RIMS Kôkyûroku Bessatsu
B40 (2013), 191–212

On the Borel summability of 0-parameter solutions of nonlinear ordinary differential equations

By

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Abstract

In this paper, we announce our recent results on the Borel summability of 0-parameter solutions of second order nonlinear ordinary differential equations with a large parameter. 0-parameter solutions are formal power series solutions with respect to a large parameter, and we establish their Borel summability for a wide class of equations including Painlevé equations. We also study the singularity structure of a 1-form ω for the Painlevé equations, which plays an important role in our analysis.

§ 0. Introduction

The main purpose of this article is to announce the results of [KKo] on the Borel summability of 0-parameter solutions of second order nonlinear ordinary differential equations with a large parameter.

The exact WKB analysis was initiated by A. Voros. He discussed WKB analysis of a Schrödinger equation

$$\left(\frac{d^2}{dx^2} - \eta^2 Q(x)\right)\psi(x,\eta) = 0 \quad (\eta: \text{ a large parameter}) \tag{0.1}$$

using the Borel resummation method ([V]). To employ the exact WKB analysis, it is important to know where the WKB solutions are Borel summable. In [KoS1] and [KoS2], such a problem was studied by considering a formal solution $S(x,\eta) = \eta S_{-1}(x) + S_0(x) + \eta^{-1} S_1(x) + \cdots$ of the Riccati equation

$$\frac{dS}{dx} + S^2 = \eta^2 Q(x) \tag{0.2}$$

Received May 15, 2012, Accepted July 30, 2012.
2010 Mathematics Subject Classification(s): 34M60, 34M30, 34M25, 40G10
Key Words: exact WKB analysis, Borel summability, Stokes curves
The research of the authors has been supported in part by GCOE 'Fostering top leaders in mathematics', Kyoto University and JSPS grants-in-aid No.21740098.
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associated with (0.1). (See also [CDK] and [DLS] for the Borel summability of WKB solutions.)

Following their results we will study in [KKo] the Borel summability of a formal solution

$$\lambda(t,\eta) = \lambda_0(t) + \eta^{-1}\lambda_1(t) + \cdots \tag{0.3}$$

of the second order nonlinear ordinary differential equations of the form

$$\frac{d^2\lambda}{dt^2} = \eta^2 \frac{P(t,\lambda)}{Q(t,\lambda)} + \frac{R_1(t,\lambda,\dot{\lambda})}{R_2(t,\lambda)}, \tag{0.4}$$

where $P(t,\lambda), Q(t,\lambda), R_2(t,\lambda) \in \mathbb{C}[t,\lambda]$, $R_1(t,\lambda,\dot{\lambda}) \in \mathbb{C}[t,\lambda,\dot{\lambda}]$ and $\dot{\lambda} = d\lambda/dt$, and P, Q, R_1, R_2 satisfy some suitable conditions. Typical examples of the above equation (0.4) are Painlevé equations with a large parameter studied in [KT]. Therefore, following the usage in [KT], we call (0.3) a 0-parameter solution of (0.4) in what follows. In our study, a 1-form

$$\omega = \sqrt{\frac{\big(\partial_\lambda P\big)(t,\lambda_0(t))}{Q(t,\lambda_0(t))}}dt \tag{0.5}$$

plays a central role when we determine regions in which a 0-parameter solution $\lambda(t,\eta)$ is Borel summable; Indeed, the most important condition of the Borel summability of $\lambda(t,\eta)$ at $t = t_0$ is that there exists a neighborhood V of t_0 such that all of the integral curves of $\operatorname{Im}\omega = 0$ which pass through V run into singular points of ω of order less than or equal to -1.

This report consists of two sections: In §1, we explain core results of [KKo]. In this report we mainly limit ourselves to the case $R_1 \equiv 0$ in (0.4) to make our arguments clear. In §2, we study the singularity structure of ω for the Painlevé equations, which is necessary to examine the Borel summability of their 0-parameter solutions.

Acknowledgment.

The authors wish to thank Professor T. Kawai, Professor Y. Takei and their students for the valuable discussions with them and their suggestions.

§ 1. 0-parameter solutions and their properties

The main purpose of this section is to give the conditions for the Borel summability of 0-parameter solutions of (0.4). For simplicity, we consider the case where $R_1 \equiv 0$, i.e.,

$$\frac{d^2\lambda}{dt^2} = \eta^2 \frac{P(t,\lambda)}{Q(t,\lambda)}. \tag{1.1}$$

To begin with, let us construct a 0-parameter solution. By multiplying (1.1) by $Q(t,\lambda)$, we obtain

$$\frac{d^2\lambda}{dt^2}Q(t,\lambda) = \eta^2 P(t,\lambda). \tag{1.2}$$

By substituting (0.3) into (1.2) and comparing both sides degree by degree with respect to η, we find that the coefficients of η^2 give

$$P(t,\lambda_0(t)) = 0. \tag{1.3}$$

Therefore we choose $\lambda_0(t)$ as a root of (1.3) and fix it in what follows. Then the lower order terms $\lambda_1, \lambda_2, \cdots$ are recursively and uniquely determined when

(1.4) $\partial_\lambda P(t,\lambda_0(t))$ is not identically 0.

Indeed, by comparing the coefficients of η of (1.2), we find

$$\big(\partial_\lambda P\big)(t,\lambda_0(t))\lambda_1(t) = 0. \tag{1.5}$$

Hence we obtain from (1.4) that

$$\lambda_1(t) \equiv 0. \tag{1.6}$$

Next, by comparing the coefficients of constant terms of η in (1.2), we find

$$\frac{d^2\lambda_0}{dt^2}Q(t,\lambda_0) = \big(\partial_\lambda P\big)(t,\lambda_0)\lambda_2(t). \tag{1.7}$$

Therefore $\lambda_2(t)$ is given by

$$\lambda_2(t) = \frac{Q(t,\lambda_0)}{\big(\partial_\lambda P\big)(t,\lambda_0)}\frac{d^2\lambda_0}{dt^2}. \tag{1.8}$$

Then, proceeding the discussion, we can inductively confirm that, by comparing the coefficients of η^{-n} $(n = 1, 2, \cdots)$, $\lambda_{n+2}(t)$ are uniquely determined by $\lambda_0(t), \cdots, \lambda_{n+1}(t)$ in such a way that

$$\lambda_{2k+1}(t) \equiv 0 \quad (k = 1, 2, \cdots). \tag{1.9}$$

In this way, we can uniquely determine a 0-parameter solution of the form

$$\lambda(t,\eta) = \sum_{k=0}^{\infty} \eta^{-2k}\lambda_{2k}(t) \tag{1.10}$$

for each root $\lambda_0(t)$ of (1.3).

Remark 1.1. We immediately find that, if $\lambda_2 \equiv 0$, then $\lambda_{2k} \equiv 0$ $(k = 2, 3, \cdots)$. Therefore, in what follows, we assume that λ_2 is not identically 0.

Since we cannot expect that the 0-parameter solution (1.10) converges, we consider its Borel sum

$$\lambda_0(t) + \int_0^\infty e^{-\eta y} \tilde{\lambda}_B(t, y) dy \tag{1.11}$$

with respect to η (see, e.g., [B]). Here $\tilde{\lambda}(t, \eta) = \lambda(t, \eta) - \lambda_0(t)$ and

$$\tilde{\lambda}_B(t, y) := \sum_{k=1}^{\infty} y^{2k-1} \frac{\lambda_{2k}(t)}{\Gamma(2k)} \tag{1.12}$$

is the Borel transform of $\tilde{\lambda}(t, \eta)$ with respect to η, and the path of integration in (1.11) is the positive real axis as usual.

Our main theorem (Theorem 1.2 below) claims that, under suitable conditions, the integral in (1.11) is well-defined, i.e., $\tilde{\lambda}(t, \eta)$ is Borel summable. Therefore our main concern is to study the analytic properties of $\tilde{\lambda}_B(t, y)$ in y-plane. To see how our assumptions naturally appear, let us see the outline of our argument before stating our main theorem.

To study the analytic properties of $\tilde{\lambda}_B(t, y)$ we study the Borel transform of (1.2):

$$\begin{aligned} &\Big(Q(t, \lambda_0(t)) \frac{\partial^2}{\partial t^2} - \big(\partial_\lambda P\big)(t, \lambda_0(t)) \frac{\partial^2}{\partial y^2}\Big) \tilde{\lambda}_B(t, y) \\ &= -\frac{d^2 \lambda_0}{dt^2} \sum_{k \geq 1} \frac{1}{k!} \big(\partial_\lambda^k Q(t, \lambda_0)\big) \tilde{\lambda}_B^{*k}(t, y) \\ &\quad - \sum_{k \geq 1} \frac{1}{k!} \big(\partial_\lambda^k Q(t, \lambda_0)\big) \frac{\partial^2 \tilde{\lambda}_B}{\partial t^2} * \tilde{\lambda}_B^{*k}(t, y) \\ &\quad + \sum_{k \geq 2} \frac{1}{k!} \big(\partial_\lambda^k P(t, \lambda_0)\big) \frac{\partial^2}{\partial y^2} \tilde{\lambda}_B^{*k}(t, y), \end{aligned} \tag{1.13}$$

where $\cdot * \cdot$ is the convolution operator defined by

$$\lambda_B * \lambda_B = \int_0^y \lambda_B(t, y - y') \lambda_B(t, y') dy' \tag{1.14}$$

and

$$\lambda_B^{*n} = \overbrace{\lambda_B * \cdots * \lambda_B}^{n}. \tag{1.15}$$

We also impose initial conditions which follows from (1.2):

$$\tilde{\lambda}_B(t, 0) = 0 \ \text{ and } \ \frac{\partial \tilde{\lambda}_B}{\partial y}(t, 0) = \lambda_2(t). \tag{1.16}$$

Remark 1.2. We may regard the left-hand side of (1.13) as the principal part in the following sense: when we define the weight of $\partial/\partial t$ and $\partial/\partial y$ by 1 and that of $\cdot * \cdot$ by -1, then the left-hand side of (1.13) has the weight 2 and the right-hand side has the weight less than 2.

