RIMS Kôkyûroku Bessatsu
B40 (2013), 163–175

Borel summability of a formal solution for Cauchy problem of some linear partial differential equations

By

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Abstract

In this paper we study the Borel summability of a certain divergent formal power series solution for some linear partial differential equations. We show the Borel summability of the formal solution under the condition that an initial value function $\phi(x)$ is an entire function of exponential order at most 2.

§ 1. Introduction

Let $(t, x) \in \mathbb{C}^2$. In this paper we consider the following Cauchy problem:

$$\text{(E)} \qquad \begin{cases} \dfrac{\partial}{\partial t}u(t,x) = \left(\dfrac{\partial}{\partial x}\right)^2 u(t,x) + \displaystyle\sum_{2\le j+\alpha\le m, j>0} a_{j,\alpha} t^{j+\alpha-2}\left(t\dfrac{\partial}{\partial t}\right)^j \left(\dfrac{\partial}{\partial x}\right)^\alpha u(t,x) \\ u(0,x) = \phi(x) \end{cases}$$

where $a_{j,\alpha} \in \mathbb{C}$ and $a_{m,0} \neq 0$ and the initial value function $\phi(x)$ is a holomorphic function in a neighbourhood of the origin.

Let us introduce the following notations. Let $D_R = \{x \in \mathbb{C};\ |x| < R\}$ and $S_{d,\theta} = \{\xi \in \mathbb{C} \setminus \{0\};\ |d - \arg\xi| < \theta\}$. $\mathcal{O}(D_R)$ (resp. $\mathcal{O}(S_{d,\theta} \times D_R)$) is the set of all holomorphic functions on D_R (resp. $S_{d,\theta} \times D_R$). $\mathcal{O}(D_R)[[t]] := \{\sum_{i=0}^{\infty} u_i(x)t^i;\ u_i(x) \in \mathcal{O}(D_R)\}$.

Let us introduce some known results. We consider the following example:

$$(1.1) \qquad \begin{cases} \dfrac{\partial}{\partial t}u(t,x) = a\left(\dfrac{\partial}{\partial x}\right)^2 u(t,x) + bt\left(t\dfrac{\partial}{\partial t}\right)^3 u(t,x) \\ u(0,x) = \phi(x) \end{cases}$$

Received February 8, 2012, Accepted October 5, 2012.
2010 Mathematics Subject Classification(s): Primary 34M30; Secondary 34M25.
Key Words: Borel summability, Linear PDE
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where a and b are any complex numbers.

Let us recall some known results. If $a=1$ and $b=0$, then the equation (1.1) is the heat equation. Then we have the following two results i) and ii).

i) Assume that the initial value function $\phi(x)$ is an entire function and satisfies with some positive constants C and K,

$$|\phi(x)| \leq Ce^{K|x|^2} \quad \text{for} \quad x \in \mathbb{C}.$$

Then the formal power series solution $\hat{u}(t,x) \in \mathcal{O}(D_R)[[t]]$ of (1.1) is holomorphic in a neighborhood of $t=0$. This is a classical result (see [2]).

ii) The following result is that of Lutz-Miyake-Schäfke in [3]. The following two statements a) and b) are equivalent: Let $d \in \mathbb{R}$ be fixed.

a). The initial value function $\phi(x)$ is analytic on $\Omega_0 = S_{d/2,\theta} \cup S_{d/2+\pi,\theta} \cup D_R$ and satisfies with some positive constants C and K,

$$|\phi(x)| \leq Ce^{K|x|^2} \quad \text{on} \quad \Omega_0.$$

b). The formal power series solution $\hat{u}(t,x) \in \mathcal{O}(D_R)[[t]]$ of (1.1) is Borel summable in a direction d.

If $a=0$, then Ōuchi treated this type some linear/nonlinear partial differential equations in [5] and [6] ('02,'06).

Set $\Xi_0 = \{re^{i\theta}; r>0, \theta \equiv -(1/2)\arg(b)\ mod\ (\pi)\}$ for the equation (1.1) with $a=0$. By the result of Ōuchi [5] we have: Assume that the initial value function $\phi(x)$ is in $\mathcal{O}(D_R)$. Then the formal solution $\hat{u}(t,x) \in \mathcal{O}(D_R)[[t]]$ is Borel summable in a direction d with $\overline{S_{d,\theta}} \cap \Xi_0 = \emptyset$.

Author considered the case $ab \neq 0$ as the mixed type equations of [3] and [5], and had a result of the Borel summability of the formal solution $\hat{u}(t,x) \in \mathcal{O}(D_R)[[t]]$ for (1.1) in [8]: Under the condition that the initial value function $\phi(x)$ is an entire function with $|\phi(x)| \leq Ce^{K|x|^2}$ on $\mathbb{C}$, the formal solution $\hat{u}(x) \in \mathcal{O}(D_R)[[t]]$ ($\forall R>0$) is Borel summable in a direction d with $\overline{S_{d,\theta}} \cap \Xi_0 = \emptyset$.

