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Vanishing theorem for holomorphic functions of exponential type and Laplace hyperfunctions

By

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Abstract

This paper consists of two part. The first part is the result concerning a vanishing theorem of cohomology groups on a pseudoconvex open subset for holomorphic functions with exponential growth at infinity given in [1]. In the second part, we show that the sheaf of locally integrable functions of exponential type is a subsheaf of the sheaf of Laplace hyperfunctions.

§ 1. Introduction

H. Komatsu ([3] – [8]) established the theory of Laplace hyperfunctions, which plays an important role in solving both linear ordinary differential equations with variable coefficients and partial differential equations. Originally, the Laplace transforms were defined for functions with the growth condition of exponential type at infinity. In 1987, H. Komatsu introduced Laplace hyperfunctions and their Laplace transforms, and it was shown that all ordinary hyperfunctions can be extended to Laplace hyperfunctions. Therefore we can treat their Laplace transforms for functions without any growth conditions in a framework of hyperfunctions.

To state the aim of our study, we briefly recall the definition of Laplace hyperfunctions with support in $[a, \infty]$ $(a \in \mathbb{R} \sqcup \{+\infty\})$ given in [3]. Let $\mathbb{D}^2$ be the radial compactification of $\mathbb{C}$ which is the disjoint union of $\mathbb{C}$ and the unit sphere S^1 in $\mathbb{R}^2$. We denote by $\mathcal{O}^{\exp}_{\mathbb{D}^2}$ the sheaf of holomorphic functions of exponential type on $\mathbb{D}^2$, where the global sections $\mathcal{O}^{\exp}_{\mathbb{D}^2}(\Omega)$ on an open set $\Omega \in \mathbb{D}^2$ is given by

$$\{f \in \mathcal{O}_{\mathbb{C}}(\Omega \cap \mathbb{C});\ \text{for any compact } K \subset \Omega \text{ there exist } C_K > 0 \text{ and } H_K > 0 \\ \text{such that } |f(z)| \le C_K e^{H_K|z|},\ z \in K \cap \mathbb{C}\}.$$

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We define the quotient space

$$\mathcal{B}^{\exp}_{[a,\,\infty]} := \mathcal{O}^{\exp}_{\mathbb{D}^2}(\mathbb{D}^2 \setminus [a,\,\infty])/\mathcal{O}^{\exp}_{\mathbb{D}^2}(\mathbb{D}^2), \tag{1.1}$$

regarding $\mathcal{O}^{\exp}_{\mathbb{D}^2}(\mathbb{D}^2)$ as a vector subspace of $\mathcal{O}^{\exp}_{\mathbb{D}^2}(\mathbb{D}^2 \setminus [a,\,\infty])$ by the natural restriction, i.e., every element of $\mathcal{O}^{\exp}_{\mathbb{D}^2}(\mathbb{D}^2 \setminus [a,\,\infty])$ that is extendable to a holomorphic function of exponential type on $\mathbb{D}^2$ is identified with 0. Each equivalence class $[F(z)]$ represented by $F \in \mathcal{O}^{\exp}_{\mathbb{D}^2}(\mathbb{D}^2 \setminus [a,\,\infty])$ is considered to be a Laplace hyperfunction $f(x)$.

The sheaf of Laplace hyperfunctions was also expected to be constructed. For that purpose, recently, N. Honda and the author [1] established a vanishing theorem of cohomology groups on a pseudoconvex open subset for holomorphic functions with exponential growth at infinity. As its benefits, the sheaf of Laplace hyperfunctions was constructed cohomologically. In this article, we introduce some results in [1] and add the results, related to embedding of the sheaf of locally integrable functions of exponential type into the sheaf of Laplace hyperfunctions.

