu(l)=0,\beta_1(l)=0,|\beta_2(l)|\ge 1\}$, then we have $n_0+n_1+n_2=k$ and

$$
k!^{s_2-1}\le (3^k n_0!n_1!n_2!)^{s_2-1}\le 3^{k(s_2-1)}n_0!^{s-1}n_1!^{\sigma_1-1}n_2!^{\sigma_2-1}.
$$

By applying this to (5.10) and by using $s\ge s_1$ we have

$$
\begin{aligned}
&\Big|\frac{\phi^{(m,\alpha_1,\alpha_2)}(t,x,z)}{m!\alpha_1!\alpha_2!}\Big| \\
&\le \frac{1}{m}\sum_{0\le p+|q_1|+|q_2|+k\le M-1} A_{p,q_1,q_2,k}p!^{s-1}|q_1|!^{\sigma_1-1}|q_2|!^{\sigma_2-1}\times n_0!^{s-1}n_1!^{\sigma_1-1}n_2!^{\sigma_2-1}\times \\
&\qquad \times (m-p-1-n_0)!^{s-1}(|\alpha_1|-|q_1|-n_1)!^{\sigma_1-1}(|\alpha_2|-|q_2|-n_2)!^{\sigma_2-1} \\
&\qquad\qquad \times \sum_{\substack{|\mu^*|=m-p-1\\ s(\beta_1^*)=\alpha_1-q_1\\ s(\beta_2^*)=\alpha_2-q_2}} \prod_{l=1}^{k} 3^{s_2-1}Y_{\mu(l),\beta_1(l),\beta_2(l)} \\
&\le (m-1)!^{s-1}|\alpha_1|!^{\sigma_1-1}|\alpha_2|!^{\sigma_2-1}\times \\
&\qquad \times \frac{1}{m}\sum_{0\le p+|q_1|+|q_2|+k\le M-1} A_{p,q_1,q_2,k}\sum_{\substack{|\mu^*|=m-p-1\\ s(\beta_1^*)=\alpha_1-q_1\\ s(\beta_2^*)=\alpha_2-q_2}} \prod_{l=1}^{k} 3^{s_2-1}Y_{\mu(l),\beta_1(l),\beta_2(l)} \\
&\le (m-1)!^{s-1}|\alpha_1|!^{\sigma_1-1}|\alpha_2|!^{\sigma_2-1}\times Y_{m,\alpha_1,\alpha_2}.
\end{aligned}
$$

This proves (1).

By using $\phi_{x_i}=H_i(t,x,z,\phi)$ $(i=1,\ldots,n)$ and $\phi_{z_j}=K_j(t,x,z,\phi)$ $(j=1,\ldots,d)$, we can prove (2) and (3) in the same way. □

Completion of the proof of Proposition 5.3. Since $Y(t,x,z)$ is a holomorphic function in a neighborhood of $(0,0,0) \in \mathbb{C}_t \times \mathbb{C}_x^n \times \mathbb{C}_z^d$, by Cauchy's inequality we can take $C > 0$ and $h > 0$ so that

$$Y_{m,\alpha_1,\alpha_2} \le Ch^{m+|\alpha_1|+|\alpha_2|}$$

holds for all $m \in \mathbb{N}$, $\alpha_1 \in \mathbb{N}^n$ and $\alpha_2 \in \mathbb{N}^d$. By combining this with Lemma 5.4 we have the result: $\phi(t,x,z) \in \mathcal{E}^{\{s,\sigma_1,\sigma_2\}}(Z)$. This proves Proposition 5.3. □

This completes the proof of Theorem 5.2. □

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