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Remarks on Strichartz estimates for Schrödinger equations on manifolds with ends

By

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Abstract

We give an account of several recent results [24, 25, 26, 27] on Strichartz estimates for the time-dependent Schrödinger equation. We consider two models: The first part of the paper is concerned with Schrödinger operators with variable coefficients and unbounded electromagnetic potentials on the Euclidean space. In the second part, we consider the Laplace-Beltrami operator on a class of non-compact manifolds with polynomially growing ends. Under several assumptions on the coefficients and the potentials at spatial infinity, we show local-in-time Strichartz estimates outside a large compact set (without the non-trapping condition). We also prove global-in-space Strichartz estimates under some geometric conditions on the Hamilton flow generated by the kinetic energy.

§ 1. Introduction

In this note we give a review of author's recent progress [24, 25, 26, 27] concerning the Strichartz estimates for Schrödinger equations with variable coefficients.

Let us start with the general framework. Consider the Schrödinger equation on a d-dimensional complete Riemannian manifold (M, g):

$$i\partial_t u = Hu; \quad u|_{t=0} = u_0 \in L^2(M), \tag{1.1}$$

where $H = -(1/2)\Delta_g + V(x)$, Δ_g is the Laplace-Beltrami operator associated to the metric g and V is a real-valued function. For instance, we assume that H is self-adjoint on $L^2(M)$. The solution to (1.1) is given by $u(t) = e^{-itH}u_0 \in C(\mathbb{R}; L^2(M))$, where e^{-itH} is a unique strongly continuous one parameter unitary group generated by H.

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We are interested in so-called the *Strichartz estimates* which are of the forms:

$$||u||_{L^p_T L^q(M)} \le C_T ||\langle H\rangle^\gamma u_0||_{L^2(M)}, \tag{1.2}$$

where $||F||_{L^p_T L^q(M)} := ||||F(t,\cdot)||_{L^q(M)}||_{L^p([T,-T])}$ with $T>0$, $\gamma\ge 0$ and (p,q) satisfies the following *admissible* condition:

$$p,q\ge 2,\quad 2/p = d(1/2-1/q),\quad (d,p,q)\neq(2,2,\infty). \tag{1.3}$$

It is widely known that Strichartz estimates play an important role in studying nonlinear Schrödinger equations (see, *e.g.*, [8]). Furthermore, if H has infinitely many positive eigenvalues $0<E_0<E_1<\cdots$, then such estimates can be applied to obtain L^p-estimates of eigenfunctions:

$$||\psi_{E_j}||_{L^p} \lesssim E_j^{\delta(d,p)},\quad 2\le p\le\infty,$$

where ψ_{E_j} is the L^2-normalized eigenfunction with the eigenvalue E_j. In particular, for $d\ge 3$, the sharp endpoint Strichartz estimate, combined with the Bernstein inequality, usually implies the sharp L^∞-estimate ([20, 21]).

To explain the purpose of the paper more precisely, we recall some known results. Let us first recall well known properties of the free propagator e^{-itH_0} on $\mathbb{R}^d$, where $H_0 = -\Delta/2$. The distribution kernel of e^{-itH_0} is given explicitly by $(2\pi it)^{-d/2}e^{i|x-y|^2/(2t)}$ and the solution $u(t)=e^{-itH_0}u_0$ thus satisfies so-called the *dispersive estimate*:

$$||u(t)||_{L^\infty(\mathbb{R}^d)} \lesssim |t|^{-d/2}||u_0||_{L^1(\mathbb{R}^d)} \tag{1.4}$$

for any $t\neq 0$, which, combined with the unitarity on L^2, implies that u enjoys the sharp global-in-time Strichartz estimates, *i.e.*, (1.2) with $T=+\infty$ and $\gamma=0$, for any admissible pair (p,q). These estimates immediately imply that, for any $u_0\in L^2$, $u(t)\in\bigcap_{q\in Q_d}L^q$ for a.e. $t\in\mathbb{R}$, where $Q_1=[2,\infty]$, $Q_2=[2,\infty)$ and $Q_d=[2,2d/(d-2)]$ for $d\ge 3$. Roughly speaking, comparing the Sobolev embedding $H^{d(1/2-1/q)}\hookrightarrow L^q$ one can recover at most one derivative loss by using Strichartz estimates. Strichartz estimates for e^{-itH_0} were first proved by Strichartz [32] for a restricted pair of (p,q) with $p=q=2(d+2)/d$, and have been generalized for (p,q) satisfying (1.3) by [15, 18].