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§ 2. Construction of the sheaf of Laplace hyperfunctions with holomorphic parameters

In this section we review a vanishing theorem for holomorphic functions of exponential type and the sheaf of Laplace hyperfunctions with holomorphic parameters. For the details and the proofs of the theorems in this section, refer the readers to [1]. We prepare some notation to give our vanishing theorem. In what follows, we fix $n \in \mathbb{N}$ and $m \in \mathbb{Z}_{\geq 0}$. We denote by $\mathbb{D}^{2n}$ the *radial compactification* $\mathbb{C}^n \sqcup S^{2n-1}\infty$ of $\mathbb{C}^n$, where S^{2n-1} is the real $(2n-1)$-dimensional unit sphere in $\mathbb{R}^{2n}$. As a fundamental system of neighborhoods of a point $z_0\infty \in S^{2n-1}\infty$, we take the following open subsets

$$G_r(\Gamma) := \left\{ z \in \mathbb{C}^n;\ |z| > r,\ \frac{z}{|z|} \in \Gamma \right\} \cup \Gamma\infty. \tag{2.1}$$

Here $r > 0$ and Γ runs through open neighborhoods of z_0 in S^{2n-1}. Let $X := \mathbb{C}^{n+m}$. We denote by $\hat{X}$ the partial radial compactification $\mathbb{D}^{2n} \times \mathbb{C}^m$ of $\mathbb{C}^{n+m}$ and by X_∞ the closed subset $\hat{X} \setminus X$ in $\hat{X}$. Let $\mathcal{O}_X$ be the sheaf of holomorphic functions on X.

Definition 2.1. For an open subset Ω in $\hat{X}$, we set

$$\begin{aligned}\mathcal{O}^{\exp}_{\hat{X}} = \{f \in \mathcal{O}_X(\Omega \cap X);\ &\text{for any } K \subset\subset \Omega \text{ there exist } C_K > 0 \text{ and } H_K > 0\\ &\text{such that } |f(z,w)| \leq C_K e^{H_K|z|},\ (z,w) \in K \cap X\},\end{aligned}$$

where $K \subset\subset \Omega$ implies that K is a compact subset in Ω. We denote by $\mathcal{O}^{\exp}_{\hat{X}}$ the associated sheaf on $\hat{X}$ of the presheaf $\{\mathcal{O}^{\exp}_{\hat{X}}(\Omega)\}_{\Omega}$. Note that the restriction of the sheaf $\mathcal{O}^{\exp}_{\hat{X}}$ to X coincides with the sheaf $\mathcal{O}_X$ of holomorphic functions on X.

Definition 2.2. For a subset A in $\hat{X}$, we define the set $\mathrm{clos}^1_\infty(A) \subset X_\infty$ as follows. A point $(z, w) \in X_\infty$ belongs to $\mathrm{clos}^1_\infty(A)$ if and only if there exist points $\{(z_k, w_k)\}_{k\in\mathbb{N}}$ in $A \cap X$ that satisfy $(z_k, w_k) \to (z, w)$ in $\hat{X}$ and $|z_{k+1}|/|z_k| \to 1$ as $k \to \infty$. We set $N^1_\infty(A) := X_\infty \setminus \mathrm{clos}^1_\infty(X \setminus A)$. An open subset U in $\hat{X}$ is said to be regular at ∞ if $N^1_\infty(U) = (U \cap X_\infty)$ is satisfied.

For a subset A in X, we denote by $\mathrm{dist}(p, A)$ the distance between a point p and A, i.e., $\mathrm{dist}(p, A) := \inf_{q\in A} |p - q|$. If A is empty, we set $\mathrm{dist}(p, A) := +\infty$. Let $p_2 : \hat{X} = \mathbb{D}^{2n} \times \mathbb{C}^m \to \mathbb{C}^m$ be the canonical projection to the second space. We also define, for $q = (z, w) \in X$,

$$\mathrm{dist}_{\mathbb{D}^{2n}}(q, A) := \mathrm{dist}(q, A \cap p_2^{-1}(p_2(q))) = \inf_{(\zeta, w)\in A} |z - \zeta|.$$