This result is not obvious. In the case of $b=0$ the formal solution $\hat{u}(t,x)$ of (1.1) converges in a neighborhood of the origin $t=0$. But in the case $a=b=1$ and $\phi(x)=e^{x^2}$, the formal solution $\hat{u}(t,x)=\sum_{i=0}^{\infty} u_i(x)t^i$ of (1.1) satisfies for $x \in \mathbb{R}$

$$(1.2)\qquad \begin{aligned} u_i(x) &\geq 2^{i-3}\frac{(\frac{i-2}{2})!^3}{(\frac{i}{2})!}\Big(\frac{\partial}{\partial x}\Big)^4\phi(x) \geq AB^i i!\Big(\frac{\partial}{\partial x}\Big)^4\phi(x) \quad i\geq 2,\ i:\text{even} \\ u_i(x) &\geq \frac{1}{2^{i-4}}\frac{(\frac{i-1}{2})!}{i!}\frac{(i-2)!^3}{(\frac{i-3}{2})!^3}\Big(\frac{\partial}{\partial x}\Big)^2\phi(x) \geq AB^i i!\Big(\frac{\partial}{\partial x}\Big)^2\phi(x) \quad i\geq 3,\ i:\text{odd}. \end{aligned}$$

By the estimate (1.2), the formal solution $\hat{u}(t,x) \in \mathcal{O}(D_R)[[t]]$ ($\forall R>0$) is a certain divergent power series. So it is our purpose to study the summability of the formal solution $\hat{u}(t,x)$ of (1.1).

Let us show the estimate (1.2). For the Cauchy problem (1.1) we have a formal solution $\hat{u}(t,x)=\sum_{i=0}^{\infty}u_i(x)t^i$ with

$$\begin{cases} u_0(x)=\phi(x) \\ iu_i(x)=(\dfrac{\partial}{\partial x})^2 u_{i-1}(x)+(i-2)^3 u_{i-2}(x), \end{cases}$$

where $u_{-i}(x)\equiv 0$ for $i\geq 1$. By the above relations the coefficients $u_i(x)$ have the following form.

$$u_i(x)=\sum_{j=0}^{i}C_{0,i,j}\big(\frac{d}{dx}\big)^{2j}\phi(x)=\sum_{j=0}^{i}C_{0,i,j}\sum_{k=0}^{j}C_{1,j,k}x^{2k}e^{x^2} \quad \text{with } C_{0,i,j}\geq 0,\ C_{1,j,k}\geq 0.$$

Then we have $u_i(x)\geq 0$ for $x\in\mathbb{R}$ and $i\geq 0$. In the case $i=2n\geq 2$ we have

$$\begin{aligned} u_i(x) &\geq \frac{(i-2)^3}{i}u_{i-2} \\ &\geq \frac{(i-2)^3}{i}\frac{(i-4)^3}{(i-2)}\cdots\frac{2^3}{4}u_2(x)=\frac{(2^{(i-2)/2})^3}{2^{(i-2)/2}}\frac{((i-2)/2)!^3}{(i/2)!}u_2(x), \end{aligned} \tag{1.3}$$

and $2u_2(x)=(d/dx)^4u_0(x)=(d/dx)^4\phi(x)$. Hence we get the estimate (1.2) by the estimate (1.3). In the case $i=2n+1$ we can show the estimate (1.2) by a similar way to the case $i=2n$.

In this paper we will study the Borel summaility of formal solutions for the Cauchy problem (E) as a general case of the Cauchy problem (1.1).

§ 2. Definition and Main result

In this section we give a definition of the Borel summability and the main theorem.

Let us give a definition of the Borel summability.

Definition 2.1. Let $\hat{u}(t,x)=\sum_{i=0}^{\infty}v_i(x)t^i\in\mathcal{O}(D_R)[[t]]$. Then the formal Borel transform $(\hat{\mathcal{B}}\hat{u})(\xi,x)$ is defined by

$$(\hat{\mathcal{B}}\hat{u})(\xi,x)=v_0(x)\delta(\xi)+\sum_{i=1}^{\infty}\frac{v_i(x)}{\Gamma(i)}\xi^{i-1}$$

where $\delta(\xi)$ means the delta function with support at $\xi=0$. In the following the notation $\xi^{-1}/\Gamma(0)$ means $\delta(\xi)$.

Definition 2.2. The formal series $\hat{u}(t,x)\in\mathcal{O}(D_R)[[t]]$ is Borel summable in a direction d if one can find some $0<r<R$ so that the following two properties hold:

1. The series $V(\xi, x) = (\hat{\mathcal{B}}\hat{u})(\xi, x) - v_0(x)\delta(\xi)$ converges for $|\xi| < \rho$ and $x \in \overline{D_r}$.