For Schrödinger operators with electromagnetic potentials

$$H=\frac{1}{2}(-i\partial_x - A(x))^2 + V(x)\quad \text{on } \mathbb{R}^d,$$

short-time dispersive and local-in-time Strichartz estimates have been extended with potentials decaying at infinity [34] or growing at infinity [14, 35]. In particular, it was shown by [14, 35] that if V is of at most quadratic type, A is of at most linear type and all derivatives of the magnetic field $B=dA$ are of short-range type, then $e^{-itH}u_0$ satisfies

(1.4) for small $t \neq 0$. Local-in-time Strichartz estimates are immediate consequences of this estimate, the L^2-boundedness and the TT^*-argument due to Ginibre-Velo [15] (see Keel-Tao [18] for the endpoint estimate). For the case with singular potentials or with supercritically growing electromagnetic potentials, we refer to [34, 36, 38, 9] and reference therein. We mention that global-in-time dispersive and Strichartz estimates for the scattering state $P_{ac}(H)u$ have been also studied under suitable decaying conditions on potentials and assumptions for the zero energy; see [17, 37, 30, 12, 10] and reference therein. We also mention that there is no result on sharp global-in-time dispersive estimates for (generic) magnetic Schrödinger operators, though [13] has recently proved dispersive estimates for the Aharonov-Bohm effect in $\mathbb{R}^2$.

On the other hand, the influence of the geometry (*e.g.*, the global behavior of the geodesic flow) on the behavior of solutions to linear and nonlinear partial differential equations has been extensively studied. From this geometric viewpoint, sharp local-in-time Strichartz estimates for Schrödinger equations with variable coefficients (or, more generally, on manifolds) have recently been investigated by many authors under several conditions on the geometry; see, *e.g.*, [31, 6, 28, 16, 4, 3, 7] and reference therein. In [31], [28], [4], the authors studied the case on the Euclidean space with nontrapping asymptotically flat metrics. The case on the nontrapping asymptotically conic manifold was studied by [16]. In [3] the author considered the case of nontrapping asymptotically hyperbolic manifold. For the trapping case, it was shown in [6] that Strichartz estimates with a loss of derivative $1/p$ hold on any compact manifolds without boundaries. They also proved that the loss $1/p$ is optimal in the case of $M = \mathbb{S}^d$, $d \geq 3$. In [4] and [3], the authors proved sharp Strichartz estimates, outside a large compact set, without the nontrapping condition. More recently, it was shown in [7] that sharp Strichartz estimates still hold for the case with hyperbolic trapped trajectories of sufficiently small fractal dimension. We mention that there are also several works on global-in-time Strichartz estimates in the case of long-range perturbations of the flat Laplacian on $\mathbb{R}^d$ ([5, 33, 23]).

As we have seen, Strichartz estimates are well studied subjects for both of potential perturbation and variable coefficient cases. We however note that the literature is more sparse for the mixed case, namely the case with variable coefficients and unbounded electromagnetic potentials. In Section 2, we give a unified approach to a combination of these two kinds of results.

In Section 3, we discuss the case on a class of non-compact manifolds with polynomially growing ends, which is regarded as a generalization of results by [16, 4]. In particular, we show that if the volume density grows polynomially at infinity and is strictly larger than that of the Euclidean space, then local-in-time Strichartz estimates, outside a large compact set, hold without the asymptotic convergence condition on the angular metric. To the best knowledge of the author, this is a first example of sharp Strichartz estimates without asymptotic convergence conditions, except for the one dimensional case.