For an open subset Ω in $\hat{X}$, we define the function ψ by

$$\psi(p) := \min\left\{\frac{1}{2}, \frac{\mathrm{dist}_{\mathbb{D}^{2n}}(p, X \setminus \Omega)}{1 + |z|}\right\} \quad (p = (z, w) \in X) \tag{2.2}$$

and we set

$$\Omega_\epsilon := \left\{p = (z, w) \in \Omega \cap X;\ \mathrm{dist}(p, X \setminus \Omega) > \epsilon,\ |w| < \frac{1}{\epsilon}\right\} \quad (\epsilon > 0). \tag{2.3}$$

Now we state our vanishing theorem.

Theorem 2.3. *Assume the following conditions* 1. *and* 2.

1. $\Omega \cap X$ *is pseudoconvex in* X *and* Ω *is regular at* ∞.
2. *At a point in* $\Omega \cap X$ *sufficiently close to* $z = \infty$ *the* $\psi(z, w)$ *is continuous and uniformly continuous with respect to the variables* w, *that is, for any* $\epsilon > 0$, *there exist* $\delta_\epsilon > 0$ *and* $R_\epsilon > 0$ *for which* $\psi(z, w)$ *is continuous on* $\Omega_{\epsilon, R_\epsilon} := (\Omega_\epsilon \cap \{|z| > R_\epsilon\})$ *and it satisfies*

$$|\psi(z, w) - \psi(z, w')| < \epsilon, \qquad (z, w), (z, w') \in \Omega_{\epsilon, R_\epsilon},\ |w - w'| < \delta_\epsilon.$$

Then we have

$$H^k(\Omega, \mathcal{O}^{\exp}_{\hat{X}}) = 0, \qquad k \neq 0. \tag{2.4}$$

Note that the condition 2. of the theorem is always satisfied if Ω is of product type. Hence we have the following corollary.

Corollary 2.4. *Let U (resp. W) be an open subset in $\mathbb{D}^{2n}$ (resp. $\mathbb{C}^m$). If $U\cap\mathbb{C}^n$ and W are pseudoconvex in $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively and if U is regular at ∞ in $\mathbb{D}^{2n}$, then we have (2.4) for $\Omega := U \times W$.*

Thanks to Theorem 2.3, we can construcut the sheaf of Laplace hyperfunctions of one variable with holomorphic parameters. From now on, we consider the case of dimention $n = 1$. Let $N = \mathbb{R} \times \mathbb{C}^m (m \geq 0)$, and let $\overline{N} = \overline{\mathbb{R}} \times \mathbb{C}^m$ be the closure of N in $\hat{X} = \mathbb{D}^2 \times \mathbb{C}^m$.

Definition 2.5. The sheaf $\mathcal{BO}_{\overline{N}}^{\exp}$ of Laplace hyperfunctions of one variable with holomorphic parameters is defined by

$$\mathcal{BO}_{\overline{N}}^{\exp} := \mathscr{H}_{\overline{N}}^{1}(\mathcal{O}_{\hat{X}}^{\exp}) \underset{\mathbb{Z}_{\overline{N}}}{\otimes} \omega_{\overline{N}}. \tag{2.5}$$

Here $\mathscr{H}_{\overline{N}}^{1}(\mathcal{O}_{\hat{X}}^{\exp})$ is the the first derived sheaf of $\mathcal{O}_{\hat{X}}^{\exp}$ with support in $\overline{N}$, the $\mathbb{Z}_{\overline{N}}$ denotes the constant sheaf on $\overline{N}$ having stalk $\mathbb{Z}$ and $\omega_{\overline{N}}$ denotes the orientation sheaf $\mathscr{H}_{\overline{N}}^{1}(\mathbb{Z}_{\hat{X}})$ on $\overline{N}$. Especially, in the case of m=0, we define the sheaf $\mathcal{B}_{\overline{\mathbb{R}}}^{\exp}$of Laplace hyperfunctions of one variable on $\overline{\mathbb{R}}$ by (2.5) with $\overline{N}$ being replaced by $\overline{\mathbb{R}}$. The restriction of $\mathcal{B}_{\overline{\mathbb{R}}}^{\exp}$ to $\mathbb{R}$ is isomorphic to the sheaf $\mathcal{B}_{\mathbb{R}}$ of ordinary hyperfunctions because of $\mathcal{O}_{\mathbb{D}^2}^{\exp}|_{\mathbb{C}} = \mathcal{O}_{\mathbb{C}}$.