To simplify left-hand side of (1.13) we employ the Liouville transformation, i.e., a coordinate transformation $(t, y) \mapsto (z, y)$ defined by

$$z(t) = \int_{t_0}^{t} \omega, \tag{1.17}$$

where $t_0 \in \mathbb{C}$ is a fixed point and

$$\omega = \sqrt{\frac{\partial_\lambda P(t, \lambda_0(t))}{Q(t, \lambda_0(t))}} dt. \tag{1.18}$$

We assume that ω is holomorphic and does not vanish in the region where we consider. Then, in (z, y)-variable, the left-hand side of (1.13) is rewritten as follows:

$$\big(\partial_\lambda P\big)(t, \lambda_0)\Big(\frac{\partial^2}{\partial z^2} + \Big(\frac{dz}{dt}\Big)^{-2}\frac{d^2 z}{dt^2}\frac{\partial}{\partial z} - \frac{\partial^2}{\partial y^2}\Big)\tilde{\lambda}_B(t(z), y). \tag{1.19}$$

Further, applying a gauge transformation

$$\big(\lambda_2(t)\big)^{-1}\tilde{\lambda}_B(t(z), y) =: \widehat{\lambda}_B(z, y), \tag{1.20}$$

we find that $\widehat{\lambda}_B(z, y)$ satisfies

$$\begin{aligned}
&\Big(\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial y^2}\Big)\widehat{\lambda}_B(z, y) \\
&= -\mathcal{L}\widehat{\lambda}_B(z, y) \\
&\quad - \frac{1}{\big(\partial_\lambda P\big)(t, \lambda_0)}\frac{1}{\lambda_2}\frac{d^2\lambda_0}{dt^2}\sum_{k\geq 1}\frac{\lambda_2^k}{k!}\big(\partial_\lambda^k Q(t, \lambda_0)\big)\widehat{\lambda}_B^{*k}(z, y) \\
&\quad - \frac{1}{\big(\partial_\lambda P\big)(t, \lambda_0)}\Big(\frac{dz}{dt}\Big)^2\sum_{k\geq 1}\frac{\lambda_2^k}{k!}\big(\partial_\lambda^k Q(t, \lambda_0)\big)\Big(\frac{\partial^2\widehat{\lambda}_B}{\partial z^2} + \mathcal{L}\widehat{\lambda}_B(z, y)\Big) * \widehat{\lambda}_B^{*k}(z, y) \\
&\quad + \frac{1}{\big(\partial_\lambda P\big)(t, \lambda_0)}\frac{1}{\lambda_2}\sum_{k\geq 2}\frac{\lambda_2^k}{k!}\big(\partial_\lambda^k P(t, \lambda_0)\big)\frac{\partial^2}{\partial y^2}\widehat{\lambda}_B^{*k}(z, y),
\end{aligned} \tag{1.21}$$

where

$$\mathcal{L} = \Big\{\Big(\frac{dz}{dt}\Big)^{-2}\frac{d^2 z}{dt^2} + 2\lambda_2^{-1}\frac{d\lambda_2}{dz}\Big\}\frac{\partial}{\partial z} + \Big(\frac{dz}{dt}\Big)^{-2}\frac{d^2 z}{dt^2}\lambda_2^{-1}\frac{d\lambda_2}{dz} + \lambda_2^{-1}\frac{d^2\lambda_2}{dz^2}. \tag{1.22}$$

This $\widehat{\lambda}_B(z,y)$ also satisfies the initial conditions

$$\widehat{\lambda}_B(z,0)=0 \quad \text{and} \quad \frac{\partial\widehat{\lambda}_B}{\partial y}(z,0)=1. \tag{1.23}$$

To study a solution of (1.21), we use

Proposition 1.1. *Let $\widehat{\lambda}_B(z,y)$ satisfy*

$$\Big(\frac{\partial^2}{\partial z^2}-\frac{\partial^2}{\partial y^2}\Big)\widehat{\lambda}_B(z,y)=\Phi(z,y) \tag{1.24}$$

and initial conditions

$$\widehat{\lambda}_B(z,0)=0 \quad \textit{and} \quad \frac{\partial\widehat{\lambda}_B}{\partial y}(z,0)=g(z), \tag{1.25}$$

where

$$\begin{aligned}\Phi(z,y)=&\sum_{k=1}^{m} f_k^{(0)}(z)\widehat{\lambda}_B^{*k}(z,y)+\sum_{k=0}^{m} f_k^{(1)}(z)\frac{\partial\widehat{\lambda}_B}{\partial z}*\widehat{\lambda}_B^{*k}(z,y)\\&+\sum_{k=1}^{m} f_k^{(2)}(z)\frac{\partial^2\widehat{\lambda}_B}{\partial z^2}*\widehat{\lambda}_B^{*k}(z,y)+\sum_{k=2}^{m} f_k^{(3)}(z)\frac{\partial^2}{\partial y^2}\widehat{\lambda}_B^{*k}(z,y)\end{aligned} \tag{1.26}$$

and m is a positive integer. Assume that

(1.27) *all $f_k^{(j)}(z)$ and $g(z)$ are holomorphic and bounded on $E_r^1=\{z\in\mathbb{C}: |\operatorname{Im} z|\le r\}$*

for a positive constant r. Then $\widehat{\lambda}_B(z,y)$ is holomorphic on

$$E_{r/2}^2=\{(z,y)\in\mathbb{C}^2 : |\operatorname{Im} z|\le r/2, |\operatorname{Im} y|\le r/2\} \tag{1.28}$$

and satisfies the following estimates for positive constants C_1 and C_2:

$$|\widehat{\lambda}_B(z,y)|\le C_1\exp[C_2|y|]. \tag{1.29}$$

Indeed, we can rewrite the differential equation to the following integral equation:

$$\widehat{\lambda}_B(z,y)=\frac{1}{2}\int_{z-y}^{z+y} g(z')dz'-\frac{1}{2}\int_0^y\int_{z-y+y'}^{z+y-y'}\Phi(z',y')dz'dy', \tag{1.30}$$

and, employing the iteration method, we can show the above proposition. (See [KKo] for the details.)

Now, our task is to examine the conditions for a 0-parameter solution so that we can apply Proposition 1.1 to it. Our first assumption is

(1.31) there exists a neighborhood U of $t = t_0$ and singular points $t = t_\pm$ of ω of order smaller than -1 such that the endpoints of a curve $\Gamma_{\check{t}}$ are t_+ and t_- for each point $\check{t}$ in U,

where $\Gamma_{\check{t}}$ denotes an integral curve of $\operatorname{Im}\omega = 0$ that passes through a point $\check{t}$. This condition guarantees that $z(t)$ extends to $\pm\infty$ along $\Gamma_{\check{t}}$ without encountering any singular point of it. Let $\widehat{U}$ denote $\bigcup_{\check{t}\in U}\Gamma_{\check{t}}$. Then we can take $r > 0$ so that $E^1_r \subset z(\widehat{U})$ and $z(t)$ is locally biholomorphic on $\widehat{U}$.

Our second assumption is

(1.32) $\widehat{U}$ does not contain $t = \infty$ in its interior.

(Cf. Remark 1.3 and Remark 1.7.)

Remark 1.3. When we take $s = 1/t$ as a coordinate variable, (1.1) is rewritten as follows:

$$\frac{d^2\lambda}{ds^2} = \eta^2\frac{P(s^{-1},\lambda)}{s^4Q(s^{-1},\lambda)} - 2\frac{1}{s}\frac{d\lambda}{ds}. \tag{1.1$'$}$$

It does not have the form of (1.1). Therefore, when we restrict our equation to the form (1.1′), we assume that the discussion is made on $\mathbb{C}$. On the other hand, since $P(t,\lambda)$ and $Q(t,\lambda)$ are polynomials, we may regard that (1.1′) has the form of (0.4). Hence, as we will mention in Remark 1.7, when we extend the following discussion to (0.4), we do not have to pay special attention to the point $\infty \in \mathbb{P}^1$.