2. There exists a $\theta > 0$ such that for any $x \in \overline{D_r}$ the function $V(\xi, x)$ can be holomorphically continued with respect to ξ into the sector $S_{d,\theta}$. Moreover for any $0 < \theta_1 < \theta$ there exist constants C, $K > 0$ such that

$$\sup_{|x|\le r} |V(\xi, x)| \le Ce^{K|\xi|} \quad \text{for} \quad \xi \in S_{d,\theta_1}.$$

Then $v_0(x) + (\mathcal{L}_d V)(t, x)$ is called the Borel summation in a direction d of $\hat{u}(t, x)$, where $\mathcal{L}_d$ is the Laplace transform that is defined by

$$(\mathcal{L}_d\phi)(t, x) := \int_0^{\infty e^{id}} \exp\Big(-\big(\frac{\xi}{t}\big)\Big)\phi(\xi, x)d\xi.$$

Let us introduce the main result. For the equation (E) set

$$A_0(\xi) = 1 - \sum_{j=2}^{m} a_{j,0}\xi^{j-1}. \tag{2.1}$$

Definition 2.3. Set $Z = \{\xi; A_0(\xi) = 0\}$. A singular direction is an argument of an element of Z. We denote by Ξ the totality of singular directions.

Remark. For the example (1.1), set $A_0(\xi) = 1 - b\xi^2$. Then $\Xi_0 = \{re^{i\theta}; r > 0, \theta \equiv -(1/2)\arg(b)\ mod\ (\pi)\}$ by $A_0(\xi) = 0$.

Theorem 2.4 (Main theorem). *Assume that the initial value function $\phi(x)$ is an entire function and satisfies*

$$|\phi(x)| \le Ce^{K|x|^2} \quad on \quad \mathbb{C}.$$

Then the Cauchy problem (E) has a formal power series solution $\hat{u}(t, x) \in \mathcal{O}(D_R)[[t]]$ ($\forall R > 0$) and the formal solution is Borel summable in a direction d with $\overline{S_{d,\theta}} \cap \Xi = \emptyset$ for a sufficiently small $\theta > 0$.

§ 3. Formal solution

In this section we will get a formal power series solution of the Cauchy problem (E) and give an estimate to all coefficients of the formal power series solution. Moreover we will introduce the Newton polygon as a tool to give the estimate.

If the initial value function $\phi(x)$ is in $\mathcal{O}(D_R)$, then for (E) we have a formal solution $\hat{u}(t, x) = \sum_{i=0}^{\infty} u_i(x)t^i$ with

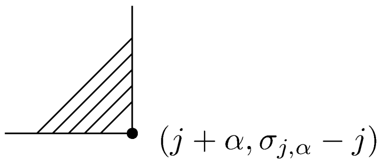
(3.1)
$$\begin{cases} u_0(x) = \phi(x) \\ iu_i(x) = (\dfrac{\partial}{\partial x})^2 u_{i-1}(x) + \displaystyle\sum_{2\le j+\alpha\le m, j>0} a_{j,\alpha}(i - (j + \alpha - 1))^j (\dfrac{\partial}{\partial x})^\alpha u_{i-(j+\alpha-1)}(x) \end{cases}$$

where $u_{-i}(x) \equiv 0$ for $i \geq 1$. By the relation (3.1) we have the formal solution $\hat{u}(t,x) \in \mathcal{O}(D_R)[[t]]$.

§ 3.1. Newton polygon

We want to give an estimate for the coefficients $u_i(x)$. Then let us define the Newton polygon for the equation (E). For an operator $t^{\sigma_{j,\alpha}}\left(\frac{\partial}{\partial t}\right)^j\left(\frac{\partial}{\partial x}\right)^\alpha$, we set

$$\Sigma(j,\alpha) := \{(a,b) \in \mathbb{R}^2; a \leq j+\alpha, b \geq \sigma_{j,\alpha} - j\}.$$



Set an operator

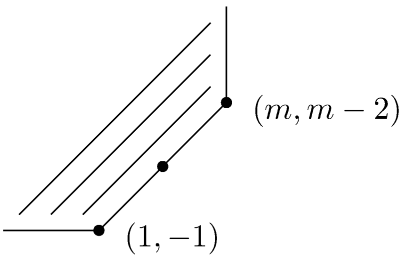
$$L = \sum_{(j,\alpha)}^{finite} a_{j,\alpha} t^{\sigma_{j,\alpha}} (\frac{\partial}{\partial t})^j (\frac{\partial}{\partial x})^\alpha.$$

Then for the operator L let us define the Newton polygon $NP(L)$ by

$$NP(L) := CH \bigcup_{(j,\alpha)}^{finite} \{\Sigma(j,\alpha); a_{j,\alpha} \neq 0\},$$

where $CH\{\cdot\}$ denotes the convex hull of a set $\{\cdot\}$.

Then the Newton polygon $NP(E)$ for the equation (E) is the following figure:



The Newton polygon $NP(E)$ has the side with a slope 1.

