RIMS Kôkyûroku Bessatsu
B42 (2013), 1–11

L^2 boundedness for the 2D exterior problems for the semilinear heat and dissipative wave equations

Takayuki Kobayashi § and Masashi Misawa † ,

Abstract

We study the Cauchy-Dirichlet problems for the semlinear heat equations and dissipative wave equations on the two dimensional exterior domain. We show the L^2-boundedness of the solutions for initial data in the Hardy space.

§1. Introduction.

Let Ω be an exterior domain in $\mathbb{R}^2$ with a compact C^2-boundary $\partial\Omega$. Without loss of generality we may assume $(0,0) \notin \overline{\Omega}$. We will consider the Cauchy-Dirichlet problem for a semlinear heat equation on the exterior domain Ω: For a function $u = u(t,x)$ defined for $(t,x) \in (0,\infty) \times \Omega$,

$$u_t - \Delta u = \frac{|u|^{p-1}u}{W} \quad \text{on} \quad (0,\infty) \times \Omega, \tag{1.1}$$

$$u = 0 \quad \text{on} \quad (0,\infty) \times \partial\Omega, \tag{1.2}$$

$$u = u_0 \quad \text{on} \quad \{t=0\} \times \Omega, \tag{1.3}$$

where $3/2 < p < \infty$ and the function $W = W(x)$ is defined by

$$W(x) = 1 + |x| \log(B|x|), \qquad x \in \mathbb{R}^2, \tag{1.4}$$

with a positive constant B such that $B|x| \geq 2$ for any $x \in \Omega$.

To explain our motive for studying (1.1), (1.2) and (1.3), we simply take a look at the Cauchy problem of the linear heat equation

$$u_t - \Delta u = 0 \quad \text{on} \quad (0,\infty) \times \mathbb{R}^2, \tag{1.5}$$

$$u = u_0 \quad \text{on} \quad \{t=0\} \times \mathbb{R}^2 \tag{1.6}$$

and we observe the following fact.

Proposition 1.1 *Let $u_0 \in L^1(\mathbb{R}^2)$ and $u = u(t,x)$ be a solution to the Cauchy problem of the linear heat equations (1.5) and (1.6). Then, the solution u satisfies*

$$\lim_{t\to\infty} t\|u(t)\|_{L^2}^2 = \frac{1}{8\pi} \left| \int_{\mathbb{R}^2} u_0(x)\,dx \right|^2. \tag{1.7}$$

Received September 30, 2012. Revised November 14, 2012.
§ Takayuki Kobayashi, Department of Mathematics, Graduate School of Science and Technology, Saga University, 1 Honjo-machi, Saga 840-8502, Japan e-mail: kobayasi@ms.saga-u.ac.jp
† Masashi Misawa, Department of Mathematics, Graduate School of Science and Technology, Kumamoto University, 2-39-1 Kurokami, Kumamoto 860-8555, Japan
e-mail: misawa@aster.sci.kumamoto-u.ac.jp

Proof. By the Fourier transforms on spatial variables of (1.5) and (1.6) and Plancherel's theorem we immediately have

$$\|u(t)\|_{L^2}^2 = \|e^{-t|\xi|^2}\hat{u}_0(\xi)\|_{L^2}^2,$$

where $\hat{u}_0(\xi)$ is the Fourier transform of the initial data u_0. Then, the desired estimation (1.7) easily follows from the change of variables $\xi = \eta/\sqrt{2t}$ and Lebesgue's dominated convergence theorem. □

From Proposition 1.1, the solution $u = u(t,x)$ of (1.5) and (1.6) may not be bounded in $L^2((0,\infty)\times \mathbb{R}^2)$ for the initial data u_0 in $L^1(\mathbb{R}^2)$, in general. Our aim of this paper is to get L^2-boundedness of the solutions to the Cauchy-Dirichlet problem for the semilinear heat equation (1.1), (1.2) and (1.3), if the initial data is taken in a suitble space in the exterior domain Ω in $\mathbb{R}^2$. The key point is the choice of the initial data u_0 in the Hardy space $\mathcal{H}^1(\Omega)$ (see the definition below, and compare with [8], [9]). In [14] and [15], Miyakawa considers the Cauchy problems for the Stokes equations and the Navier-Stokes equations and shows the $\mathcal{L}(\mathcal{H}^1,\mathcal{H}^1)$-boundedness for the solutions. Ogawa in [20], Ogawa and Shimizu in [21] prove that the inequality