§ 2. Schrödinger equations with variable coefficients and unbounded potentials

In this section we consider Schrödinger operators with variable coefficients and electromagnetic potentials on $\mathbb{R}^d$, $d \ge 1$:

$$H = \frac{1}{2}(-i\partial_j - A_j(x))g^{jk}(x)(-i\partial_k - A_k(x)) + V(x), \quad x \in \mathbb{R}^d,$$

with the Einstein summation convention. We suppose the following:

Assumption 2.1. *$g^{jk}, A_j, V \in C^\infty(\mathbb{R}^d;\mathbb{R})$. $(g^{jk})_{j,k}$ is symmetric and uniformly elliptic: $g^{jk}(x)\xi_j\xi_k \ge c_0|\xi|^2$, $x, \xi \in \mathbb{R}^d$, with some positive constant c_0. Moreover, there exists $\mu \ge 0$ such that for any $\alpha \in \mathbb{Z}_+^d := \mathbb{N}^d \cup \{0\}$,*

$$\begin{aligned} |\partial_x^\alpha (g^{jk}(x) - \delta_{jk})| &\le C_\alpha \langle x\rangle^{-\mu-|\alpha|}, \\ |\partial_x^\alpha A_j(x)| &\le C_\alpha \langle x\rangle^{1-\mu-|\alpha|}, \\ |\partial_x^\alpha V(x)| &\le C_\alpha \langle x\rangle^{2-\mu-|\alpha|}, \quad x \in \mathbb{R}^d, \end{aligned}$$

where $\langle x\rangle$ stands for $\sqrt{1+|x|^2}$.

Under Assumption 2.1, H is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$ (see, *e.g.*, [11]) and we denote its self-adjoint extension on $L^2(\mathbb{R}^d)$ by the same symbol H.

Let $k(x,\xi) = \frac{1}{2}g^{jk}(x)\xi_j\xi_k$ be the classical kinetic energy associated to g^{jk}. Consider the Hamilton flow generated by k, that is the solution to the Hamilton system

$$\dot{x}(t) = \frac{\partial k}{\partial \xi}(x(t),\xi(t)),\ \dot{\xi}(t) = -\frac{\partial k}{\partial x}(x(t),\xi(t)); \quad (x(0),\xi(0)) = (x_0,\xi_0).$$

We then impose the following geometric conditions:

Assumption 2.2.

- *(Nontrapping condition) For any initial data $x_0, \xi_0 \in \mathbb{R}^d$ with $\xi_0 \neq 0$, $|x(t)| \to +\infty$ as $t \to \pm\infty$.*
- *(Convexity near infinity) There exists $f \in C^\infty(\mathbb{R}^d)$ satisfying $f \ge 1$ and $f \to +\infty$ as $|x| \to +\infty$ such that $f \in L^\infty(\mathbb{R}^d)$ for any $|\alpha| \ge 2$ and that*

  $$H_k^2 f(x,\xi) \ge ck(x,\xi)$$

  on $\{(x,\xi) \in \mathbb{R}^{2d};\ f(x) \ge R\}$, for some constants $c, R > 0$.

Note that if $\partial_x g^{jk} = o(|x|^{-1})$ as $|x| \to +\infty$, then the convexity condition holds. In particular, Assumption 2.1 with $\mu > 0$ implies the convexity near infinity. For more example satisfying Assumption 2.2, we refer to [11].

§ 2.1. Main results

We now state main results in this section. In the sequel, $\mathbf{1}_A$ denotes the characteristic function designated by A.