§ 3.2. Formal Gevrey estimate

We give an estimate for the coefficients $u_i(x)$. The estimate is characterized by the Newton polygon and is called the formal Gevrey estimate. For some linear or non linear partial differential equations, the formal Gevrey estimate of a formal solution is given by many mathematicians (M. Miyake, S. Ōuchi, etc). We omit a proof of the following lemma. We refer the details to Ōuchi [4].

Lemma 3.1. *Assume that for the Cauchy problem (E), the initial value function* $\phi(x)$ *is in* $\mathcal{O}(D_R)$. *Then for the formal solution* $\hat{u}(t,x) = \sum_{i=0}^{\infty} u_i(x)t^i$, *there exist positive constants* A *and* B *such that*

$$|u_i(x)| \leq AB^i\Gamma(i+1) \quad on \quad D_r \quad for \quad 0 < r < R. \tag{3.2}$$

§ 4. Preparatory lemmas

In order to prove Theorem 2.4, we give a definition of the convolution and two lemmas (Lemma 1.4 and 3.1) in [5]. For two lemmas we omit the proof.

Definition 4.1. Let $\phi_i(\xi,x) \in \mathcal{O}(S_{d,\theta} \times D_R)$ $(i = 1,2)$ satisfying $|\phi_i(\xi,x)| \leq C|\xi|^{\epsilon-1}$ for $\epsilon > 0$. Then the convolution of $\phi_1(\xi,x)$ and $\phi_2(\xi,x)$ is defined by

$$(\phi_1 * \phi_2)(\xi,x) = \int_0^{\xi} \phi_1(\xi-\eta,x)\phi_2(\eta,x)d\eta.$$

Let us introduce two lemmas in [5].

Lemma 4.2 (Lemma 1.4 in [5]). *Assume that functions* $\phi_i(\xi,x) \in \mathcal{O}(S_{d,\theta} \times D_R)$ *satisfy*

$$|\phi_i(\xi,x)| \leq c_i \frac{|\xi|^{s_i-1}}{\Gamma(s_i)} \quad on \quad S_{d,\theta} \times D_R$$

with $s_i > 0$ *for* $i = 1,2$. *Then the convolution* $\phi_1 * \phi_2$ *satisfies*

$$|(\phi_1 * \phi_2)(\xi,x)| \leq c_1c_2 \frac{|\xi|^{s_1+s_2-1}}{\Gamma(s_1+s_2)} \quad on \quad S_{d,\theta} \times D_R.$$

Lemma 4.3 (Lemma 3.1 in [5]). *For a series* $\hat{v}(t,x) = \sum_{n=1}^{\infty} v_n(x)t^n$ *set* $(\hat{\mathcal{B}}\hat{v})(\xi,x) = V(\xi,x)$. *For* $1 \leq k \leq \delta$ *we have*

$$\hat{\mathcal{B}}(t^{\delta}(t\frac{\partial}{\partial t})^k\hat{v})(\xi,x) = \sum_{s=1}^{k} C_{k,s} \frac{\xi^{\delta-(s+1)}}{\Gamma(\delta-s)} * (\xi^s V(\xi,x))$$

where numbers $C_{k,s}$ $(1 \leq s \leq k)$ *are constant numbers in [5] and satisfy*

$$C_{1,1} = 1, \quad C_{k,s} = -sC_{k-1,s} + C_{k-1,s-1} \quad for\ k \geq 2, \tag{4.1}$$

with $C_{k,0} = C_{k-1,k} = 0$. *By the relation* (4.1) *we have* $C_{k,k} = 1$ *for* $k \geq 1$.

§ 5. Proof of Main theorem

In this section we will give a proof of Theorem 2.4. Firstly we shall construct a convolution equation from the Cauchy problem (E). Secondly we shall show that the convolution equation has a unique solution $V_{loc}(\xi, x)$ which is holomorphic in a neighborhood of $(\xi, x) = (0, 0)$ (Lemma 5.2). Moreover we shall show that the convolution equation has an analytic solution $V_S(\xi, x)$ on $S_{d,\theta} \times D_r$ $(0 < r < R)$ with the exponential growth estimate of order at most 1 there (Corollary 5.5). Finally we shall give that $V_{loc}(\xi, x) = V_S(\xi, x)$ on $S_{d,\theta} \cap \{\xi; |\xi| < \rho\} \times D_r$ (Proposition 5.6).

Let us construct a convolution equation from the Cauchy problem (E). We substitute $u(t, x) = \phi(x) + v(t, x)$ into the equation (E), where the function $v(t, x)$ is a new unknown function. Then we get the following equation:

$$\frac{\partial}{\partial t} v(t,x) = \varphi(x) + (\frac{\partial}{\partial x})^2 v(t,x) + \sum_{2 \le j+\alpha \le m, j>0} a_{j,\alpha} t^{j+\alpha-2} \big(t\frac{\partial}{\partial t}\big)^j \big(\frac{\partial}{\partial x}\big)^\alpha v(t,x) \tag{5.1}$$

where $\varphi(x) = (\frac{\partial}{\partial x})^2 \phi(x)$.