$$\int_0^t \|\nabla u\|_{\mathcal{H}^1(\mathbb{R}^2)}^2 ds \le C\|u_0\|_{\mathcal{H}^1(\mathbb{R}^2)}^2$$

holds for the solutions u of (1.5) and (1.6). From this inequality, we can directly derive the L^2-boundedness for the solutions of (1.5) and (1.6) in $\mathbb{R}^2$. In fact, by the continuous imbeddings $W_1^1 \subset L^2$ and $\mathcal{H}^1 \subset L^1$ in $\mathbb{R}^2$, we deduce that

$$\begin{aligned}\int_0^t \|u\|_{L^2(\mathbb{R}^2)}^2 ds &\le C\int_0^t \|\nabla u\|_{L^1(\mathbb{R}^2)}^2 ds\\ &\le C\int_0^t \|\nabla u\|_{\mathcal{H}^1(\mathbb{R}^2)}^2 ds\\ &\le C\,\|u_0\|_{\mathcal{H}^1(\mathbb{R}^2)}^2.\end{aligned}$$

Comparing with those estimates above, our main method to get this L^2-boundedness is based on only two ingredients, one is the usual Morawetz energy estimates (see [16, 17]) and another one is the Fefferman-Stein inequality, implying the duality between Hardy space and BMO space. Therefore, our method is also easily applied for the nonlinear dissipative wave equations in an exterior domain in $\mathbb{R}^2$, showing the L^2-boundeness of the solutions. Using this method, Misawa, Okamura and Kobayashi obtain the L^2-boundeness of the solutions and the decay estimates of the solutions when $2 < p < \infty$. For the nonlinear term $|u|^{p-1}u$, the number $p = 2 (= 1 + 2/n$ with $n = 2)$ is the well known Fujita exponent, which is the threshold number between the global in time existence for the small initial data and blowing up in a finite time (see [5] etc.). In this paper, we treat the nonlinear term in (1.1) in the case that $3/2 < p < \infty$ and obtain the L^2-boundedness for the solutions, which is somehow improvement of the results in [13]. The Fujita exponent for (1.1) seems to be $p = 3/2$, because the decay rate of the nonlinear term in (1.1) is faster than that of the nonlinear term $|u|^{p-1}u$.

We will start with the definition of function spaces (refer to [4]) in order to state our main results.

Definition 1.2 (Hardy space) *The Hardy space consists of functions f in $L^1(\mathbb{R}^n)$ such that*

$$\|f\|_{\mathcal{H}^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \sup_{r>0} |\phi_r * f(x)| dx$$

is finite, where $\phi_r(x) = r^{-n}\phi(r^{-1}x)$ for $r>0$ and ϕ is a smooth function on $\mathbb{R}^n$ with compact support in an unit ball with center of the origin $B_1(0) = \{x \in \mathbb{R}^n; |x|<1\}$.

The definition dose not depend on choice of a function ϕ.

We will use the Hardy space on the exterior domain Ω and BMO space.

Definition 1.3 (Hardy space on Ω) *$f \in L^1(\Omega)$ is said to be in $\mathcal{H}^1(\Omega)$ if and only if $\widetilde{f} \in \mathcal{H}^1(R^n)$, where f is extended to all of $\mathbb{R}^n$ by letting $f = 0$ outside Ω and its extension of f is denoted by $\widetilde{f}$. A norm on $\mathcal{H}^1(\Omega)$ is defined to be*

$$\|f\|_{\mathcal{H}^1(\Omega)} = \|\widetilde{f}\|_{\mathcal{H}^1(\mathbb{R}^n)}.$$

Definition 1.4 (functions of bounded mean oscillation) *Let f be a locally integrable in $\mathbb{R}^n$, that is $f \in L^1_{loc}(\mathbb{R}^n)$. We say that f is of bounded mean oscillation (abbreviated as BMO) if*

$$\|f\|_{BMO(\mathbb{R}^n)} = \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|} \int_B |f - (f)_B| dx<\infty,$$

where the supremum ranges over all finite ball $B \subset \mathbb{R}^n$, $|B|$ is the n-dimensional Lebesgue measure of B, and $(f)_B$ denotes the integral mean of f over B, namely $(f)_B = \frac{1}{|B|} \int_B f(x)dx$.