By taking the form (1.8) of λ_2 into account, it suffices to confirm the holomorphy and the boundedness of the following terms on $\widehat{U}$:

$$\frac{\big(\partial^k_\lambda P\big)(t,\lambda_0)\lambda_2^{k-1}}{\big(\partial_\lambda P\big)(t,\lambda_0)} \text{ and } \frac{\big(\partial^k_\lambda Q\big)(t,\lambda_0)\lambda_2^{k}}{Q(t,\lambda_0)} \quad (k \geq 1). \tag{1.33}$$

Indeed, under the assumptions (1.31) and (1.32) (and modifying the gauge transformation (1.20) if necessary), we may assume that the coefficients of $\mathcal{L}$ are holomorphic and bounded on $\widehat{U}$.

To guarantee the holomorphy of all terms in (1.33) on $\widehat{U}$, we impose the third assumption:

(1.34) the discriminant $\operatorname{Disc}_P(t)$ of $P(t,\lambda)$ and the resultant $\operatorname{Res}_{(P,Q)}(t)$ of $P(t,\lambda)$ and $Q(t,\lambda)$ do not vanish on $\widehat{U}$.

Note that the condition (1.34) is violated at finitely many points on $\widehat{U}$ if $\mathrm{Disc}_P(t)$ and $\mathrm{Res}_{(P,Q)}(t)$ are not identically equal to 0. However, if the terms (1.33) are holomorphic there, then Theorem 1.2 below holds even though (1.34) is violated.

To give the last assumption to ensure the boundedness of the terms (1.33), we prepare some notations. Under the assumption (1.34), by shrinking U if necessary, it suffices to show the boundedness of them at the singular points $t_\pm$. For simplicity, we assume that $t_+ \in \mathbb{C}$ and $\lambda_0(t)$ behaves as

$$\lambda_0(t) = \beta_+(t-t_+)^{\alpha_+} + o\big((t-t_+)^{\alpha_+}\big) \tag{1.35}$$

with $\alpha_+ \in \mathbb{Q}$ and $\beta_+ \neq 0$ when t tends to t_+. Let $F(t,\lambda) = F_n(t)\lambda^n + \cdots + F_0(t) \in \mathbb{C}[t,\lambda]$ be a polynomial and assume that $F_k(t)$ $(k = 0, 1, \cdots, n)$ behave as

$$F_k(t) = F_k^{(0)}(t-t_+)^{\nu_k} + o\big((t-t_+)^{\nu_k}\big) \tag{1.36}$$

with $F_k^{(0)} \neq 0$ and $\nu_k \in \mathbb{Z}_{\geq 0} = \{0, 1, 2, \cdots\}$. (When $F_k \equiv 0$, we consider that $\nu_k = +\infty$.) Then, we define an index $\mathrm{ind}_{\lambda_0}^{t_+}(F)$ (relevant to $\lambda_0(t)$) by

$$\mathrm{ind}_{\lambda_0}^{t_+}(F) = \min_{0 \leq k \leq n} \big\{k\alpha_+ + \nu_k\big\} \tag{1.37}$$

and a polynomial $D_F^{t_+}(\lambda)$ by

$$D_F^{t_+}(\lambda) = \sum_k F_k^{(0)} \lambda^k, \tag{1.38}$$

where the sum is taken over k that give the minimum in (1.37), i.e., $k\alpha_+ + \nu_k = \mathrm{ind}_{\lambda_0}^{t_+}(F)$. In the same way, we can define an index $\mathrm{ind}_{\lambda_0}^{t_-}(F)$ and a polynomial $D_F^{t_-}(\lambda)$ at $t = t_-$ for

$$\lambda_0(t) = \beta_-(t-t_-)^{\alpha_-} + o\big((t-t_-)^{\alpha_-}\big). \tag{1.35$'$}$$

We note that the constant β_+ in (1.35) (resp., β_- in (1.35′)) is given by one of the roots of $D_P^{t_+}(\lambda) = 0$ (resp., $D_P^{t_-}(\lambda) = 0$).

Our last assumption is

$$D_{\partial_\lambda P}^{t_\pm}(\beta_\pm) \neq 0 \text{ and } D_Q^{t_\pm}(\beta_\pm) \neq 0 \text{ hold.} \tag{1.39}$$

This condition (1.39) entails that the order of $\partial_\lambda P\big(t, \lambda_0(t)\big)$ (resp., $Q\big(t, \lambda_0(t)\big)$) at $t = t_\pm$ coincides with the index $\mathrm{ind}_{\lambda_0}^{t_\pm}(\partial_\lambda P)$ (resp., $\mathrm{ind}_{\lambda_0}^{t_\pm}(Q)$). We also note that the first condition $D_{\partial_\lambda P}^{t_\pm}(\beta_\pm) \neq 0$ is equivalent to that the leading term $\beta_\pm(t-t_\pm)^{\alpha_\pm}$ of $\lambda_0(t)$ at $t = t_\pm$ is different from that of the other roots of $P(t,\lambda) = 0$. In this sense, if (1.39) holds at $t = t_\pm$, we call $t = t_\pm$ a nondegenerate singular point. Further, we can derive the boundedness of the terms (1.33) at $t = t_\pm$ from (1.31) and (1.39).

Remark 1.4. When $t_+ = \infty$, by taking $s = t^{-1}$ as a coordinate variable, we can define the index $\mathrm{ind}^{t_+}_{\lambda_0}(F)$ and the polynomial $D^{t_+}_F(\lambda)$ in the same manner as above.

Remark 1.5. If the order of the singular points $t = t_\pm$ of ω is strictly less than -1, we can modify the condition (1.39). See [KKo] for details.

Now we state our main theorem:

Theorem 1.2. *Let $\lambda_0(t)$ be a root of* (1.3) *and assume that* (1.31), (1.32), (1.34) *and* (1.39) *hold. Then the* 0*-parameter solution $\lambda(t,\eta)$ of* (1.1) *that has $\lambda_0(t)$ as its initial part is Borel summable on $\widehat{U}$. More precisely, the Borel transform $\tilde{\lambda}_B(t,y)$ of $\lambda(t,\eta) - \lambda_0(t)$ satisfies the following estimates on $\widehat{U} \times \{y \in \mathbb{C} : |\mathrm{Im} y| \le r\}$ for positive constants r, C_1 and C_2:*

$$|\tilde{\lambda}_B(t,y)| \le C_1\bigl(|\lambda_2(t)| + 1\bigr)\exp[C_2|y|]. \tag{1.40}$$

Remark 1.6. We give a remark here on our results of the Borel summability of 0-parameter solutions in the case when $R_1 \not\equiv 0$ in (0.4). In this case, $\lambda_2(t)$ is given by

$$\lambda_2(t) = \frac{Q(t,\lambda_0)}{\bigl(\partial_\lambda P\bigr)(t,\lambda_0)}\Bigl(\frac{d^2\lambda_0}{dt^2} - \frac{R_1(t,\lambda_0,\dot{\lambda}_0)}{R_2(t,\lambda_0)}\Bigr). \tag{1.8$'$}$$

In addition to the assumptions of Theorem 1.2, if the following terms (1.41) and (1.42) are holomorphic and bounded on $\widehat{U}$, we obtain the same results as Theorem 1.2:

$$\frac{\bigl(\partial^k_\lambda R_2\bigr)(t,\lambda_0)\lambda_2^k}{R_2(t,\lambda_0)} \quad\text{and}\quad \frac{Q(t,\lambda_0)}{\bigl(\partial_\lambda P\bigr)(t,\lambda_0)}\frac{d^2\lambda_0}{dt^2}\frac{\bigl(\partial^k_\lambda R_2\bigr)(t,\lambda_0)\lambda_2^{k-1}}{R_2(t,\lambda_0)} \tag{1.41}$$

for $k \ge 0$ and

$$\frac{Q(t,\lambda_0)}{\bigl(\partial_\lambda P\bigr)(t,\lambda_0)}\frac{\bigl(\partial^{k_1}_\lambda \partial^{k_2}_{\dot{\lambda}} R_1\bigr)(t,\lambda_0,\dot{\lambda}_0)\lambda_2^{k_1-1}\dot{\lambda}_2^{k_2}}{R_2(t,\lambda_0)} \tag{1.42}$$

for $\{k_1, k_2 \ge 0\} \setminus \{k_1 = k_2 = 0\}$. (See [KKo] for details.)

Remark 1.7. In parallel with Remark 1.3, when we take $s = 1/t$ as a coordinate variable, (0.4) is rewritten as follows:

$$\frac{d^2\lambda}{ds^2} = \eta^2\frac{P(s^{-1},\lambda)}{s^4 Q(s^{-1},\lambda)} - 2\frac{1}{s}\frac{d\lambda}{ds} + \frac{R_1(s^{-1},\lambda,-s^2 d\lambda/ds)}{s^4 R_2(s^{-1},\lambda)}. \tag{0.4$'$}$$

We may regard that (0.4′) has the form of (0.4). Therefore, when the terms corresponding to (1.33), (1.41) and (1.42) for (0.4′) are holomorphic and bounded at $s = 0$, we can extend Theorem 1.2 to the case where $\widehat{U}$ contains $t = \infty$ in its interior.

§ 2. Singularity structure of ω for the Painlevé equations

In the previous section, we gave a condition which guarantees the Borel summability of a 0-parameter solution of (0.4) (cf. Theorem 1.2 and Remark 1.6). In view of the assumption (1.31), we lead to the notion of Stokes geometry such as *turning points* and *Stokes curves*.