Set $\hat{v}(t, x) = \hat{u}(t, x) - \phi(x)$ where $\hat{u}(t, x)$ is the formal soution of the Cauchy problem (E). Then the formal series $\hat{v}(t, x)$ is a formal solution of the equation (5.1).

Set $V(\xi, x) = (\hat{\mathcal{B}}\hat{v})(\xi, x)$. By Lemma 3.1 we have the following result.

Lemma 5.1. *The series $V(\xi, x)$ converges on $\{\xi \in \mathbb{C} : |\xi| < \rho\} \times D_R$ $(\forall R > 0)$ for a sufficiently small $\rho > 0$.*

Let us find a formal convolution equation that $V(\xi, x)$ satisfies. We multiply each term of the equation (5.1) by t^2, and apply the formal Borel transform. Then we get the following convolution equation by Lemma 4.3:

$$\begin{aligned} \xi V(\xi,x) &= \xi\varphi(x) + \xi * (\frac{\partial}{\partial x})^2 V(\xi,x) \\ &\quad + \sum_{2 \le j+\alpha \le m, j>0} a_{j,\alpha} \sum_{s=1}^{j} C_{j,s} \frac{\xi^{j+\alpha-(s+1)}}{\Gamma(j+\alpha-s)} * (\xi^s (\frac{\partial}{\partial x})^\alpha V(\xi,x)). \end{aligned} \tag{5.2}$$

Lemma 5.2. *The function $V(\xi, x)$ in Lemma 5.1 is a solution of the convolution equation (5.2) on $\{\xi \in \mathbb{C} : |\xi| < \rho\} \times D_R$ for a sufficiently small $\rho > 0$.*

Let $V_{loc}(\xi, x)$ be the solution in Lemma 5.2. We will show that the solution $V_{loc}(\xi, x)$ is holomorhically extensible to $S_{d,\theta} \times D_r$ for $0 < r < R$. In order to show that, we construct another solution $V_S(t, x)$ of the convolution equation (5.2) on $S_{d,\theta} \times D_r$. Moreover we show $V_{loc}(\xi, x) = V_S(\xi, x)$ on $S_{d,\theta}(\rho) \times D_r$ where $S_{d,\theta}(\rho) = S_{d,\theta} \cap \{\xi \in \mathbb{C} : |\xi| < \rho \ll 1\}$.

Set $A(\xi) := \xi - \sum_{j=2}^{m} a_{j,0}\xi^j$. Then we have $A(\xi) = \xi A_0(\xi)$ where $A_0(\xi)$ is defined in (2.1). By

$$C_{j,j} \frac{\xi^{-1}}{\Gamma(0)} * (\xi^j V(\xi,x)) = \delta(\xi) * (\xi^j V(\xi,x)) = \xi^j V(\xi,x)$$

we can rewrite the equation (5.2) as follows;

$$\text{(CE)}\ A(\xi)V(\xi,x)=\xi\varphi(x)+\xi*(\frac{\partial}{\partial x})^2V(\xi,x)+\sum_{2\le j\le m}B_j(V)+\sum_{2\le j+\alpha\le m,j,\alpha>0}B_{j,\alpha}(V)$$

where

$$B_j(V):=a_{j,0}\sum_{s=1}^{j-1}C_{j,s}\frac{\xi^{j-(s+1)}}{\Gamma(j-s)}*(\xi^sV(\xi,x))\quad\text{for } j=2,\dots,m,\ \text{and}$$

$$B_{j,\alpha}(V):=a_{j,\alpha}\sum_{s=1}^{j}C_{j,s}\frac{\xi^{j+\alpha-(s+1)}}{\Gamma(j+\alpha-s)}*(\xi^s(\frac{\partial}{\partial x})^\alpha V(\xi,x))\quad\text{for } 2\le j+\alpha\le m \text{ and } j,\alpha>0.$$

Let us construct a solution of the equation (CE) on $S_{d,\theta}$. Then for the equation (CE) we determine a sequence $\{V_k(\xi,x)\}_{k=0}^{\infty}$ by the following recurrences:
(5.3)

$$\begin{cases} A(\xi)V_0(\xi,x)=\xi\varphi(x) \\ A(\xi)V_k(\xi,x)=\xi*(\frac{\partial}{\partial x})^2V_{k-2}(\xi,x) \\ \qquad +\sum_{2\le j\le m}B_j(V_{k-2}(\xi,x))+\sum_{2\le j+\alpha\le m,j,\alpha>0}B_{j,\alpha}(V_{k-\alpha}(\xi,x))\quad\text{for}\quad k\ge 1 \end{cases}$$

where $V_{-k}(\xi,x)\equiv 0$ for $k\ge 1$. Set $\Omega_{d,\theta}(\rho)=\{|\xi|<\rho\}\cup S_{d,\theta}$ with $\overline{S_{d,\theta}}\cap\Xi=\emptyset$. Then we see that $V_k(\xi,x)\in\mathcal{O}(\Omega_{d,\theta}(\rho)\times D_R)$ and $V_S(\xi,x):=\sum_{k=0}^{\infty}V_k(\xi,x)$ is a formal solution of the equation (CE).