The class of functions of BMO, modulo constants, is a Banach space with the norm $\|\cdot\|_{BMO}$ defined above.

We will prepare the decisive Fefferman-Stein inequality, which means the duality between $\mathcal{H}^1(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$, $(\mathcal{H}^1(\mathbb{R}^n))^* = BMO(\mathbb{R}^n)$. For the proof, see [4].

Lemma 1.5 (Fefferman-Stein inequality) *There is a positive constant C depending only on n such that if $f \in \mathcal{H}^1(\mathbb{R}^n)$ and $g \in BMO(\mathbb{R}^n)$, then*

$$\left| \int_{\mathbb{R}^n} fg dx \right| \le C\|f\|_{\mathcal{H}^1(\mathbb{R}^n)} \|g\|_{BMO(\mathbb{R}^n)}.$$

Also, we shall use the following Poincaré inequality in $\mathbb{R}^2$, which is proved by the definition of BMO and the usual Poincaré inequality in $\mathbb{R}^2$. For the detail of the proof, see [13] etc.

Lemma 1.6 (Poincaré inequality) *For $f \in H^1(\mathbb{R}^2)$, the following inequality holds.*

$$\|f\|_{BMO(\mathbb{R}^2)} \leq C\|\nabla f\|_{L^2(\mathbb{R}^2)}. \tag{1.8}$$

Now we recall the existence of a solution defined in the following. We shall use the usual $L^2(\Omega)$ space with the norm $\|f\|_{L^2(\Omega)} = \|f\|_2$, the usual Sobolev spaces $H^1(\Omega)$ and $H^1_0(\Omega)$ with norms

$$\|f\|_{H^1(\Omega)} = \sqrt{\|f\|_2^2 + \|\nabla f\|_2^2}, \qquad \|f\|_{H^1_0(\Omega)} = \|\nabla f\|_2,$$

the usual notation for $f, g \in L^2(B)$ with a domain $B \subset \mathbb{R}^2$

$$(f,g)_{L^2(B)} = \int_B fg dx, \quad \|f\|_{L^2(B)} = \sqrt{(f,f)_{L^2(B)}}$$

and their abbreviation $(f,g) = (f,g)_{L^2(\Omega)}$ for $f, g \in L^2(\Omega)$.

Definition 1.7 *A function u defined on $(0,\infty)\times\Omega$ is called to be a weak solution to* (1.1), (1.2) *and* (1.3) *if u belongs to the class $C([0,\infty); H^1_0(\Omega)) \cap C^1([0,\infty); L^2(\Omega))$ and, satisfies the equations* (1.1) *in the distribution sense and the initial conditions* (1.2), (1.3) *in the trace sense, respectively.*

Our main result in this paper is the decay estimate for the nonlinear heat equations.

Theorem 1.8 *Let $3/2 < p < \infty$ and initial data $u_0 \in H^1(\Omega) \cap \mathcal{H}^1(\Omega)$ to satisfy*

$$\|u_0\|_{H^1(\Omega)} + \|u_0\|_{\mathcal{H}^1(\Omega)} << 1.$$

Then, there exists a weak solution $u \in C([0,\infty); H^1_0(\Omega)) \cap C^1([0,\infty); L^2(\Omega))$ to the problem (1.1), (1.2) *and* (1.3) *such that*

$$\|u(t)\|_2 \leq C(1+t)^{-1/2}, \quad \|\nabla u(t)\|_2 \leq C(1+t)^{-1} \tag{1.9}$$

and

$$\int_0^t \|u(t)\|_2^2 \leq C$$

hold for any $t \geq 0$ where the positive constants C is independent of t.

§2. Proof of Theorem 1.8.