For an open set $\Omega \subset \overline{\mathbb{R}}$ and a pseudoconvex open subset $T \subset \mathbb{C}^m$, by taking a complex neighborhood V of Ω in $\mathbb{D}^2$, we have

$$\mathcal{BO}_{\overline{N}}^{\exp}(\Omega \times T) = H_{\Omega\times T}^{1}(V \times T, \mathcal{O}_{\hat{X}}^{\exp}) = \frac{\mathcal{O}_{\hat{X}}^{\exp}((V \setminus \Omega) \times T)}{\mathcal{O}_{\hat{X}}^{\exp}(V \times T)}. \tag{2.6}$$

According to the excision theorem, we may replace V by any complex open set containing Ω. Indeed, the following fact is shown by Theorem 2.3.

Theorem 2.6. *The closed set $\overline{\mathbb{R}}$ in $\mathbb{D}^2$ is purely 1- codimensional relative to the sheaf $\mathcal{O}_{\mathbb{D}^2}^{\exp}$. More generally, the closed set $\overline{N}$ in $\hat{X}$ is purely 1- codimensional relative to the sheaf $\mathcal{O}_{\hat{X}}^{\exp}$, i.e., $\mathscr{H}_{\overline{N}}^{k}(\mathcal{O}_{\hat{X}}^{\exp}) = 0$ for $k \neq 1$.*

Therefore we see that the global sections of the sheaf $\mathcal{BO}_{\overline{N}}^{\exp}$ can be written in terms of cohomology groups. Similarly we have

$$\Gamma_{[a,\,\infty]}(\overline{\mathbb{R}},\, \mathcal{B}_{\overline{\mathbb{R}}}^{\exp}) = \frac{\mathcal{O}_{\mathbb{D}^2}^{\exp}(\mathbb{D}^2 \setminus [a,\,\infty])}{\mathcal{O}_{\mathbb{D}^2}^{\exp}(\mathbb{D}^2)}. \tag{2.7}$$

Hence the set $\mathcal{B}_{[a,\,\infty]}^{\exp}$ defined by H. Komatsu coincides with $\Gamma_{[a,\infty]}(\overline{\mathbb{R}},\, \mathcal{B}_{\overline{\mathbb{R}}}^{\exp})$ in our framework.

The following vanishing theorem is closely related to the flabiness of $\mathcal{BO}_{\overline{N}}^{\exp}$.

Theorem 2.7. *Let U be an open subset in $\mathbb{D}^2$, and W a pseudoconvex open subset in $\mathbb{C}^m$. Then we have*

$$H^k(U \times W, \mathcal{O}^{\exp}_{\hat{X}}) = 0, \qquad k \neq 0.$$

Now we state the theorem for the flabiness and the unique continuation property of $\mathcal{BO}^{\exp}_{\overline{N}}$.

Theorem 2.8. *Let $\Omega_1 \subset \Omega_2 \subset \overline{\mathbb{R}}$ and $W_1 \subset W_2 \subset \mathbb{C}^m$ be open subsets. Then we have*

(i) If W_1 is a Stein open subset in $\mathbb{C}^m$, then $\mathcal{BO}^{\exp}_{\overline{N}}(\Omega_2 \times W_1) \to \mathcal{BO}^{\exp}_{\overline{N}}(\Omega_1 \times W_1)$ is surjective, i.e., the sheaf $\mathcal{BO}^{\exp}_{\overline{N}}$ is flabby with respect to the variable of hyperfunction.