For each $V_k(\xi,x)$ let us give an estimate. To estimate functions $V_k(\xi,x)$ we need the following lemma, which can be found in [2] and [7].

Lemma 5.3. *The following two statements are equivalent:*
(i) A function $\phi(x)$ is an entire function and satisfies that there exist positive constants C and K such that

$$|\phi(x)|\le Ce^{K|x|^2}\quad on\quad \mathbb{C}.$$

(ii) For any $R>0$, there exist positive constants D and E depending on R such that

$$||(\frac{\partial}{\partial x})^i\phi||_R\le DE^i\Gamma(\frac{i}{2}+1)$$

for all $i\ge 0$ where $||\cdot||_R=\sup_{|x|\le R}|\cdot|$.

The following proposition is important to show the main theorem.

Proposition 5.4. *For the initial value function $\varphi(x)=(\frac{\partial}{\partial x})^2\phi(x)$, we assume*

$$||(\frac{\partial}{\partial x})^i\varphi||_R\le DE^i\Gamma(\frac{i}{2}+1)$$

for all $i \geq 0$. Then we have that for $k = 0, 1, \ldots,$ there exist positive constants D_0 and K such that

$$||(\frac{\partial}{\partial x})^i V_k||_R \leq D_0 E^{i+k} K^k \Gamma(\frac{i+k}{2}+1)\frac{|\xi|^{\frac{k}{2}}}{k!^{\frac{1}{2}}(k+1)!^{\frac{1}{2}}} \quad for \quad \xi \in \Omega_{d,\theta}(\rho)$$

for all $i \geq 0$.

We give a proof of Proposition 5.4 in Section 6.

By Proposition 5.4, we have the following corollary.

Corollary 5.5. *For the solution $V_S(\xi, x) = \sum_{k\geq 0} V_k(\xi, x)$, we have that there exist positive constants c and D_1 such that*

$$||V_S||_R \leq D_1 e^{c|\xi|} \quad for \quad \xi \in \Omega_{d,\theta}(\rho).$$

Proof. By Proposition 5.4 there exist positive constants D' and E' such that

$$||V_k||_R \leq D' E'^k \frac{|\xi|^{\frac{k}{2}}}{\Gamma(\frac{k}{2}+1)} \quad \text{for} \quad \xi \in \Omega_{d,\theta}(\rho). \tag{5.4}$$

By the estimate (5.4) we get Corollary 5.5. □

Let us show the uniqueness of the solution of (CE) on $\Omega_{d,\theta}(\rho) \times D_R$.

Proposition 5.6. *Let $V_{loc}(\xi, x)$ be the solution in Lemma 5.2 and $V_S(\xi, x)$ be the solution in Corollary 5.5. Then we have $V_{loc}(\xi, x) = V_S(\xi, x)$ on $S_{d,\theta}(\rho) \times D_R$ for $0 < \rho \ll 1$.*

We give a proof of Proposition 5.6 in Section 6.

Let us give a proof of the main theorem.

Proof. By Lemma 5.2, Corollary 5.5 and Proposition 5.6, the formal solution $\hat{u}(t, x)$ of the Cauchy problem (E) is Borel Summable in a direction d. □

§ 6. Proof of Proposition 5.4 and 5.6

In this section we shall give a proof on Proposition 5.4 and 5.6.

Proof of Proposition 5.4.

Proof. We can give a proof by Lemma 4.2. For $A_0(\xi)$ in (2.1) we have

$$|\{A_0(\xi)\}^{-1}| \leq C_0(|\xi|^{m-1}+1)^{-1} \quad \text{for} \quad \xi \in \Omega_{d,\theta}(\rho). \tag{6.1}$$

If $k=0$, by the relation (5.3) and the estimate (6.1) we have

$$||(\frac{\partial}{\partial x})^i V_0||_R \leq C_0 D E^i \Gamma(\frac{i}{2}+1) \quad \text{for} \quad \xi \in \Omega_{d,\theta}(\rho).$$

Then by taking $D_0 = C_0 D$ we get the estimate in Proposition 5.4 on $k=0$.