Theorem 2.3 (Subcritical case [25, 26]). (1) *Assume that Assumption 2.1 with* $\mu > 0$. *Then, there exists* $R_0 > 0$ *such that for any* $T > 0$, $p \geq 2$, $q < \infty$, $2/p = d(1/2 - 1/q)$ *and* $R \geq R_0$, *we have*

$$||\mathbf{1}_{\{|x|>R\}} e^{-itH} u_0||_{L^p([-T,T];L^q(\mathbb{R}^d))} \leq C_T ||u_0||_{L^2(\mathbb{R}^d)}, \tag{2.1}$$

where $C_T > 0$ *may be taken uniformly with respect to* R.
(2) *Assume that Assumption 2.1 with* $\mu \geq 0$. *Then, for any* $T > 0$, $p \geq 2$, $q < \infty$, $2/p = d(1/2 - 1/q)$ *and* $r > 0$, *we have*

$$||\mathbf{1}_{\{|x|<r\}} e^{-itH} u_0||_{L^p([-T,T];L^q(\mathbb{R}^d))} \leq C_{T,r} ||\langle H \rangle^{\frac{1}{2p}} u_0||_{L^2(\mathbb{R}^d)}. \tag{2.2}$$

Moreover, if we assume in addition that Assumption 2.2, then

$$||\mathbf{1}_{\{|x|<r\}} e^{-itH} u_0||_{L^p([-T,T];L^q(\mathbb{R}^d))} \leq C_{T,r} ||u_0||_{L^2(\mathbb{R}^d)}. \tag{2.3}$$

In particular, combining with (2.1) *we obtain global-in-space estimates:*

$$||e^{-itH} u_0||_{L^p([-T,T];L^q(\mathbb{R}^d))} \leq C_{T,r} ||u_0||_{L^2(\mathbb{R}^d)},$$

provided that $\mu > 0$.

For the general case, we obtain an almost optimal result:

Theorem 2.4 (Critical case [26]). *Let* $\mu \geq 0$ *and assume that Assumptions 2.1 and 2.2. Then, for any* $\varepsilon > 0, T > 0$, $p \geq 2$, $q < \infty$ *and* $2/p = d(1/2 - 1/q)$,

$$||e^{-itH} u_0||_{L^p([-T,T];L^q(\mathbb{R}^d))} \leq C_{T,\varepsilon} ||\langle H \rangle^{\varepsilon} u_0||_{L^2(\mathbb{R}^d)}.$$

Note that if $A \equiv 0$ and $V \gtrsim \langle x \rangle^{2-\mu}$, then H is uniformly elliptic. Then, using the parametrix of H, we see that $||\langle H \rangle^{\gamma} u_0||_{L^p} \approx ||\langle D \rangle^{2\gamma} u_0||_{L^p} + ||\langle x \rangle^{(2-\mu)\gamma} u_0||_{L^p}$ for $p \in (1, \infty)$ and $\gamma \geq 0$.

There are some remarks.

Remark 2.5. *(1) The estimates of forms* (2.1), (2.2) *and* (2.3) *have been proved by [31, 4] when* $A \equiv 0$ *and* V *is of long-range type. Therefore, Theorem 2.3 is regarded as a generalization of their results for the case with unbounded potential perturbations.* *(2) The only restriction for admissible pairs, in comparison to the free case, is to exclude* $(p, q) = (4, \infty)$ *for* $d = 1$*, which is due to the use of the Littlewood-Paley decomposition.* *(3) The missing derivative loss* $\langle H \rangle^{\varepsilon}$ *in Theorem 2.4 is due to the use of the following local smoothing effect (due to Doi [11]):*

$$||\langle x \rangle^{-1/2-\varepsilon} \langle D \rangle^{1/2} e^{-itH} \varphi||_{L^2([-T,T];L^2(\mathbb{R}^d))} \le C_{T,\varepsilon} ||\varphi||_{L^2(\mathbb{R}^d)}.$$

It is known that this estimate does not holds when $\varepsilon = 0$ *even for* $H = H_0$*. We would expect that Theorem 2.3 still holds true for the case with critical electromagnetic potentials in the following sense:*

$$\langle x \rangle^{-1} |\partial_x^\alpha A_j(x)| + \langle x \rangle^{-2} |\partial_x^\alpha V(x)| \le C_{\alpha\beta} \langle x \rangle^{-|\alpha|},$$

at least if g^{jk} *satisfies the bound in Assumption 2.1 with* $\mu > 0$*. However, this is beyond our techniques.*