Definition 2.1. We call $t = t_0$ a turning point of a 0-parameter solution of (0.4) when the order of a 1-form ω defined by (1.18) at $t = t_0$ is greater than -1, i.e., ω behaves as

$$\omega = \begin{cases} \big(C_0(t-t_0)^\gamma + o((t-t_0)^\gamma)\big)dt & \text{at } t = t_0 \in \mathbb{C} \\ \big(C_0 t^{-\gamma-2} + o(t^{-\gamma-2})\big)dt & \text{at } t = \infty \end{cases} \tag{2.1}$$

with $C_0 \neq 0$ and $\gamma > -1$. Especially, when

$$\partial_\lambda P(t_0, \beta_0) = 0, \tag{2.2}$$

$$\partial_\lambda^2 P(t_0, \beta_0) \neq 0, \tag{2.3}$$

$$\partial_t P(t_0, \beta_0) \neq 0, \tag{2.4}$$

$$Q(t_0, \beta_0) \neq 0 \tag{2.5}$$

hold for a root β_0 of $P(t_0, \beta_0) = 0$, we call $t = t_0$ a simple turning point of the corresponding 0-parameter solution. Further, the integral curves of Im $\omega = 0$ that emanate from turning points are called Stokes curves.

Remark 2.1. In s-variable with $s = t^{-1}$, the behavior (2.1) of ω at $t = \infty$ is rewritten as follows:

$$\omega = \big(-C_0 s^\gamma + o(s^\gamma)\big)ds \quad \text{at } s = 0. \tag{2.6}$$

Remark 2.2. Our definition of turning points and Stokes curves coincides with that of [KT] for the Painlevé equations. In general, turning points of the Painlevé equations except for $t = 0$ of $\mathrm{P_{III}}$, $t = 0$ of $\mathrm{P_V}$ and $t = 0, 1, \infty$ of $\mathrm{P_{VI}}$ are simple turning points. However, when parameters of the Painlevé equations satisfy some relations, these simple turning points become "double turning points". See [T2, Proposition 2.4] for precise conditions.

It is clear from the definition that the assumption (1.31) does not hold at a point on Stokes curves. Therefore, Stokes curves play such a role that they become boundaries of the regions in which the assumption (1.31) is expected to hold (these regions are called *Stokes regions*). Integral curves of $\mathrm{Im}\,\omega = 0$ are usually studied by numerical computation and we do not consider it here further.

Specifying the location of turning points and verifying the other assumptions of Theorem 1.2 can be done in an algebraic manner. An important step towards them is to study the singularity structure of a 1-form ω defined by (0.5). To illustrate the analysis of the singularity structure of ω, we study the Painlevé equations with a large parameter introduced by [KT] in the remaining part of this section. As we will see below, singular points of ω for the Painlevé equations of order smaller than -1 satisfy (1.39) when parameters in the equations are taken generically. Further, we will remark on the properties of their turning points. (See also [T2].)

Example 2.2 (the first Painlevé equation). We consider the first Painlevé equation:

$$(\mathrm{P_I}) \qquad \frac{d^2\lambda}{dt^2} = \eta^2(6\lambda^2 + t).$$

The 1-form ω_{I} defined by (1.18) for $(\mathrm{P_I})$ is given by

$$\omega_{\mathrm{I}} = \sqrt{12\lambda(t)}dt, \tag{2.7}$$

and the roots of $P_{\mathrm{I}}(t,\lambda) = 6\lambda^2 + t$ are $\lambda^{(l)}(t) = (-1)^l\sqrt{-1/6}\ t^{1/2}$ $(l = 1, 2)$. Since the discriminant $\mathrm{Disc_I}(t)$ of $P_{\mathrm{I}}(t,\lambda)$ is

$$\mathrm{Disc_I}(t) = 144t, \tag{2.8}$$

we find that ω_{I} is holomorphic and does not vanish except for $t = 0$ and ∞.

First, we focus on the behavior of ω_{I} at $t = 0$. Obviously, $t = 0$ is a simple turning point of $\lambda^{(l)}(t)$ $(l = 1, 2)$. Then, the index (1.37) for $\partial_\lambda P_{\mathrm{I}}$ relevant to these $\lambda^{(l)}(t)$ at $t = 0$ and the polynomial (1.38) are respectively given by

$$\mathrm{ind}^0_{\lambda^{(l)}}(\partial_\lambda P_{\mathrm{I}}) = \frac{1}{2} \tag{2.9}$$

and

$$D^{0,(l)}_{P_{\mathrm{I}}}(\beta) = 6\beta^2 + 1 = 0 \quad (l = 1, 2). \tag{2.10}$$

Since $D^{0,(l)}_{P_{\mathrm{I}}}(\beta)$ has no multiple root, $D^{0,(l)}_{\partial_\lambda P_{\mathrm{I}}}(\pm\sqrt{-1/6}) \neq 0$, and hence, the order $\gamma^{(l)}_0$ of ω_{I} for $\lambda^{(l)}$ $(l = 1, 2)$ at $t = 0$ is given by

$$\gamma^{(l)}_0 = \frac{1}{2}\mathrm{ind}^0_{\lambda^{(l)}}(\partial_\lambda P_{\mathrm{I}}) = \frac{1}{4}. \tag{2.11}$$

Second, let us consider the behavior of ω_{I} at $t = \infty$. Since $\lambda^{(l)}(s) = (-1)^l\sqrt{-1/6}\ s^{-1/2}$ with $s = t^{-1}$, the index $\mathrm{ind}^\infty_{\lambda^{(l)}}(\partial_\lambda P_{\mathrm{I}})$ at $t = \infty$ is given by

$$\mathrm{ind}^\infty_{\lambda^{(l)}}(\partial_\lambda P_{\mathrm{I}}) = -\frac{1}{2} \quad (l = 1, 2). \tag{2.12}$$

Since ω_{I} is represented as

$$\omega_{\mathrm{I}} = -\sqrt{12\lambda(s)}s^{-2}ds \tag{2.13}$$

in s-variable, we find the order $\gamma_{\infty}^{(l)}$ of ω_{I} for $\lambda^{(l)}$ $(l = 1, 2)$ at $t = \infty$ is given by

$$\gamma_{\infty}^{(l)} = \frac{1}{2}\mathrm{ind}_{\lambda^{(l)}}^{\infty}(\partial_{\lambda}P_{\mathrm{I}}) - 2 = -\frac{9}{4} \quad (l = 1, 2). \tag{2.14}$$

	$l = 1$	2
$\alpha^{(l)}$	$-1/2$	$-1/2$
$\beta^{(l)}$	$-\sqrt{-1/6}$	$\sqrt{-1/6}$
$\gamma_{\infty}^{(l)}$	$-9/4$	$-9/4$

Table 1. The leading term $\beta^{(l)}t^{-\alpha^{(l)}}$ of $\lambda^{(l)}(t)$ and the order $\gamma_{\infty}^{(l)}$ of ω_{I} at $t = \infty$.

Remark 2.3. We find that the above discussion indicates the Borel summability of 0-parameter solutions of $(\mathrm{P_I})$ except on the Stokes curves emanating from $t = 0$, and hence, we can take the Borel sum of them. On the other hand, as is discussed in [T1], $t = 0$ actually behaves as a turning point and 0-parameter solutions of $(\mathrm{P_I})$ are not Borel summable on these Stokes curves. Hence, when we consider the analytic continuation of the Borel sum of a 0-parameter solution across a Stokes curve, Stokes phenomena occur, and a so-called "1-parameter solution" appears. We can also show the generalized Borel summability of it. See [K] for the details. Here, we mention that a similar kind of formal solutions called "transseries solutions" are studied in [C], which are the formal exponential series solutions at an irregular singular point of nonlinear ordinary differential equations. Further, the generalized Borel summability of transseries solutions is discussed there.