For $k \geq 1$ we use an inductive method on k. Let us give an estimate for each term of the right hand side in the relation (5.3). For the first term, by the induction's assumption we have

$$||(\frac{\partial}{\partial x})^i V_{k-2}||_R \leq D_0 E^{i+k-2} K^{k-2} \Gamma(\frac{i+k-2}{2}+1) \frac{|\xi|^{\frac{k-2}{2}}}{(k-2)!^{\frac{1}{2}}(k-1)!^{\frac{1}{2}}} \tag{6.2}$$

for all $i \geq 0$. Then we have

$$||(\frac{\partial}{\partial x})^{i+2} V_{k-2}||_R \leq D_0 E^{i+k} K^{k-2} \Gamma(\frac{i+k}{2}+1) \frac{|\xi|^{\frac{k-2}{2}}}{(k-2)!^{\frac{1}{2}}(k-1)!^{\frac{1}{2}}}$$

for all $i \geq 0$. Here we use the following inequality

$$\frac{\Gamma(\frac{k-2}{2}+1)}{(k-2)!^{\frac{1}{2}}(k-1)!^{\frac{1}{2}}} \frac{1}{\Gamma(\frac{k-2}{2}+3)} = 2^2 \frac{1}{(k-2)!^{\frac{1}{2}}(k-1)!^{\frac{1}{2}}} \frac{1}{(k+2)k} \leq 2^2 \frac{1}{k!^{\frac{1}{2}}(k+1)!^{\frac{1}{2}}}.$$

Then by Lemma 4.2 we get

$$\begin{aligned} ||\xi * (\frac{\partial}{\partial x})^{i+2} V_{k-2}||_R &\leq D_0 E^{i+k} K^{k-2} \Gamma(\frac{i+k}{2}+1) \frac{\Gamma(\frac{k-2}{2}+1)}{(k-2)!^{\frac{1}{2}}(k-1)!^{\frac{1}{2}}} \frac{|\xi|^{\frac{k-2}{2}+3-1}}{\Gamma(\frac{k-2}{2}+3)} \\ &\leq D_0 E^{i+k} K^{k-2} \Gamma(\frac{i+k}{2}+1) 2^2 \frac{|\xi|^{\frac{k}{2}+1}}{k!^{\frac{1}{2}}(k+1)!^{\frac{1}{2}}}. \end{aligned} \tag{6.3}$$

Let us give an estimate for the term $B_j(V_{k-2})$. By the estimate (6.2) and Lemma 4.2 for $s=1,\ldots,j-1$ we get

$$\begin{aligned} &||\frac{\xi^{j-(s+1)}}{\Gamma(j-s)} * (\xi^s (\frac{\partial}{\partial x})^i V_{k-2})||_R \\ \leq& D_0 E^{i+k-2} K^{k-2} \Gamma(\frac{i+k-2}{2}+1) \frac{\Gamma(s+\frac{k-2}{2}+1)}{(k-2)!^{\frac{1}{2}}(k-1)!^{\frac{1}{2}}} \frac{|\xi|^{j+\frac{k-2}{2}}}{\Gamma(j+\frac{k-2}{2}+1)} \\ \leq& D_0 E^{i+k-2} K^{k-2} \Gamma(\frac{i+k-2}{2}+1) \frac{1}{(k-2)!^{\frac{1}{2}}(k-1)!^{\frac{1}{2}}} \frac{|\xi|^{j+\frac{k-2}{2}}}{(j+\frac{k-2}{2})} \\ \leq& D_0 E^{i+k-2} K^{k-2} \frac{|\xi|^{j+\frac{k-2}{2}}}{k!^{\frac{1}{2}}(k+1)!^{\frac{1}{2}}} \Gamma(\frac{i+k}{2}) \frac{(k-1)^{\frac{1}{2}} k (k+1)^{\frac{1}{2}}}{j+\frac{k-2}{2}}. \end{aligned}$$

For $2 \leq j \leq m$ we have

$$\frac{(k-1)^{\frac{1}{2}} k (k+1)^{\frac{1}{2}}}{j+\frac{k-2}{2}} \leq 2k.$$

Therefore we get

$$(6.4)\quad ||(\frac{\partial}{\partial x})^i B_j(V_{k-2})||_R \le |a_{j,0}| \sum_{s=1}^{j-1} |C_{j,s}| \frac{4}{E^2} D_0 E^{i+k} K^{k-2} \Gamma(\frac{i+k}{2}+1) \frac{|\xi|^{j+\frac{k}{2}-1}}{k!^{\frac{1}{2}}(k+1)!^{\frac{1}{2}}}$$

for all $i \ge 0$.

Let us give an estimate for the term $B_{j,\alpha}(V_{k-\alpha})$. By induction's assumption we have

$$||(\frac{\partial}{\partial x})^i V_{k-\alpha}||_R \le D_0 E^{i+k-\alpha} K^{k-\alpha} \Gamma(\frac{i+k-\alpha}{2}+1) \frac{|\xi|^{\frac{k-\alpha}{2}}}{(k-\alpha)!^{\frac{1}{2}}(k-\alpha+1)!^{\frac{1}{2}}}$$

for all $i \ge 0$. Then we have

$$||\xi^s (\frac{\partial}{\partial x})^{i+\alpha} V_{k-\alpha}||_R \le D_0 E^{i+k} K^{k-\alpha} \Gamma(\frac{i+k}{2}+1) \frac{|\xi|^{s+\frac{k-\alpha}{2}}}{(k-\alpha)!^{\frac{1}{2}}(k-\alpha+1)!^{\frac{1}{2}}}$$