(ii) If W_1 and W_2 be non-empty connected open subsets in $\mathbb{C}^m$, then $\mathcal{BO}^{\exp}_{\overline{N}}(\Omega_1 \times W_2) \to \mathcal{BO}^{\exp}_{\overline{N}}(\Omega_1 \times W_1)$ is injective, i.e., the sheaf $\mathcal{BO}^{\exp}_{\overline{N}}$ has a unique continuation property with respect to variables of holomorphic paramerters.

§ 3. Embedding of locally integrable functions of exponential type

In this section, we construct the sheaf morphism from the sheaf of locally integrabl functions of exponential type to the sheaf of Laplace hyperfunctions and show that it is injective. We also see that the Laplace transformation as a Laplace hyperfunction and an ordinary function coincide on the space of locally integrable functions of exponential type.

Definition 3.1. Let Ω be an open subset in $\overline{\mathbb{R}}$. The set $\mathcal{L}^{\exp}_{\mathrm{loc}}(\Omega)$ of locally integrable functions of exponential type on Ω consists of a locally integrable function $f(x)$ on $\Omega \cap \mathbb{R}$ which satisfies, for any compact set K in Ω,

$$\int_{K\cap\mathbb{R}} |f(x)|e^{-H_K|x|}dx < \infty \tag{3.1}$$

with a constant H_K. We denote by $\mathcal{L}^{\exp}_{\mathrm{loc}}$ the associated sheaf on $\overline{\mathbb{R}}$ of the presheaf $\{\mathcal{L}^{\exp}_{\mathrm{loc}}(\Omega)\}_\Omega$.

Note that, if $\Omega \subset \mathbb{R}$, the estimate (3.1) is always satisfied. Hence the restriction of $\mathcal{L}^{\exp}_{\mathrm{loc}}$ to $\mathbb{R}$ is isomorphic to the sheaf $\mathcal{L}^1_{\mathrm{loc}}$ of locally integrable functions on $\mathbb{R}$.

Let us construct a sheaf morphism ι from the sheaf $\mathcal{L}^{\exp}_{\mathrm{loc}}$ to the sheaf $\mathcal{B}^{\exp}_{\overline{\mathbb{R}}}$. It sufficies to give a morphism $\iota_K : \Gamma_K(\overline{\mathbb{R}}, \mathcal{L}^{\exp}_{\mathrm{loc}}) \to \Gamma_K(\overline{\mathbb{R}}, \mathcal{B}^{\exp}_{\overline{\mathbb{R}}})$ for any compact set K in $\overline{\mathbb{R}}$ by the lemma 3.2 given in [9]. Moreover, by considering a partition of support, it is enough to give morphisms for any $K \subseteq [0, \infty]$ or $K \subseteq [-\infty, 0]$.

Lemma 3.2 ([9]). *Let X be a locally compact topological space with a countable base of open sets, and let $\mathcal{F}$ and $\mathcal{G}$ be soft sheaves on X. If the morphism $h_c : \Gamma_c(X, \mathcal{F}) \to \Gamma_c(X, \mathcal{G})$ satisfies*

$$\operatorname{supp} h_c(f) \subset \operatorname{supp} f, \qquad f \in \Gamma_c(X, \mathcal{F}), \tag{3.2}$$

then we can extend h_c to the sheaf morphism $h : \mathcal{F} \to \mathcal{G}$ uniquely. Moreover, if h_c also satisfies

$$\operatorname{supp} h_c(f) = \operatorname{supp} f, \qquad f \in \Gamma_c(X, \mathcal{F}), \tag{3.3}$$

then h is injective.