First we recall a special case of the so-called multiplicative inequality and the Hardy inequality. Thoughout this section, the usual function space $L^q(\Omega)$ of $q-$ integrable functions on Ω for $q \geq 1$ is also used with its norm

$$\|f\|_q = \|f\|_{L^q(\Omega)} = \left(\int_\Omega |f|^q dx\right)^{1/q}.$$

Lemma 2.1 (The Sobolev and Hardy inequalities) *For any $f \in H_0^1(\Omega)$,*

$$\|f\|_q \leq C_s \|\nabla f\|_2^\theta \|f\|_2^{1-\theta} \tag{2.1}$$

holds for any $q \geq 2$ and a positive number $\theta < 1$ such that $\theta = 1 - 2/q$, where a positive constant C_s depends only on q. It holds that

$$\left\|\frac{f}{W}\right\|_2 \leq C_h \|\nabla f\|_2 , \tag{2.2}$$

where C_h is a uniform positive constant.

Remark. In the case that $\Omega = \mathbb{R}^2$, the Hardy type inequality (2.2) without a weight is not valid for $f \in C_0^\infty(\mathbb{R}^2)$, in general. In fact, take the function $\varphi(x) \in H^1(\mathbb{R}^2)$ so that $\varphi(x) = \varphi(r)$, $(r = |x|)$ and

$$\varphi(r) = \begin{cases} 1 & 0 \leq r \leq 1, \\ r^{-\varepsilon} & 1 \leq r \leq R, \\ -R^{-1-\varepsilon}(r - 2R) & R \leq r \leq 2R, \\ 0 & r \geq 2R. \end{cases}$$

Then, for sufficiently small ε, choosing R suffiently large depending on ε, one gets the estimates

$$\begin{aligned} \|\varphi(x)\|_{L^2(|x|<1)} &= \sqrt{\pi}, \\ \|\nabla\varphi(x)\|_{L^2(\mathbb{R}^2)} &\leq \sqrt{\pi\tilde{\varepsilon}}. \end{aligned}$$

The positive number $\tilde{\varepsilon}$ can be taken small if ε is small and R is large and so, (2.2) without a weight does not hold for $\Omega = \mathbb{R}^2$.

Therefore, in this paper, we only consider the exterior problem.

Proof of Theorem 1.8. We proceed the continuity method for the proof of our decay estimates. Let $u \in C([0,T); H_0^1(\Omega)) \cap C^1([0,T); L^2(\Omega))$ be a weak solution to the problem (1.1), (1.2) and (1.3) on a time interval $[0,T)$ for a positive number T. By the local in time existence and the continuity in $H_0^1(\Omega)$ of the solution $u(t)$ in $C([0,T); H_0^1(\Omega)) \cap C^1([0,T); L^2(\Omega))$, we may suppose that, for some positive number K,

$$\begin{aligned} &\|u(t)\|_2 \leq K\, I_0\, (1+t)^{-1/2}, \\ &\|\nabla u(t)\|_2 \leq K\, I_0\, (1+t)^{-1} \end{aligned} \tag{2.3}$$

hold for any $t \in [0,T)$, where we put

$$I_0 = \|u_0\|_{H^1(\Omega)} + \|u_0\|_{\mathcal{H}^1(\Omega)}. \tag{2.4}$$

We denote the nonlinear terms by $F(u(t)) = |u(t)|^{p-1}u(t)/W(x)$ and then, we have

$$\frac{1}{2}\frac{d}{dt}||u(t)||_2^2 + ||\nabla u(t)||_2^2 = (F(u(t)), u(t)), \tag{2.5}$$

and

$$||u_t(t)||_2^2 + \frac{1}{2}\frac{d}{dt}||\nabla u(t)||_2^2 = (F(u(t)), u_t(t)). \tag{2.6}$$

From (2.5), we find that

$$\frac{1}{2}||u(t)||_2^2 + \int_0^t ||\nabla u(s)||_2^2 ds = \frac{||u_0||_2^2}{2} + \int_0^t (F(u(s)), u(s))ds, \tag{2.7}$$

and

$$\begin{aligned} &\frac{(1+t)}{2}||u(t)||_2^2 + \int_0^t (1+s)||\nabla u(s)||_2^2 ds - \frac{1}{2}\int_0^t ||u(s)||_2^2 ds \\ = \ &\frac{||u_0||_2^2}{2} + \int_0^t (1+s)(F(u(s)), u(s))ds. \end{aligned} \tag{2.8}$$