§ 2.2. Strategy of the proof

We here explain the idea of the proof and refer to [25, 26] for the details. The general strategy is based on microlocal techniques and the Littlewood-Paley theory using the semiclassical spectral multiplier $f(H)$. We however note that, since our Hamiltonian H is not bounded below, the Littlewood-Paley estimate using H, which is of the form

$$||v||_{L^q} \lesssim ||v||_{L^2} + \Big(\sum_{j=0}^{\infty} ||f(2^{-2j} H) v||_{L^q}^2 \Big)^{1/2}, \quad f \in C_0^\infty(\mathbb{R} \setminus \{0\}), \tag{2.4}$$

seems to be false for $q \neq 2$ in general. To overcome this difficulty, we consider a partition of unity on the phase space $\mathbb{R}^{2d}$: $\psi_\varepsilon(x, \xi) + \chi_\varepsilon(x, \xi) = 1$, where ψ_ε is supported in $\{(x, \xi); \langle x \rangle < \varepsilon |\xi|\}$ for some $\varepsilon > 0$ and satisfies $\partial_x^\alpha \partial_\xi^\beta \psi_\varepsilon = O(\langle x \rangle^{-|\alpha|} \langle \xi \rangle^{-|\beta|})$. Let $p(x, \xi)$ be the full symbol of H (modulo lower order term):

$$p(x, \xi) = \frac{1}{2} g^{jk}(x)(\xi_j - A_j(x))(\xi_k - A_k(x)) + V(x).$$

It is easy to see that the symbol $p(x, \xi)$ is uniformly elliptic on $\operatorname{supp} \psi_\varepsilon$:

$$C^{-1} |\xi|^2 \le p(x, \xi) \le C |\xi|^2, \quad (x, \xi) \in \operatorname{supp} \psi_\varepsilon,$$

provided that $\varepsilon > 0$ is small enough. Therefore, H is essentially elliptic and hence $h^2 H - z$ has a semiclassical parametrix on the range of $\mathrm{Op}(\psi_\varepsilon)$, where $\mathrm{Op}(\psi_\varepsilon) := \psi_\varepsilon(x, D)$ is the standard pseudodifferential operator (PDO for short) with the symbol ψ_ε and $h \in (0,1]$ is the semiclassical parameter. Combining with the Helffer-Sjöstrand formula, namely $f(h^2H) = -\frac{1}{2\pi i}\int_{\mathbb{C}} \partial_{\bar z}\widetilde{f}(z)(h^2H-z)^{-1}dzd\bar z$, where $\widetilde{f}$ is an almost analytic extension of f (see, *e.g.*, [22]), we can see that if $f \in C_0^\infty(\mathbb{R})$ and $\operatorname{supp} f \Subset (0,\infty)$, then $\mathrm{Op}(\psi_\varepsilon)f(h^2H)$ is a semiclassical pseudodifferential operator (h-PDO) with a symbol supported in

$$\operatorname{supp}\psi_{\varepsilon/h} \cap \operatorname{supp} f\circ p_h \subset \{|x|<1/h,\ |\xi|\in I\},$$

with some $I \Subset (0,\infty)$ modulo some error term whose kernel is rapidly decaying with respect to h, where $p_h(x,\xi) := h^2p(x,\xi/h)$. Using the same argument as that in [6], we then obtain the Littlewood-Paley estimates on a range of $\mathrm{Op}(\psi_\varepsilon)$:

$$||\,\mathrm{Op}(\psi_\varepsilon)v||_{L^q} \le C_q||v||_{L^2} + C_q\bigg(\sum_{h=2^{-j},j\ge0}||\,\mathrm{Op}_h(a_h)f(h^2H)v||_{L^q}^2\bigg)^{1/2},$$

where $2\le q<\infty$, $\{f(h^2\cdot); h=2^{-j}, j\ge0\}$ is a 4-adic partition of unity on $[1,\infty)$, a_h is a h-dependent symbol, supported in $\{|x|<1/h,\ |\xi|\in I\}$, satisfying $\partial_x^\alpha\partial_\xi^\beta a_h(x,\xi) = O(\langle x\rangle^{-|\alpha|}\langle\xi\rangle^{-|\beta|})$ and $\mathrm{Op}_h(a_h) := a_h(x,hD)$ denotes the corresponding h-PDO.