Let K be a compact set in $[0, \infty]$ or $[-\infty, 0]$ and let $f \in \Gamma_K(\overline{\mathbb{R}}, \mathcal{L}^{\exp}_{\mathrm{loc}})$ satisfying (3.1) for a constant H_K. For an arbitrary constant $A \geq H_K$, we set

$$F^{\pm}(z) := \frac{e^{\pm Az}}{2\pi\sqrt{-1}} \int_{-\infty}^{\infty} \frac{f(t)e^{-A|t|}}{t-z} dt. \tag{3.4}$$

As $f(x)e^{-A|x|}$ is integrable on $\mathbb{R}$, the functions $F^{\pm}$ give a holomolphic functions of exponential type on $\mathbb{D}^2 \setminus K$. If $K \subseteq [0, \infty]$, we define the morphism $\iota_K : \Gamma_K(\overline{\mathbb{R}}, \mathcal{L}^{\exp}_{\mathrm{loc}}) \to \Gamma_K(\overline{\mathbb{R}}, \mathcal{B}^{\exp}_{\overline{\mathbb{R}}})$ by $\iota_K(f) = [F^+]$, where F^+ is given by (3.4). Note that $\iota_K(f)$ does not depend on a choice of A. As a matter of fact, we have the following equation

$$\frac{e^{Az}}{2\pi\sqrt{-1}} \int_{-\infty}^{\infty} \frac{f(t)e^{-A|t|}}{t-z} dt - \frac{e^{Bz}}{2\pi\sqrt{-1}} \int_{-\infty}^{\infty} \frac{f(t)e^{-B|t|}}{t-z} dt = \frac{1}{2\pi\sqrt{-1}} \int_{-\infty}^{\infty} f(t)dt \int_A^B e^{-w(|t|-z)} dw \tag{3.5}$$

for constants $B > A \geq H_K$ and the right hand side of (3.5) is an entire function of exponential type. Hence ι_K is well-defined and clearly satisfies $\operatorname{supp} \iota_K(f) \subset \operatorname{supp} f$. If $K \subseteq [-\infty, 0]$, we define ι_K by $\iota_K(f) = [F^-]$ in the similar way as the case of K in $[0, \infty]$. Note that, for any compact set K in $\mathbb{R}$, we have

$$[F^+] = [F^-] = \left[\frac{1}{2\pi\sqrt{-1}} \int_{-\infty}^{\infty} \frac{f(t)}{t-z} dt\right] \tag{3.6}$$

in $\Gamma_K(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ and $\Gamma_{\{\infty\}}(\overline{\mathbb{R}}, \mathcal{L}^{\exp}_{\mathrm{loc}}) = \Gamma_{\{-\infty\}}(\overline{\mathbb{R}}, \mathcal{L}^{\exp}_{\mathrm{loc}}) = \emptyset$. Therefore we can define the morphism ι_K for any compact set K in $\overline{\mathbb{R}}$ (using a partition of support, if necessary). The morphism $\iota_c : \Gamma_c(\overline{\mathbb{R}}, \mathcal{L}^{\exp}_{\mathrm{loc}}) \to \Gamma_c(\overline{\mathbb{R}}, \mathcal{B}^{\exp}_{\overline{\mathbb{R}}})$ defined by $\{\iota_K\}_K$ satisfies (3.2). Hence ι_c is extended to the sheaf morphism $\iota : \mathcal{L}^{\exp}_{\mathrm{loc}} \to \mathcal{B}^{\exp}_{\overline{\mathbb{R}}}$ uniquely. The details are as follows; let U be an open subset in $\overline{\mathbb{R}}$. For a locally integrable function of exponential type $f \in \mathcal{L}^{\exp}_{\mathrm{loc}}(U)$, we first decompose it into a locally finite sum of locally integrable functions of exponential type with compact support:

$$f = \sum_{\lambda} f_\lambda. \tag{3.7}$$

Then the morphism $\iota_U : \mathcal{L}^{\exp}_{\rm loc}(U) \to \mathcal{B}^{\exp}_{\overline{\mathbb{R}}}(U)$ is defined by

$$\iota_U(f) = \sum_{\lambda} \iota_c(f_\lambda). \tag{3.8}$$

This is defined independently of a choice of a locally finite decomposition (3.7) of $f \in \mathcal{L}^{\exp}_{\rm loc}(U)$.