Also, form (2.6), we obtain that

$$\begin{aligned} \int_0^t (1+s)^2||u_s(s)||_2^2 ds + \frac{(1+t)^2}{2}||\nabla u(s)||_2^2 - \int_0^t (1+s)||\nabla u(s)||_2^2 ds \\ = \frac{||\nabla u_0||_2^2}{2} + \int_0^t (1+s)^2(F(u(s)), u_s(s))ds. \end{aligned} \tag{2.9}$$

Set

$$w(t) = w(t,x) = \int_0^t u(s,x)ds.$$

We find that w satisfies

$$w_t(t,x) - \Delta w(t,x) = u_0(x) + \int_0^t F(u(s))ds, \tag{2.10}$$

$$w(t,x)|_{\partial\Omega} = 0, w(0,x) = 0, \tag{2.11}$$

so that

$$||w_t(t)||_2^2 + \frac{1}{2}\frac{d}{dt}||\nabla w(t)||_2^2 = (u_0, w_t(t)) + (\int_0^t F(u(s))ds, w_t(t)).$$

Hence, noting that $w_t = u$, one has

$$\int_0^t ||u(s)||_2^2 ds + \frac{1}{2}||\nabla w(t)||_2^2 = (u_0, w(t)) + \int_0^t (\int_0^s (F(u(\tau))d\tau, w_s(s))ds. \tag{2.12}$$

We now make an estimation of the first term of the right hand side in (2.12), that is the one of main part in our argument. For this purpose, we extend functions u_0 and w to all of $\mathbb{R}^2$ by letting $u_0 = 0$ and $w(t) = 0, 0 \le t < \infty$, outside Ω and denote their extension by $\tilde{u}_0$ and $\tilde{w}(t)$, respectively. Employing Lemma 1.5, we deduce

$$|(u_0, w(t))| = |(\tilde{u}_0, \tilde{w}(t))| \le C\|\tilde{u}_0\|_{\mathcal{H}^1(\mathbb{R}^2)}\|\tilde{w}(t)\|_{BMO(\mathbb{R}^2)}.$$

From

$$\|\tilde{w}(t)\|_{BMO(\mathbb{R}^2)} \le C\|\nabla w(t)\|_2$$

by Lemma 1.6, it follows that

$$|(u_0, w(t))| \le C\|u_0\|_{\mathcal{H}^1(\Omega)}\|\nabla w(t)\|_2. \tag{2.13}$$

Next, we estimate the nonlinear terms in (2.7), (2.8), (2.9) and (2.12), by using the a priori decay estimates (2.3). From Lemmas 2.1 and 1.8, it follows that

$$\begin{aligned} ||u(t)/W^{\frac{1}{p+1}}||_{p+1}^{p+1} &= \int_\Omega |u|^p \cdot |u|/W(x)dx \\ &\le \||u|^p\|_2\|u/W\|_2 \\ &= \|u\|_{2p}^p\|u/W\|_2 \\ &\le C_s^pC_h||\nabla u(t)||_2^p||u(t)||_2 \\ &\le C_s^pC_hK^{p+1}I_0^{p+1}(1+t)^{-p-1/2}. \end{aligned} \tag{2.14}$$

Then, in (2.7) and (2.8), we have by (2.14), for $\alpha = 0, 1$,

$$\begin{aligned} |\int_0^t (1+s)^\alpha(F(u(s)), u(s))ds| &= \int_0^t (1+s)^\alpha||u(s)/W^{\frac{1}{p+1}}||_{p+1}^{p+1}ds \\ &\le C_s^pC_hK^{p+1}I_0^{p+1}\int_0^t (1+s)^{\alpha-p-1/2}ds. \end{aligned} \tag{2.15}$$

We will estimate the second term in the right hand side of (2.9). Integrating by parts on t, one gets

$$\begin{aligned} &\int_0^t (1+s)^2(F(u(s)), u_s(s))ds \\ =\ & \frac{1}{p+1}\int_0^t (1+s)^2\frac{d}{ds}||u(s)/W^{\frac{1}{p+1}}||_{p+1}^{p+1}ds \\ =\ & \frac{1}{p+1}\{(1+t)^2||u(t)/W^{\frac{1}{p+1}}||_{p+1}^{p+1} - ||u_0/W^{\frac{1}{p+1}}||_{p+1}^{p+1}\} \\ & -\frac{2}{p+1}\int_0^t (1+s)||u(s)/W^{\frac{1}{p+1}}||_{p+1}^{p+1}ds. \end{aligned} \tag{2.16}$$