Example 2.3 (the second Painlevé equation)**.** Next, we consider the second Painlevé equation

$$\frac{d^2\lambda}{dt^2} = \eta^2(2\lambda^3 + t\lambda + c). \tag{$\mathrm{P_{II}}$}$$

We discuss on the singular points of

$$\omega_{\mathrm{II}} = \sqrt{6\lambda^2(t) + t}\, dt \tag{2.15}$$

with a root $\lambda(t)$ of $P_{\mathrm{II}}(t, \lambda) = 2\lambda^3 + t\lambda + c$. The discriminant $\mathrm{Disc}_{\mathrm{II}}(t)$ of $P_{\mathrm{II}}(t, \lambda)$ is given by

$$\mathrm{Disc}_{\mathrm{II}}(t) = 8(2t^3 + 27c^2). \tag{2.16}$$

Therefore, when $c \neq 0$, $\mathrm{Disc}_{\mathrm{II}}(t) = 0$ has three distinct roots, i.e., $t = t_j := 3\theta^j (c^2/2)^{1/3}$ $(j = 0, 1, 2)$ with $\theta = \exp[2\pi\sqrt{-1}/3]$. In what follows, we assume that $c \neq 0$. We examine the behavior of the roots of $P_{\mathrm{II}}(t, \lambda) = 0$ and ω_{II} for the roots at $t = t_j$. We first note that three roots of $P_{\mathrm{II}}(t, \lambda) = 0$ behave as $\lambda_j^{(l)}(t) = \beta_j^{(l)} + o(1)$ $(l = 1, 2, 3)$ at $t = t_j$, where $\{\beta_j^{(l)}\}_{l=1}^3$ are the roots of

$$D_{P_{\mathrm{II}}}^{t_j,(l)}(\beta) = 2\beta^3 + t_j\beta + c \quad (l = 1, 2, 3). \tag{2.17}$$

Since $\mathrm{Disc}_{\mathrm{II}}(t_j) = 0$, two of them coincide. Let $\beta_j^{(1)} = \beta_j^{(2)}$ be such roots. Then, we immediately find that $t = t_j$ is a simple turning point of $\lambda_j^{(l)}(t)$ $(l = 1, 2)$. Since $\partial_\beta D_{P_{\mathrm{II}}}^{t_j,(1)}(\beta_j^{(1)}) = 6(\beta_j^{(1)})^2 + t_j = 0$, the Newton polygon of $\tilde{P}_{\mathrm{II}}(t, \tilde{\lambda}) := P_{\mathrm{II}}(t, \beta_j^{(1)} + \tilde{\lambda}) = 2\tilde{\lambda}^3 + 6\beta_j^{(1)}\tilde{\lambda}^2 + (t - t_j)\tilde{\lambda} + \beta_j^{(1)}(t - t_j)$ at $t = t_j$ is given by Figure 1.

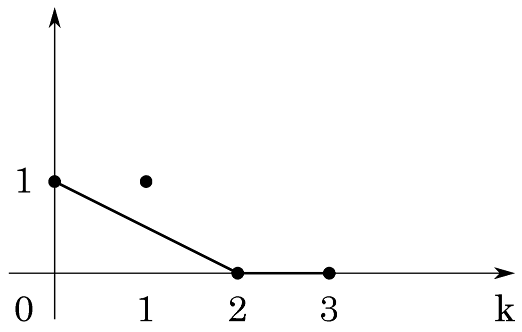

Figure 1. Newton Polygon of $\tilde{P}_{\mathrm{II}}(t, \tilde{\lambda})$ at $t = t_j$.

Therefore, two of the roots $\tilde{\lambda}_j^{(1)}(t)$ and $\tilde{\lambda}_j^{(2)}(t)$ of $\tilde{P}_{\mathrm{II}}(t, \tilde{\lambda})$ behave as

$$\tilde{\lambda}_j^{(l)}(t) = \tilde{\beta}_j^{(l)}(t - t_j)^{1/2} + o((t - t_j)^{1/2}) \quad (l = 1, 2), \tag{2.18}$$

where $\tilde{\beta}_j^{(l)}$ are the two distinct roots of

$$D_{\tilde{P}_{\mathrm{II}}}^{t_j,(l)}(\tilde{\beta}) = 6\tilde{\beta}^2 + 1 = 0 \quad (l = 1, 2), \tag{2.19}$$

and hence,

$$\mathrm{ind}_{\tilde{\lambda}_j^{(l)}}^{t_j}(\partial_{\tilde{\lambda}}\tilde{P}_{\mathrm{II}}) = \min\left\{1, \frac{1}{2}, 2 \cdot \frac{1}{2}\right\} = \frac{1}{2} \quad (l = 1, 2). \tag{2.20}$$

Therefore, the order $\gamma_j^{(l)}$ of ω_{II} for $\lambda_j^{(l)}(t)$ $(l = 1, 2)$ at $t = t_j$ is given by

$$\gamma_j^{(l)} = \frac{1}{2}\mathrm{ind}_{\tilde{\lambda}_j^{(l)}}^{t_j}(\partial_{\tilde{\lambda}}\tilde{P}_{\mathrm{II}}) = \frac{1}{4} \quad (l = 1, 2). \tag{2.21}$$

Remark 2.4. Since $D^{t_j,(1)}_{\partial_\lambda P_{\rm II}}(\beta^{(1)}_j) = \partial_\beta D^{t_j,(1)}_{P_{\rm II}}(\beta^{(1)}_j) = 0$, we find that $t = t_j$ $(j = 1, 2, 3)$ are degenerate singular points, and hence,

$$\gamma^{(l)}_j > \frac{1}{2}\mathrm{ind}^{t_j}_{\lambda^{(1)}_j}(\partial_\lambda P_{\rm II}) = 0. \tag{2.22}$$

However, by considering $\tilde{\lambda}^{(l)}_j(t)$ $(l = 1, 2)$ and $\tilde{P}_{\rm II}(t, \tilde{\lambda})$ instead of $\lambda^{(l)}_j(t)$ $(l = 1, 2)$ and $P_{\rm II}(t, \lambda)$, we can reduce these singular points to nondegenerate ones as above. Then, we can measure the order $\gamma^{(l)}_j$ by the index $\mathrm{ind}^{t_j}_{\tilde{\lambda}^{(l)}_j}(\partial_{\tilde{\lambda}} \tilde{P}_{\rm II})$ as (2.21).

On the other hand, since $\partial_\beta D^{t_j,(3)}_{P_{\rm II}}(\beta^{(3)}_j) \neq 0$, we find that the other root $\lambda^{(3)}_j(t)$ is holomorphic at $t = t_j$, and hence, $\omega_{\rm II}$ for $\lambda^{(3)}_j(t)$ is also holomorphic and does not vanish there.

Now, let us focus on the singular points of $\omega_{\rm II}$ at $t = \infty$. We find that three roots of $\tilde{P}_{\rm II}(t, \tilde{\lambda})$ behave as Table 2 below.

	$l = 1$	2	3
$\alpha^{(l)}_\infty$	1	$-1/2$	$-1/2$
$\beta^{(l)}_\infty$	$-c$	$\sqrt{-1/2}$	$-\sqrt{-1/2}$
$\gamma^{(l)}_\infty$	$-5/2$	$-5/2$	$-5/2$

Table 2. The leading term $\beta^{(l)}_\infty t^{-\alpha^{(l)}_\infty}$ of $\lambda^{(l)}_\infty(t)$ and the order $\gamma^{(l)}_\infty$ of $\omega_{\rm II}$ at $t = \infty$.

Table 2 indicates that $D^{\infty,(l)}_{P_{\rm II}}(\beta)$ $(l = 1, 2, 3)$ has no multiple root. Hence, the order $\mathrm{ord}^\infty_{\lambda^{(l)}_\infty}(\partial_\lambda P_{\rm II})$ of $\partial_\lambda P_{\rm II}(t, \lambda^{(l)}_\infty(t))$ at $t = \infty$ is given by

$$\mathrm{ord}^\infty_{\lambda^{(l)}_\infty}(\partial_\lambda P_{\rm II}) = \mathrm{ind}^\infty_{\tilde{\lambda}^{(l)}_\infty}(\partial_\lambda P_{\rm II}) = \min\{2\alpha^{(l)}_\infty, -1\}. \tag{2.23}$$

Therefore, we find that the order $\gamma^{(1)}_\infty$ of $\omega_{\rm II}$ for $\lambda^{(1)}_\infty(t)$ at $t = \infty$ is given by

$$\gamma^{(1)}_\infty = \frac{1}{2}\mathrm{ind}^\infty_{\lambda^{(1)}_\infty}(\partial_\lambda P_{\rm II}) - 2 = -\frac{5}{2}. \tag{2.24}$$

On the other hand, the order $\gamma^{(l)}_\infty$ of $\omega_{\rm II}$ for the other roots is given by

$$\gamma^{(l)}_\infty = \frac{1}{2}\mathrm{ind}^\infty_{\lambda^{(l)}_\infty}(\partial_\lambda P_{\rm II}) - 2 = -\frac{5}{2} \quad (l = 2, 3). \tag{2.25}$$

Example 2.4 (the third Painlevé equation)**.** Let us consider the third Painlevé equation

$$\frac{d^2\lambda}{dt^2} = \frac{1}{\lambda}\Big(\frac{d\lambda}{dt}\Big)^2 - \frac{1}{t}\frac{d\lambda}{dt} + 8\eta^2\Big[2c_\infty\lambda^3 + \frac{c'_\infty}{t}\lambda^2 - \frac{c'_0}{t} - 2\frac{c_0}{\lambda}\Big]. \tag{$\mathrm{P_{III}}$}$$

In what follows, we assume that $c_\infty, c'_\infty, c'_0$ and c_0 are not equal to 0. Let ω_{III} be a 1-form defined by

$$\omega_{\mathrm{III}} = \sqrt{\frac{8\big(8c_\infty t\lambda^3(t) + 3c'_\infty\lambda^2(t) - c'_0\big)}{t\lambda(t)}}\,dt \tag{2.26}$$

with a root $\lambda(t)$ of $P_{\mathrm{III}}(t,\lambda) = 8\big(2c_\infty t\lambda^4 + c'_\infty\lambda^3 - c'_0\lambda - 2c_0 t\big)$. Since the discriminant $\mathrm{Disc}_{\mathrm{III}}(t)$ of $P_{\mathrm{III}}(t,\lambda)$ and the resultant $\mathrm{Res}_{\mathrm{III}}(t)$ of $P_{\mathrm{III}}(t,\lambda)$ and $Q_{\mathrm{III}}(t,\lambda) = t\lambda$ are respectively given by