Let us show that the sheaf morphism $\iota : \mathcal{L}^{\exp}_{\rm loc} \to \mathcal{B}^{\exp}_{\overline{\mathbb{R}}}$ is injective. For that purpose, we prove that ι_K is injective for any compact K in $[0, \infty]$ or $[-\infty, 0]$.

Proposition 3.3. *Let K be a compact set in $[0, \infty]$ or $[-\infty, 0]$. Let $f \in \Gamma_K(\overline{\mathbb{R}}, \mathcal{L}^{\exp}_{\rm loc})$ and let $F^{\pm}(z)$ be the function defined by (3.4). Then $F^{\pm}(x+\sqrt{-1}\varepsilon) - F^{\pm}(x-\sqrt{-1}\varepsilon)$ converge to $f(x)$ almost everywhere as $\varepsilon \to 0$.*

Proof. We have

$$\begin{aligned}
&F^{\pm}(x+\sqrt{-1}\varepsilon) - F^{\pm}(x-\sqrt{-1}\varepsilon)\\
&= \frac{e^{\pm A(x+\sqrt{-1}\varepsilon)}}{2\pi\sqrt{-1}}\int_{-\infty}^{\infty}\frac{f(t)e^{-A|t|}}{t-(x+\sqrt{-1}\varepsilon)}dt - \frac{e^{\pm A(x-\sqrt{-1}\varepsilon)}}{2\pi\sqrt{-1}}\int_{-\infty}^{\infty}\frac{f(t)e^{-A|t|}}{t-(x-\sqrt{-1}\varepsilon)}dt\\
&= \frac{e^{\pm Ax}\cos A\varepsilon}{\pi}\int_{-\infty}^{\infty} f(t)e^{-A|t|}\frac{\varepsilon}{(t-x)^2+\varepsilon^2}dt\\
&\quad \pm \frac{e^{\pm Ax}\sin A\varepsilon}{\pi}\int_{-\infty}^{\infty} f(t)e^{-A|t|}\frac{t-x}{(t-x)^2+\varepsilon^2}dt.
\end{aligned}\tag{3.9}$$

The first term in the right hand side of (3.9) is the convolution of L^1-function $f(x)e^{-A|x|}$ with the Poisson kernel, which is well-known to converge to $f(x)e^{-A|x|}$ almost everywhere. On the other hand, the second term in the right hand side of (3.9) converges to 0 as $\varepsilon \to 0$. Indeed, setting

$$\Phi(t) := \int_0^t f(s+x)e^{-A|s+x|}ds, \tag{3.10}$$

we have

$$\begin{aligned}
\left|\int_{-\infty}^{\infty} f(t)e^{-A|t|}\frac{t-x}{(t-x)^2+\varepsilon^2}dt\right| &= \left|\int_{-\infty}^{\infty} f(t+x)e^{-A|t+x|}\frac{t}{t^2+\varepsilon^2}dt\right|\\
&\leqq \left|\left[\Phi(t)\frac{t}{t^2+\varepsilon^2}\right]_{t=-\infty}^{t=\infty} - \int_{-\infty}^{\infty}\Phi(t)\frac{\varepsilon^2-t^2}{(t^2+\varepsilon^2)^2}dt\right|\\
&= \left|\int_{-\infty}^{\infty}\Phi(t)\frac{t^2-\varepsilon^2}{(t^2+\varepsilon^2)^2}\right|
\end{aligned}\tag{3.11}$$