for all $i \ge 0$. For $2 \le j+\alpha \le m$ and $j, \alpha > 0$ we get

$$\begin{aligned} & ||\frac{\xi^{j+\alpha-(s+1)}}{\Gamma(j+\alpha-s)} * \{\xi^s (\frac{\partial}{\partial x})^{i+\alpha} V_{k-\alpha}\}||_R \\ \le & D_0 E^{i+k} K^{k-\alpha} \Gamma(\frac{i+k}{2}+1) \frac{\Gamma(s+\frac{k-\alpha}{2}+1)}{(k-\alpha)!^{\frac{1}{2}}(k-\alpha+1)!^{\frac{1}{2}}} \frac{|\xi|^{j+\alpha+\frac{k-\alpha}{2}}}{\Gamma(j+\alpha+\frac{k-\alpha}{2}+1)} \end{aligned}$$

by Lemma 4.2. For $s = 1, \ldots, j$ we have

$$\frac{\Gamma(s+\frac{k-\alpha}{2}+1)}{(k-\alpha)!^{\frac{1}{2}}(k-\alpha+1)!^{\frac{1}{2}}} \frac{1}{\Gamma(j+\alpha+\frac{k-\alpha}{2}+1)} \le 2^\alpha \frac{1}{k!^{\frac{1}{2}}(k+1)!^{\frac{1}{2}}}.$$

Therefore we get

$$(6.5)\quad ||(\frac{\partial}{\partial x})^i B_{j,\alpha}(V_{k-\alpha})||_R \le 2^\alpha |a_{j,\alpha}| \sum_{s=1}^{j} |C_{j,s}| D_0 E^{i+k} K^{k-\alpha} \Gamma(\frac{i+k}{2}+1) \frac{|\xi|^{j+\frac{k+\alpha}{2}}}{k!^{\frac{1}{2}}(k+1)!^{\frac{1}{2}}}$$

for all $i \ge 0$. For $|\xi|^{j+(k+\alpha)/2}$ we remark $1+\frac{k}{2} \le j+\frac{k+\alpha}{2} \le j+\alpha+\frac{k}{2} \le m+\frac{k}{2}$.

We take a sufficiently large $K > 0$ so that

$$1 \ge C_0\Big(\frac{4}{K^2} + 4\sum_{2\le j\le m} |a_{j,0}| \sum_{s=1}^{j-1} |C_{j,s}| \frac{1}{E^2K^2} + \sum_{2\le j+\alpha\le m, j,\alpha>0} 2^\alpha |a_{j,\alpha}| \sum_{s=1}^{j} |C_{j,s}| \frac{1}{K^\alpha}\Big).$$

Hence we complete a proof of Proposition 5.4 by the estimates (6.3), (6.4) and (6.5). □

Proof of Proposition 5.6

Proof. Set $U(\xi,x) = V_{loc}(\xi,x) - V_S(\xi,x)$. The the function $U(\xi,x)$ is a solution of the following equation:

(6.6)
$$A(\xi)U(\xi,x) = \xi * (\frac{\partial}{\partial x})^2 U(\xi,x) + \sum_{2\le j\le m} B_j(U(\xi,x)) + \sum_{2\le j+\alpha\le m, j,\alpha>0} B_{j,\alpha}(U(\xi,x)).$$

Further by Lemma 5.2 and Corollary 5.5, we have

$$||U||_R \le D_4 \quad \text{for } \xi \in S_{d,\theta}(\rho)$$

for some $D_4 > 0$. Then by the Cauchy's integral theorem we have

$$||(\frac{\partial}{\partial x})^i U||_r \le D_4 (E_r)^i i! \quad \text{for } \xi \in S_{d,\theta}(\rho) \tag{6.7}$$

where $E_r = 2/(R-r)$ for $0 < r < R$.

We estimate the left hand side in the equation (6.6) by substituting the estimate (6.7) into the right hand side in (6.6). By a similar way to Proposition 5.4, we get for all $k \ge 0$,

$$||U||_r \le D_4 (E_r K_r)^k |\xi|^{k/2} \quad \text{for } \xi \in S_{d,\theta}(\rho)$$

where $K_r > 0$ satisfies

$$1 \ge C_0\Big(\frac{4}{K_r^2} + 2\sum_{2\le j\le m} |a_{j,0}| \sum_{s=1}^{j-1} |C_{j,s}| \frac{1}{E_r^2 K_r^2} + \sum_{2\le j+\alpha\le m, j,\alpha>0} 2^\alpha |a_{j,\alpha}| \sum_{s=1}^{j} |C_{j,s}| \frac{1}{K_r^\alpha}\Big).$$

By letting $k \to \infty$ then we have $U(\xi,x) \equiv 0$ on $S_{d,\theta}(\rho) \times D_r$ for $0 < \rho \ll 1$. □

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