Thus we see from (2.14), (2.15) and (2.16) that

$$\begin{aligned} &|\int_0^t (1+s)^2(F(u(s)), u_s(s))ds| \\ \le\ & C_s^pC_hK^{p+1}I_0^{p+1}\{\int_0^t (1+s)^{1/2-p}ds + (1+t)^{3/2-p}\} + \frac{1}{p+1}||u_0/W^{\frac{1}{p+1}}||_{p+1}^{p+1}. \end{aligned} \tag{2.17}$$

The last term coming from the nonlinear term in (2.12) is estimated by Fubini ’s theorem and an integration by parts on s

$$\begin{aligned} &\int_0^t (\int_0^s F(u(\tau))d\tau, w_s(s))ds \\ =\ & (\int_0^t F(u(s))ds, w(t)) - \int_0^t (F(u(s)), w(s))ds \\ =\ & I_1 + I_2. \end{aligned} \tag{2.18}$$

The first term of (2.18) is bounded by

$$\begin{aligned} |I_1| &= |\int_\Omega \int_0^t |u(s)|^{p-1}u(s)ds \cdot \frac{w(t)}{W(x)}dx| \quad (2.19)\\ &\le \int_0^t ||u(s)||_{2p}^p ds \cdot ||\frac{w(t)}{W}||_2 \\ &\le C_s^p C_h K^p I_0^p \int_0^t (1+s)^{1/2-p}ds||\nabla w(t)||_2 \end{aligned}$$

where we use Jensen's inequality in the first inequality, and Lemma 2.1 in the second inequality and, lastly, we substitute (2.3). The second term of (2.18) is estimated as

$$\begin{aligned} |I_2| &\le |\int_0^t (|u(s)|^{p-1}u(s), \frac{w(s)}{W})ds| \\ &\le C_h \int_0^t ||u(s)||_{2p}^p ||\nabla w(s)||_2 ds \\ &\le C_s^p C_h K^p I_0^p \int_0^t (1+s)^{1/2-p}ds \sup_{0<s<t} ||\nabla w(s)||_2, \quad (2.20) \end{aligned}$$

where we firstly use the Schwarz's inequality, secondly, the Jensen inequality, thirdly, Lemma 2.1 and, lastly, (2.3). Finally, gathering the estimates (2.13), (2.15)-(2.20) with energy equations (2.7), (2.8), (2.9) and (2.12), we see that if $p > 3/2$, then

$$X(t) \le \ell_p(I_0) + C_p X(t)^q, \quad (q = q(p) > 1)$$

where $\ell_p(I_0) \to 0$ as $I_0 \to 0$ and

$$\begin{aligned} X(t) = \sup_{0<s<t} \{(1+s)||u(s)||_2^2 + (1+s)^2||\nabla u(s)||_2^2 + ||\nabla \int_0^s u(\tau)d\tau||_2^2\} \\ + \int_0^t (1+s)||\nabla u(s)||_2^2 ds + \int_0^t (1+s)^2 ||u_s(s)||_2^2 ds + \int_0^t ||u(s)||_2^2 ds. \end{aligned}$$

Thus, we arrive at our desired estimation. □

§3. The nonlinear dissipative wave equations.

In this section, we study a decay estimation for a nonlinear dissipative wave equation on the two dimensional exterior domain Ω, that is an application of our decay results for the nonlinear heat equations as before. Our nonlinear wave equation of dissipative type considered here is

$$u_{tt} - \Delta u + u_t = \frac{|u|^{p-1}u}{W} \quad \text{on} \quad (0,\infty)\times\Omega, \quad (3.1)$$

$$u = 0 \quad \text{on} \quad (0,\infty)\times\partial\Omega, \quad (3.2)$$

$$u = u_0, \quad u_t = u_1 \quad \text{on} \quad \{t=0\}\times\Omega, \quad (3.3)$$

where $3/2 < p < \infty$.