Since $\Phi(t)$ is continuous function at 0 and $\Phi(0) = 0$, for any $\delta > 0$ there exist some $l_\delta > 0$ such that $|\Phi(t)| \leqq \delta$ $(|t| < l_\delta)$. Then the right hand side of (3.11) is bounded by

$$\int_{|t|\leqq l_\delta}\frac{|\Phi(t)|}{t^2+\varepsilon^2}dt + \int_{|t|\geqq l_\delta}\frac{|\Phi(t)|}{t^2}dt \leqq \frac{\delta}{\epsilon}\int_{-\infty}^{\infty}\frac{dt}{1+t^2} + M\int_{|t|\geqq l_\varepsilon}\frac{dt}{t^2} = \frac{\delta\pi}{\varepsilon} + \frac{2M}{l_\delta}, \tag{3.12}$$

where $M := \displaystyle\int_{-\infty}^{\infty} |f(t)|e^{-A|t|}dt$. Hence we have

$$\left|\frac{e^{\pm Ax}\sin A\varepsilon}{\pi}\int_{-\infty}^{\infty} f(t)e^{-A|t|}\frac{t-x}{(t-x)^2+\varepsilon^2}dt\right| \leqq \frac{Ae^{\pm Ax}}{\pi}(\delta\pi + \frac{2M\varepsilon}{l_\delta}) \tag{3.13}$$

for any $\varepsilon > 0$ and $\delta > 0$. This implies that the second term in the right hand side of (3.9) converges to 0 as $\varepsilon \to 0$ □

The morphism ι_c satisfies (3.3). Hence we get the following theorem by Lemma 3.2.

Theorem 3.4. *The sheaf morphism $\iota : \mathcal{L}_{\mathrm{loc}}^{\mathrm{exp}} \to \mathcal{B}_{\overline{\mathbb{R}}}^{\mathrm{exp}}$ is injective and we can regard $\mathcal{L}_{\mathrm{loc}}^{\mathrm{exp}}$-functions as Laplace hyperfunctions.*

Let us recall the definition of the Laplace transforms of Laplace hyperfunctions.

Definition 3.5 ([3])**.** The Laplace transform $\widehat{f}(\lambda)$ of a Laplace hyperfunction $f(x) = [F] \in \mathcal{B}_{[a,\,\infty]}^{\mathrm{exp}}$ is defined by the integral

$$\widehat{f}(\lambda) := \int_C e^{-\lambda z}F(z)dz. \tag{3.14}$$

Here the path C of the integration is composed of a ray from $e^{i\alpha}\infty(-\pi/2 < \alpha < 0)$ to a point $c < a$ and a ray from c to $e^{i\beta}\infty(0 < \beta < \pi/2)$.

Theorem 3.6. *Let K be a compact set in $[0,\,\infty]$ and let $f \in \Gamma_K(\overline{\mathbb{R}},\, \mathcal{L}_{\mathrm{loc}}^{\mathrm{exp}})$. The Laplace transform $\widehat{\iota(f)}$ of the Laplace hyperfunction $\iota(f)$ coincides with the ordinary Laplace transform of f .*

Proof. Let $\iota(f)$ be a Laplace hyperfunction represented by (3.4). We have

$$\widehat{\iota(f)}(\lambda) = \int_C e^{-\lambda z}F(z)dz = \int_C e^{-\lambda z}\frac{e^{Az}}{2\pi\sqrt{-1}}\int_{-\infty}^{\infty}\frac{f(t)e^{-At}}{t-z}dtdz. \tag{3.15}$$

If $\mathrm{Re}\lambda > A$, we may change the order of integration above by Fubini's theorem and the integlation path C can be replaced by a simple closed curve surrounding t. Hence $\widehat{\iota(f)}(\lambda)$ is equal to

$$\int_{-\infty}^{\infty} f(t)e^{-At}\frac{1}{2\pi\sqrt{-1}}\int_C \frac{e^{-(\lambda-A)z}}{t-z}dzdt = \int_{-\infty}^{\infty} e^{-\lambda x}f(x)dx. \tag{3.16}$$

This completes the proof. □

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