(2.27)
$$\begin{aligned}\mathrm{Disc}_{\mathrm{III}}(t) = N_1 c_\infty t\big((c'_\infty)^3(c'_0)^3 - (27c_\infty^2(c'_0)^4 + 27(c'_\infty)^4c_0^2 - 6c_\infty(c'_\infty)^2(c'_0)^2c_0)t^2 \\ + 768c_\infty^2c'_\infty c'_0c_0^2t^4 - 4096c_\infty^3c_0^3t^6\big)\end{aligned}$$

and

$$\mathrm{Res}_{\mathrm{III}}(t) = N_2c_0t^5 \tag{2.28}$$

with some integers N_1 and N_2, ω_{III} may have six singular points $\{t_j\}_{j=1}^6$ except for $t=0$ and ∞ in general. Indeed, the discriminant of $\mathrm{Disc}_{\mathrm{III}}(t)$ is written as

$$Nc_\infty^{25}(c'_\infty)^9(c'_0)^9c_0^{12}\big(c_\infty(c'_0)^2 - (c'_\infty)^2c_0\big)^8\big(c_\infty(c'_0)^2 + (c'_\infty)^2c_0\big)^4 \tag{2.29}$$

with some integer N, and hence, $\mathrm{Disc}_{\mathrm{III}}(t)$ has seven distinct roots when it does not vanish. Since $\mathrm{Disc}_{\mathrm{III}}(t_j) = 0$ $(j = 1,2,\cdots,6)$, two of the roots $\beta_j^{(1)}$ and $\beta_j^{(2)}$ of $P_{\mathrm{III}}(t_j,\beta) = 0$ coincide. Then, we find that $t = t_j$ are simple turning points and that two of the roots $\tilde{\lambda}_j^{(1)}(t)$ and $\tilde{\lambda}_j^{(2)}(t)$ of $\tilde{P}_{\mathrm{III}}(t,\tilde{\lambda}) := P_{\mathrm{III}}(t,\beta_j^{(1)} + \tilde{\lambda})$ behave as

$$\tilde{\lambda}_j^{(l)}(t) = \tilde{\beta}_j^{(l)}(t-t_j)^{1/2} + o((t-t_j)^{1/2}), \tag{2.30}$$

where $\tilde{\beta}_j^{(l)}$ are the two distinct roots of

$$D_{\tilde{P}_{\mathrm{III}}}^{t_j,(l)}(\tilde{\beta}) = \frac{1}{2}\partial_\lambda^2 P_{\mathrm{III}}(t_j,\beta_j^{(1)})\tilde{\beta}^2 + \partial_t P_{\mathrm{III}}(t_j,\beta_j^{(1)}) = 0 \quad (l = 1,2). \tag{2.31}$$

Indeed, $\partial_\lambda^2 P_{\mathrm{III}}(t_j,\beta_j^{(1)})$ and $\partial_t P_{\mathrm{III}}(t_j,\beta_j^{(1)})$ do not vanish when $c_\infty(c'_0)^2 - (c'_\infty)^2c_0 \neq 0$, and hence, we can read the behavior of $\tilde{\lambda}_j^{(l)}(t)$ $(l = 1,2)$ at $t = t_j$ from Figure 2.

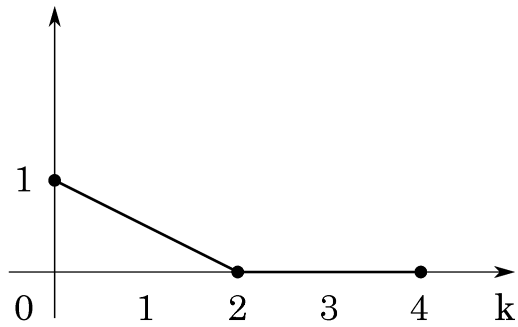

Figure 2. Newton Polygon of $\tilde{P}_{\rm III}(t,\tilde{\lambda})$ at $t=t_j$.

Since $\mathrm{Res}_{\rm III}(t_j)\neq 0$ $(j=1,2,\cdots,6)$, the order of $Q_{\rm III}(t,\lambda_j^{(l)}(t))$ $(l=1,2,3,4)$ at $t=t_j$ coincide with $\mathrm{ind}^{t_j}_{\lambda_j^{(l)}}(Q_{\rm III})=0$. Therefore, the order $\gamma_j^{(l)}$ of $\omega_{\rm III}$ for $\lambda_j^{(l)}(t)$ $(l=1,2)$ at $t=t_j$ is given by

$$\gamma_j^{(l)}=\frac{1}{2}\big(\mathrm{ind}^{t_j}_{\tilde{\lambda}_j^{(l)}}(\partial_{\tilde{\lambda}}\tilde{P}_{\rm III})-\mathrm{ind}^{t_j}_{\lambda_j^{(l)}}(Q_{\rm III})\big)=\frac{1}{4}\quad(l=1,2). \tag{2.32}$$

On the other hand, we immediately see that the other roots $\lambda_j^{(l)}(t)$ $(l=3,4)$ are holomorphic at $t=t_j$ and $\omega_{\rm III}$ for these roots is also holomorphic and does not vanish there when $c_\infty(c_0')^2+(c_\infty')^2c_0\neq 0$. Otherwise, one more multiple root $\beta_j^{(3)}(=\beta_j^{(4)})$ appears. However, applying the same reasoning as above to the pair $\lambda_j^{(3)}(t)$ and $\lambda_j^{(4)}(t)$, we find that $\tilde{\lambda}_j^{(l)}(t)=\lambda_j^{(l)}(t)-\beta_j^{(l)}$ behave as (2.30) and $\gamma_j^{(l)}=1/4$ for $l=3$ and 4.

Now, we focus on the behavior of $\omega_{\rm III}$ at $t=0$. We find four roots of $P_{\rm III}(t,\lambda)$ behave at $t=0$ as Table 3 below.

	$l=1$	2	3	4
$\alpha_0^{(l)}$	1	0	0	-1
$\beta_0^{(l)}$	$-2c_0/c_0'$	$\sqrt{c_0'/c_\infty'}$	$-\sqrt{c_0'/c_\infty'}$	$-c_\infty'/2c_\infty$
$\gamma_0^{(l)}$	-1	$-1/2$	$-1/2$	-1

Table 3. The leading term $\beta_0^{(l)}t^{\alpha_0^{(l)}}$ of $\lambda_0^{(l)}(t)$ and the order $\gamma_0^{(l)}$ of $\omega_{\rm III}$ at $t=0$.

Since Table 3 indicates that the leading terms of these four roots are different, we immediately see that the orders of $\partial_\lambda P_{\rm III}(t,\lambda_0^{(l)}(t))$ and $Q_{\rm III}(t,\lambda_0^{(l)}(t))$ are simply given by

$$\operatorname{ord}^0_{\lambda^{(l)}_0}(\partial_\lambda P_{\mathrm{III}}) = \operatorname{ind}^0_{\lambda^{(l)}_0}(\partial_\lambda P_{\mathrm{III}}) = \min\{1+3\alpha^{(l)}_0, 2\alpha^{(l)}_0, 0\} \tag{2.33}$$

and

$$\operatorname{ord}^0_{\lambda^{(l)}_0}(Q_{\mathrm{III}}) = \operatorname{ind}^0_{\lambda^{(l)}_0}(Q_{\mathrm{III}}) = 1+\alpha^{(l)}_0. \tag{2.34}$$

Hence, the order $\gamma^{(l)}_0$ of ω_{III} for $\lambda^{(l)}_j(t)$ $(l=1,2,3,4)$ at $t=0$ is given by

$$\gamma^{(l)}_0 = \frac{1}{2}\big(\operatorname{ind}^0_{\lambda^{(l)}_0}(\partial_\lambda P_{\mathrm{III}}) - \operatorname{ind}^0_{\lambda^{(l)}_0}(Q_{\mathrm{III}})\big). \tag{2.35}$$

Finally, we study the behavior of ω_{III} at $t=\infty$. The behavior of the roots of $P_{\mathrm{III}}(t,\lambda)$ at $t=\infty$ is given in Table 4 below.

	$l=1$	2	3	4
$\alpha^{(l)}_\infty$	0	0	0	0
$\beta^{(l)}_\infty$	$\sqrt[4]{c_0/c_\infty}$	$\sqrt{-1}\sqrt[4]{c_0/c_\infty}$	$-\sqrt[4]{c_0/c_\infty}$	$-\sqrt{-1}\sqrt[4]{c_0/c_\infty}$
$\gamma^{(l)}_\infty$	-2	-2	-2	-2

Table 4. The leading term $\beta^{(l)}_\infty t^{-\alpha^{(l)}_\infty}$ of $\lambda^{(l)}_\infty(t)$ and the order $\gamma^{(l)}_\infty$ of ω_{III} at $t=\infty$.