The idea of the proof of Theorem 2.3 (1) then is as follows. In view of the above Littlewood-Paley type estimates, the proof is reduced to that of Strichartz estimates for $\mathbf{1}_{\{|x|>R\}}\mathrm{Op}_h(a_h)e^{-itH}$ and $\mathrm{Op}(\chi_\varepsilon)e^{-itH}$. For $\mathbf{1}_{\{|x|>R\}}\mathrm{Op}_h(a_h)e^{-itH}$, we use the semiclassical Isozaki-Kitada (IK for short) parametrix, which originally comes from long-range scattering theory with time-independent modifiers. We however note that because of the unboundedness of potentials with respect to x, it is difficult to construct directly such approximations. To overcome this difficulty, we introduce a modified Hamiltonian $\widetilde{H}$ due to [38] so that $\widetilde{H}=H$ for $|x|\le L/h$ and $\widetilde{H}=K$ for $|x|\ge 2L/h$ for some constant $L\ge1$, where $K=-\sum_{j,k}\partial_jg^{jk}\partial_k/2$ is the kinetic energy part of H. Then, $\widetilde{H}^h = h^2\widetilde{H}$ can be regarded as a "long-range perturbation" of the semiclassical free Schrödinger operator $H_0^h = h^2H_0$. Indeed, if we denote the corresponding classical symbol by $\widetilde{p}_h(x,\xi)$ *i.e.*, $\widetilde{p}_h(x,\xi) = p_h(x,\xi)$ for $|x|\le L/h$ and $\widetilde{p}_h(x,\xi) = k(x,\xi)$ for $|x|\ge 2L/h$, then

$$|\partial_x^\alpha\partial_\xi^\beta(p_h(x,\xi)-|\xi|^2/2)| \le C_{L\alpha\beta}\langle x\rangle^{-\mu-|\alpha|}\langle\xi\rangle^{2-|\beta|},\quad h\in(0,1].$$

Let $a_h^\pm$ be symbols supported in $\{R/2<|x|<1/h,\ |\xi|\in I,\ \pm\hat x\cdot\hat\xi>1/2\}$, respectively, so that $\mathbf{1}_{\{|x|>R\}}a_h = a_h^+ + a_h^-$, where $\hat x = x/|x|$. Rescaling $t\mapsto th$, we first construct the semiclassical IK parametrices for $e^{-it\widetilde{H}^h/h}\mathrm{Op}_h(a_h^\pm)^*$ of the forms

$$e^{-it\widetilde{H}^h/h}\mathrm{Op}_h(a_h^\pm)^* = J_h(S_h^\pm, b_h^\pm)e^{-itH_0^h/h}J_h(S_h^\pm, c_h^\pm)^* + O(h^N),\quad 0\le\pm t\le 1/h,$$

respectively, where $S_h^\pm$ solve the Eikonal equation associated to $\widetilde{p}_h$:

$$\widetilde{p}_h(x,\partial_xS_h^\pm) = |\xi|^2/2\quad \text{on a neighborhood of } \operatorname{supp} a_h^\pm,$$

$b_h^\pm$ and $c_h^\pm$ are supported in a neighborhood of $\operatorname{supp} a_h^\pm$, respectively, and $J_h(S_h^\pm, b_h^\pm)$ and $J_h(S_h^\pm, c_h^\pm)$ are associated semiclassical Fourier integral operators (h-FIOs):

$$J_h(S_h^\pm, w)u(x) = (2\pi h)^{-d} \int e^{i(S_h^\pm(x,\xi)-y\cdot\xi)/h} w(x,\xi)u(y)dyd\xi.$$