For the linear dissipative wave equations, in a view of the energy methods, Ikehata and Matsuyama [9] was able to show a boundedness in $L^2((0,\infty)\times\Omega)$ of solutions if initial data u_0+u_1 satisfies $\|d(u_0+u_1)\|_2 < \infty$ for $d(x) = 1+|x|\log(B|x|)$. In this paper and [13], under the condition that u_0+u_1 belongs to the Hardy space $\mathcal{H}^1$ on Ω (see the definition above), we prove an $L^2((0,\infty)\times\Omega)$-boundedness. Here we notice that any member of the Hardy space on $\mathbb{R}^n$ has an integrability of $L\log(L)$ on $\mathbb{R}^n$ (refer to [11]). We also study deacy estimates of total energy $E(t) = \frac{1}{2}(\|\nabla u(t)\|_2^2 + \|u_t(t)\|_2^2)$ and $\|u(t)\|_2$ for the dissipative wave equations if initial data $u_0+u_1 \in \mathcal{H}^1(\Omega)$. Then they decay like t^{-2} and $t^{-1/2}$, respectively, of which the decay rate may be optimal in comparison with that of the heat kernel and the proof is also based on a boundedness of its energy of solutions (refer to [18, 19], [7, 9, 10] and also see [12], [2], [8] for the Cauchy problem).

In [7], the same decay estimates is obtained for the semilinear dissipative wave equation without the weight W in the nonlinear term and with initial data of compact support. Instead the equation has the weight and the Hardy space regularity of initial data is assumed here. In two dimensions, such polynomial nonlinearity as in (3.1) is subcritical for the energy class by the Sobolev type inequality and the natural critical nonlinearity is an exponential one (see [6] and [1] for the defocusing and focusing cases, respectively). Here we would like to emphasize an effect of the Hardy space regularity of initial data. The nonlinear wave equations on two dimensional exterior domains will be studied elsewhere, as well as the nonlinear term of critical growth for the energy class (for the Cauchy problem see, for example, [1, 6], [22] and the references therein). The existence immediately follows from the local in time existence result and the usual extension argument, once the decay estimates of L^2 and total energy of solutions are obtained for the nonlinear equation (3.1), (3.2) and (3.3) (see [7, Proposition 2.2] and refer to [23], [22, Chapter 6]). Thus we will study the decay estimates of $L^2(\Omega)$, total energy $E(t)$ and $L^2((0,\infty)\times\Omega)$ of solutions. We proceed our decay estimations along those in section 3 of [13] for the nonlinear dissipative wave equations for the (3.1), (3.2) and (3.3). Of course, the main task here is to control the nonlinear terms.

Our result on decay estimations for the nonlinear dissipative wave equations is the following:

Theorem 3.1 *Let $3/2 < p < \infty$ and let initial data $(u_0,u_1) \in H^1(\Omega)\times L^2(\Omega)$ satisfy $u_0+u_1 \in \mathcal{H}^1(\Omega)$ and*

$$\|u_0\|_{H^1(\Omega)} + \|u_1\|_2 + \|u_0+u_1\|_{\mathcal{H}^1(\Omega)} << 1.$$

Then, there exists a weak solution $u \in C([0,\infty); H_0^1(\Omega)) \cap C^1([0,\infty); L^2(\Omega))$ to the problem (3.1)*,* (3.2) *and* (3.3) *such that*

$$\|u(t)\|_2 \le C(1+t)^{-1/2}, \quad \|u_t(t)\|_2, \|\nabla u(t)\|_2 \le C(1+t)^{-1} \tag{3.4}$$

and

$$\int_0^t \|u(t)\|_2^2 \le C$$

hold for any $t \ge 0$ where the positive constants C is independent of t.

To prove Theorem 3.1, we use the same methods as in the proof of Theorem 1.8. The main difference between the proof of this paper and that of [13] is the estimation (2.14) for the nonlinear terms. By using the estimation (2.14) in this paper, we can show Theorem 3.1. (see [13]).