Since the leading terms of these four roots are different, we find

$$\operatorname{ord}^\infty_{\lambda^{(l)}_\infty}(\partial_\lambda P_{\mathrm{III}}) = \operatorname{ind}^\infty_{\lambda^{(l)}_\infty}(\partial_\lambda P_{\mathrm{III}}) = \min\{-1+3\alpha^{(l)}_\infty, 2\alpha^{(l)}_\infty, 0\}, \tag{2.36}$$

$$\operatorname{ord}^\infty_{\lambda^{(l)}_\infty}(Q_{\mathrm{III}}) = \operatorname{ind}^\infty_{\lambda^{(l)}_\infty}(Q_{\mathrm{III}}) = -1+\alpha^{(l)}_\infty \tag{2.37}$$

and

$$\gamma^{(l)}_\infty = \frac{1}{2}\big(\operatorname{ind}^\infty_{\lambda^{(l)}_\infty}(\partial_\lambda P_{\mathrm{III}}) - \operatorname{ind}^\infty_{\lambda^{(l)}_\infty}(Q_{\mathrm{III}})\big) - 2, \tag{2.38}$$

where $\gamma^{(l)}_\infty$ $(l=1,2,3,4)$ are the order of ω_{III} for $\lambda^{(l)}_j(t)$ at $t=\infty$.

Example 2.5 (the fourth Painlevé equation). We consider the fourth Painlevé equation

$$\frac{d^2\lambda}{dt^2} = \frac{1}{2\lambda}\Big(\frac{d\lambda}{dt}\Big)^2 - \frac{2}{\lambda} + 2\eta^2\Big[\frac{3}{4}\lambda^3 + 2t\lambda^2 + (t^2+4c_1)\lambda - \frac{4c_0}{\lambda}\Big]. \tag{$\mathrm{P_{IV}}$}$$

In what follows, we assume that $c_0 \neq 0$. Let us study the singularity structure of

$$\omega_{\mathrm{IV}} = \sqrt{\frac{3\lambda^3(t)+6t\lambda^2(t)+2(t^2+4c_1)\lambda(t)}{\lambda(t)}}dt \tag{2.39}$$

with a root $\lambda(t)$ of $P_{\mathrm{IV}}(t,\lambda) = \big(3\lambda^4 + 8t\lambda^3 + 4(t^2+4c_1)\lambda^2 - 16c_0\big)/4$. Since the discriminant $\mathrm{Disc}_{\mathrm{IV}}(t)$ of $P_{\mathrm{IV}}(t,\lambda)$ is a polynomial of degree 8 and the resultant $\mathrm{Res}_{\mathrm{IV}}(t)$ of $P_{\mathrm{IV}}(t,\lambda)$ and $Q_{\mathrm{IV}}(t,\lambda) = \lambda$ are given by $\mathrm{Res}_{\mathrm{IV}}(t) = Nc_0$ with some integers N, ω_{IV} may have eight singular points $\{t_j\}_{j=1}^{8}$ except for $t=\infty$ in general. Indeed, the discriminant of $\mathrm{Disc}_{\mathrm{IV}}(t)$ is written as

$$Nc_0^{19}(-4c_1^2+c_0)^8(4c_1^2+3c_0)^2 \tag{2.40}$$

with some integer N, and hence, $\mathrm{Disc}_{\mathrm{IV}}(t)$ has eight distinct roots when it does not vanish. Since $\partial_\lambda^2 P_{\mathrm{IV}}(t_j,\beta_j)$ and $\partial_t P_{\mathrm{IV}}(t_j,\beta_j)$ do not vanish when $-4c_1^2+c_0 \neq 0$ for a multiple root β_j of $P_{\mathrm{IV}}(t_j,\beta)=0$, we find that these singular points $\{t_j\}_{j=1}^{8}$ are simple turning points.

Now, we focus on the singular point of ω_{IV} at $t=\infty$. The leading term of the roots of $P_{\mathrm{IV}}(t,\lambda)$ at $t=\infty$ is given in Table 5 below.

	$l=1$	2	3	4
$\alpha_\infty^{(l)}$	1	1	-1	-1
$\beta_\infty^{(l)}$	$2\sqrt{c_0}$	$-2\sqrt{c_0}$	$-2/3$	-2
$\gamma_\infty^{(l)}$	-3	-3	-3	-3

Table 5. The leading term $\beta_\infty^{(l)} t^{-\alpha_\infty^{(l)}}$ of $\lambda_\infty^{(l)}(t)$ and the order $\gamma_\infty^{(l)}$ of ω_{IV} at $t=\infty$.

Since the leading terms of these four roots are different, we find

$$\mathrm{ord}^\infty_{\lambda_\infty^{(l)}}(\partial_\lambda P_{\mathrm{IV}}) = \mathrm{ind}^\infty_{\lambda_\infty^{(l)}}(\partial_\lambda P_{\mathrm{IV}}) = \min\{3\alpha_\infty^{(l)}, -1+2\alpha_\infty^{(l)}, -2+\alpha_\infty^{(l)}\}, \tag{2.41}$$

$$\mathrm{ord}^\infty_{\lambda_\infty^{(l)}}(Q_{\mathrm{IV}}) = \mathrm{ind}^\infty_{\lambda_\infty^{(l)}}(Q_{\mathrm{IV}}) = \alpha_\infty^{(l)}, \tag{2.42}$$

and hence, we obtain the order $\gamma_\infty^{(l)}$ of ω_{IV} at $t=\infty$.

Example 2.6 (the fifth Painlevé equation)**.** Let us consider the fifth Painlevé equation:

$$\begin{aligned}(\mathrm{P_V})\qquad \frac{d^2\lambda}{dt^2} =& \Big(\frac{1}{2\lambda}+\frac{1}{\lambda-1}\Big)\Big(\frac{d\lambda}{dt}\Big)^2 - \Big(\frac{1}{t}\Big)\frac{d\lambda}{dt} + \frac{(\lambda-1)^2}{t^2}\Big(2\lambda - \frac{1}{2\lambda}\Big)\\ &+\eta^2\frac{2\lambda(\lambda-1)^2}{t^2}\Big[(c_0+c_\infty) - c_0\frac{1}{\lambda^2} - c_2\frac{t}{(\lambda-1)^2} - c_1 t^2\frac{\lambda+1}{(\lambda-1)^3}\Big].\end{aligned}$$

In what follows, we assume that $\tilde{c}_\infty := c_0 + c_\infty$, c_0, c_1 and c_2 are not equal to 0. In general,

$$\omega_{\mathrm{V}} = \sqrt{\frac{\partial_\lambda P_{\mathrm{V}}(t,\lambda(t))}{Q_{\mathrm{V}}(t,\lambda(t))}}dt \tag{2.43}$$

has nine singular points $\{t_j\}_{j=1}^9$ except for $t = 0$ and ∞, where

(2.44)
$$\begin{aligned}P_{\mathrm{V}}(t,\lambda) =&2(c_0+c_\infty)\lambda^2(\lambda-1)^3 - 2c_0(\lambda-1)^3 - 2c_2t\lambda^2(\lambda-1) - 2c_1t^2\lambda^2(\lambda+1)\\ =&2\tilde{c}_\infty\lambda^5 - 6\tilde{c}_\infty\lambda^4 + 2(2c_0+3c_\infty - c_2t - c_1t^2)\lambda^3\\ &+ 2(2c_0 - c_\infty + c_2t - c_1t^2)\lambda^2 - 6c_0\lambda + 2c_0,\end{aligned}$$

$Q_{\mathrm{V}}(t,\lambda) = t^2\lambda(\lambda-1)$ and $\lambda(t)$ is a root of $P_{\mathrm{V}}(t,\lambda)$. Further, we find that $\{t_j\}_{j=1}^9$ are simple turning points.

Now, we focus on the singular points at $t = 0$ and ∞. We first note that, at $t = 0$, $P_{\mathrm{V}}(t,\lambda)$ is factorized as

$$P_{\mathrm{V}}(0,\beta) = 2(\beta-1)^3((c_\infty+c_0)\beta^2 - c_0). \tag{2.45}$$

Since $P_{\mathrm{V}}(0,\beta)$ has a multiple root $\beta = 1$, we consider

$$\begin{aligned}\tilde{P}_{\mathrm{V}}(t,\tilde{\lambda}) :=&P_{\mathrm{V}}(t,1+\tilde{\lambda})\\ =&2\tilde{c}_\infty\tilde{\lambda}^5 + 4\tilde{c}_\infty\tilde{\lambda}^4 + 2(c_\infty - c_2t - c_1t^2)\tilde{\lambda}^3\\ &- 4(c_2t+2c_1t^2)\tilde{\lambda}^2 - 2(c_2t+5c_1t^2)\tilde{\lambda} - 4c_1t^2\end{aligned} \tag{2.46}$$

instead of $P_{\mathrm{V}}(t,\lambda)$. When $c_\infty \neq 0$, the leading term of the roots of $\tilde{P}_{\mathrm{V}}(t,\tilde{\lambda})$ at $t = 0$ is given in Table 6 below.

	$l=1$	2	3	4	5
$\tilde{\alpha}_0^{(l)}$	1	1/2	1/2	0	0
$\tilde{\beta}_0^{(l)}$	$-2c_1/c_2$	$\sqrt{c_2/c_\infty}$	$-\sqrt{c_2/c_\infty}$	$-1+\sqrt{c_0/\tilde{c}_\infty}$	$-1-\sqrt{c_0/\tilde{c}_\infty}$
$\gamma_0^{(l)}$	-1	$-3/4$	$-3/4$	-1	-1

Table 6. The leading term $\tilde{\beta}_0^{(l)}t^{\tilde{\alpha}_0^{(l)}}$ of $\tilde{\lambda}_0^{(l)}(t)$ and the order $\gamma_0^{(l)}$ of ω_{V} at $t = 0$.

Then, the order $\gamma_0^{(l)}$ of ω_{V} at $t = 0$ immediately follows from the following relations:

$$\mathrm{ord}^0_{\lambda_0^{(l)}}(\partial_\lambda P_{\mathrm{V}}) = \mathrm{ind}^0_{\tilde{\lambda}_0^{(l)}}(\partial_{\tilde{\lambda}}\tilde{P}_{\mathrm{V}}) = \min\{4\tilde{\alpha}_0^{(l)}, 3\tilde{\alpha}_0^{(l)}, 2\tilde{\alpha}_0^{(l)}, 1+\tilde{\alpha}_0^{(l)}, 1\}, \tag{2.47}$$

$$\mathrm{ord}^0_{\lambda_0^{(l)}}(Q_{\mathrm{V}}) = \mathrm{ind}^0_{\tilde{\lambda}_0^{(l)}}(\tilde{Q}_{\mathrm{V}}) = \min\{2\tilde{\alpha}_0^{(l)}, \tilde{\alpha}_0^{(l)}\} + 2, \tag{2.48}$$

where $\tilde{Q}_{\mathrm{V}}(t,\tilde{\lambda}) := Q_{\mathrm{V}}(t,1+\tilde{\lambda})$.

Finally, we display Table 7 below.

	$l=1$	2	3	4	5
$\alpha_\infty^{(l)}$	1	1	0	-1	-1
$\beta_\infty^{(l)}$	$\sqrt{c_0/c_1}$	$-\sqrt{c_0/c_1}$	-1	$\sqrt{c_1/\tilde{c}_\infty}$	$-\sqrt{c_1/\tilde{c}_\infty}$
$\gamma_\infty^{(l)}$	-2	-2	-2	-2	-2

Table 7. The leading term $\beta_\infty^{(l)} t^{-\alpha_\infty^{(l)}}$ of $\lambda_\infty^{(l)}(t)$ and the order $\gamma_\infty^{(l)}$ of $\omega_{\rm V}$ at $t=\infty$.

We can read the order $\gamma_\infty^{(l)}$ of $\omega_{\rm V}$ at $t=\infty$ from the Table 7 and the following relations:

$$\operatorname{ord}^\infty_{\lambda_\infty^{(l)}}(\partial_\lambda P_{\rm V}) = \operatorname{ind}^\infty_{\lambda_\infty^{(l)}}(\partial_\lambda P_{\rm V}) = \min\{4\alpha_\infty^{(l)}, 3\alpha_\infty^{(l)}, -2+2\alpha_\infty^{(l)}, -2+\alpha_\infty^{(l)}, 0\}, \tag{2.49}$$

$$\operatorname{ord}^\infty_{\lambda_\infty^{(l)}}(Q_{\rm V}) = \operatorname{ind}^\infty_{\lambda_\infty^{(l)}}(Q_{\rm V}) = \min\{2\alpha_\infty^{(l)}, \alpha_\infty^{(l)}\} - 2. \tag{2.50}$$

Example 2.7 (the sixth Painlevé equation)**.** Finally, we consider the sixth Painlevé equation:

$$\begin{aligned}\frac{d^2\lambda}{dt^2} =& \frac{1}{2}\Big(\frac{1}{\lambda}+\frac{1}{\lambda-1}+\frac{1}{\lambda-t}\Big)\Big(\frac{d\lambda}{dt}\Big)^2 - \Big(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{\lambda-t}\Big)\frac{d\lambda}{dt}\\ &+ \frac{2\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\Big[1-\frac{\lambda^2-2t\lambda+t}{4\lambda^2(\lambda-1)^2}\\ &+\eta^2\Big\{(c_0+c_1+c_t+c_\infty) - c_0\frac{t}{\lambda^2} + c_1\frac{t-1}{(\lambda-1)^2} - c_t\frac{t(t-1)}{(\lambda-t)^2}\Big\}\Big]\end{aligned} \tag{$P_{\rm VI}$}$$

In what follows, we assume that $\tilde{c}_\infty := c_0+c_1+c_t+c_\infty$, c_0, c_1 and c_t are not equal to 0. In general,

$$\omega_{\rm V} = \sqrt{\frac{\partial_\lambda P_{\rm V}(t,\lambda(t))}{Q_{\rm V}(t,\lambda(t))}}dt \tag{2.51}$$

has nine singular points $\{t_j\}_{j=1}^9$ except for $t=0,1$ and ∞, where

$$\begin{aligned}P_{\rm VI}(t,\lambda) =& 2(c_0+c_1+c_t+c_\infty)\lambda^2(\lambda-1)^2(\lambda-t)^2 - 2c_0t(\lambda-1)^2(\lambda-t)^2\\ &+2c_1(t-1)\lambda^2(\lambda-t)^2 - 2c_tt(t-1)\lambda^2(\lambda-1)^2\\ =& 2\tilde{c}_\infty\lambda^6 - 4\tilde{c}_\infty(1+t)\lambda^5\\ &+2(-c_1+\tilde{c}_\infty+(-c_0+c_1+c_t+4\tilde{c}_\infty)t+(-c_t+\tilde{c}_\infty)t^2)\lambda^4\\ &+4((c_0-\tilde{c}_\infty+c_1-c_t)t+(c_0-c_1+c_2-\tilde{c}_\infty)t^2)\lambda^3\\ &+2((-c_0+c_t)t+(-4c_0-c_1-c_t+\tilde{c}_\infty)t^2+(-c_0+c_1)t^3)\lambda^2\\ &+4c_0(t^2+t^3)\lambda - 2c_0t^3,\end{aligned} \tag{2.52}$$

$Q_{\mathrm{VI}}(t,\lambda)=t^2(t-1)^2\lambda(\lambda-1)(\lambda-t)$ and $\lambda(t)$ is a root of $P_{\mathrm{VI}}(t,\lambda)$. Further, we find that $\{t_j\}_{j=1}^{9}$ are simple turning points. Since we can discuss the singular points $t=1$ and ∞ in a similar manner to $t=0$ (e.g., by considering $P(\tilde{t},\tilde{\lambda}):=P(1+\tilde{t},1+\tilde{\lambda})$ at $t=1$), we focus on the singular point at $t=0$. Let us see Table 8 below, where $\{\beta_0^{(2l-)},\beta_0^{(2l)}\}$ $(l=1,2,3)$ respectively are two distinct roots of

$$D^{0,(1)}_{P_{\mathrm{VI}}}(\beta)=2(-c_0+c_t)\beta^2+4c_0\beta-2c_0, \tag{2.53}$$

$$D^{0,(2)}_{P_{\mathrm{VI}}}(\beta)=2(-c_1+\tilde{c}_\infty)\beta^2+2(-c_0+c_t) \tag{2.54}$$

and

$$D^{0,(3)}_{P_{\mathrm{VI}}}(\beta)=2\tilde{c}_\infty\beta^2-4\tilde{c}_\infty\beta+2(-c_1+\tilde{c}_\infty). \tag{2.55}$$

	$l=1$	2	3	4	5	6
$\alpha_0^{(l)}$	1	1	1/2	1/2	0	0
$\beta_0^{(l)}$	$\beta_0^{(1)}$	$\beta_0^{(2)}$	$\beta_0^{(3)}$	$\beta_0^{(4)}$	$\beta_0^{(5)}$	$\beta_0^{(6)}$
$\gamma_0^{(l)}$	-1	-1	$-3/4$	$-3/4$	-1	-1

Table 8. The leading term $\beta_0^{(l)}t^{\alpha_0^{(l)}}$ of $\lambda_0^{(l)}(t)$ and the order $\gamma_0^{(l)}$ of ω_{VI} at $t=0$.

Since $\beta_0^{(1)}$, $\beta_0^{(2)}$, $\beta_0^{(5)}$ and $\beta_0^{(6)}$ are not equal to 1, we find that the following relations hold:

$$\mathrm{ord}^0_{\lambda_0^{(l)}}(\partial_\lambda P_{\mathrm{V}})=\mathrm{ind}^0_{\lambda_0^{(l)}}(\partial_\lambda P_{\mathrm{VI}})=\min\{5\alpha_0^{(l)},4\alpha_0^{(l)},3\alpha_0^{(l)},1+2\alpha_0^{(l)},1+\alpha_0^{(l)},2\}, \tag{2.56}$$

$$\mathrm{ord}^0_{\lambda_0^{(l)}}(Q_{\mathrm{VI}})=\mathrm{ind}^0_{\lambda_0^{(l)}}(Q_{\mathrm{VI}})=\min\{3\alpha_0^{(l)},2\alpha_0^{(l)},1+\alpha_0^{(l)}\}+2. \tag{2.57}$$

Then, the order $\gamma_0^{(l)}$ of ω_{VI} at $t=0$ immediately follows.

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