Acknowledgments. The work of Takayuki Kobayashi is partially supported by JSPS KAKENHI Grant Number #22540202, #24340025 and #24340028. The work of Mashashi Misawa is partially supported by JSPS KAKENHI Grant Number #21540222 and #24540215.

References

[1] C.O. Alves, M. M. Cavalcanti, On existence, uniform decay rates and blow up for solutions of the 2-D wave equation with exponential source, Calc. Var. Partial Differential Equations **34** (2009), 377-411.

[2] W. Dan, Y. Shibata, On a local energy decay of a dissipative wave equation, Funkcial. Ekvac. **38** (1995), 545-568.

[3] L. C. Evans, Partial Differential Equations, Graduate Studies in Math., **19**. Amer. Math. Soc., Providence, RI, (1998).

[4] C. Fefferman, E. M. Stein, H^p spaces of several variables, Acta Math. **192** (1972), no. 3-4, 137-193.

[5] H. Fujita, On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo Sect. I **13** (1966), 109-124

[6] S. Ibrahim, M. Majdoub, N. Masmoudi, Global solutions for a semilinear, two-dimensional Klein-Gordon equation with exponential-type nonlinearlity, Comm. Pure Appl. Math. **59** (2006), no. 11, 1639-1658.

[7] R. Ikehata, Two dimensional exterior mixed problem for semilinear damped wave equations, J. Math. Anal. Appl. **301** (2005), 366-377.

[8] R. Ikehata, Y. Miyaoka, T. Nakatake, Decay estimates of solutions for dissipative wave equations in $\mathbb{R}^n$ with lower power nonlinearities, J. Math. Soc. Japan **56**, no. 2, (2004), 365-373.

[9] R. Ikehata, T. Matsuyama, L^2-behaviour of solutions to the linear heat and wave equations in exterior domains, Sci. Math. Jpn. **55** (2002), no.1, 33-42.

[10] R. Ikehata, T. Matsuyama, Remarks on the behaviour of solutions to the linear wave equations in unbounded domains, Proc. School of Science, Tokai Univ. **36** (2001), 1-13.

[11] T. Iwaniec, G. Martin, Geometric function theory and non-linear analysis, Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, (2001).

[12] S. Kawashima, M. Nakao, K. Ono, On the decay property of solutions to the Cauchy problem of the semilinear wave equation with a dissipative term, J. Math. Soc. Japan **47** (1995), 617-653.

[13] M. Misawa, S. Okamura and T. Kobayashi, Decay property for the linear wave equations in two dimensional exterior domains, Diffrerntial and Integral Equations **24** (2011), 941-964.

[14] T. Miyakawa, Hardy spaces of solenoidal vector fields, with applications to the Navier-Stokes equations, Kyushu J. Math. **50** (1996), 1-64.

[15] T. Miyakawa, Application of Hardy space techniques to the time-decay problem for incompressible Navier-Stokes flows in R^n, Funkcial Ekvac. **41** (1998), 384-434.

[16] C. Morawetz, Exponential decay of solutions of the wave equations, Comm. Pure Appl. Math. **19** (1966), 439-444.

[17] C. Morawetz, The limiting amplitude principle, Comm. Pure Appl. Math. **15** (1962), 349-361.

[18] M. Nakao, Energy decay for the linear and semilinear wave equations in exterior domains with some localized dissipations, Math. Z. **238** (2001), 781-797.

[19] M. Nakao, Stabilization of local energy in an exterior domain for the wave equation with a localized dissipation, J. Differential Equations **148** (1998), 388-406.

[20] T. Ogawa, Drift-diffusion system in the critical case, RIMS Kôkyûroku Bessatu B14, "Harmonic Analysis and Nonlinear Partial Differential Equations" (2009), 87-102.

[21] T. Ogawa and S. Shimizu, The drift-diffusion system in two-dimensional critical Hardy space, J. Functional Analysis **255** (2008), 1107-1138.

[22] J. Shatah, M. Struwe, Geometric Wave Equations, Courant Lecture Notes in Math. **2**, New York University, Courant Institute of Mathematical Sciences, New York; Amer. Math. Soc., Providence, RI, 1998.

[23] W. A. Strauss, Nonlinear Wave Equations, CBMS Reg. Conf. Ser. Math., **73**, Amer. Math. Soc., Providence, RI, 1989.