The method of the construction is similar to as that of Robert [29]. On the other hand, we can see that if $L \ge 1$ is large enough, then the Hamilton flow generated by $\widetilde{p}_h$ with initial conditions in $\operatorname{supp} a_h^\pm$ cannot escape from $\{|x| \le L/h\}$ for $0 < \pm t \le 1/h$, *i.e.*,

$$\pi_x \left(\exp tH_{\widetilde{p}_h}(\operatorname{supp} a_h^\pm)\right) \subset \{|x| \le L/h\}, \quad 0 < \pm t \le 1/h.$$

Since $\widetilde{p}_h = p_h$ for $|x| \le L/h$, we have $\exp tH_{\widetilde{p}_h}(\operatorname{supp} a_h^\pm) = \exp tH_{p_h}(\operatorname{supp} a_h^\pm)$ for any $0 < \pm t \le 1/h$, respectively. We thus can expect (at least formally) that the corresponding two quantum evolutions are approximately equivalent modulo some smoothing operator. By using the Duhamel formula and the semiclassical IK parametrix, we can prove the following rigorous justification of this formal consideration:

$$||(e^{-itH^h/h} - e^{-it\widetilde{H}^h/h}) \operatorname{Op}_h(a_h^\pm)^*||_{L^2\to L^2} \le C_M h^M, \quad 0 \le \pm t \le 1/h,\ M \ge 0,$$

where $H^h = h^2 H$. By using such approximations for $e^{-itH^h/h} \operatorname{Op}_h(a_h^\pm)^*$, we prove local-in-time dispersive estimates for $\operatorname{Op}_h(a_h^\pm)e^{-itH} \operatorname{Op}_h(a_h^\pm)^*$:

$$|| \operatorname{Op}_h(a_h^\pm)e^{-itH} \operatorname{Op}_h(a_h^\pm)^*||_{L^1\to L^\infty} \le C|t|^{-d/2}, \quad 0 < h \ll 1,\ 0 < |t| < 1.$$

Strichartz estimates then follow from these estimates and the TT^*-argument.

The estimates for $\operatorname{Op}(\chi_\varepsilon)e^{-itH}$ follow from the short-time dispersive estimate:

$$|| \operatorname{Op}(\chi_\varepsilon)e^{-itH} \operatorname{Op}(\chi_\varepsilon)^*||_{L^1\to L^\infty} \le C_\varepsilon |t|^{-d/2}, \quad 0 < |t| < t_\varepsilon \ll 1.$$

To prove this, we first construct the WKB parametrix for $e^{-itH} \operatorname{Op}(\chi_\varepsilon)^*$ of the form:

$$e^{-itH} \operatorname{Op}(\chi_\varepsilon)^* = J(\Psi, a) + O_{H^{-\gamma}\to H^\gamma}(1), \quad |t| < t_\varepsilon,\ \gamma > d/2,$$

where the phase function $\Psi = \Psi(t, x, \xi)$ is a solution to a time-dependent Hamilton-Jacobi equation associated to $p(x, \xi)$ and $J(\Psi, a)$ is the corresponding Fourier integral operator. In the construction, the following fact plays an important rule:

$$|\partial_x^\alpha \partial_\xi^\beta p(x,\xi)| \le C_{\alpha\beta}, \quad (x,\xi) \in \operatorname{supp} \chi_\varepsilon,\ |\alpha + \beta| \ge 2.$$

(Note that if $(g^{jk})_{j,k}$ depends on x then these bounds do not hold without such a restriction of the phase space.) Using these bounds, we construct the phase function $\Psi(t, x, \xi)$ such that

$$|\partial_x^\alpha \partial_\xi^\beta (\Psi(t,x,\xi) - x\cdot\xi + p(x,\xi))| \le C_{\alpha\beta} |t|^2 \langle x\rangle^{2-|\alpha+\beta|}.$$

We then can follow a classical argument (due to, *e.g.*, [19]) and construct the FIO $J(\Psi, a)$. By the composition formula, $\mathrm{Op}(\chi_\varepsilon)J(\Psi, a)$ is also a FIO and dispersive estimates for this operator follow from the standard stationary phase method. Finally, using an Egorov type lemma, we prove that the remainder, $\mathrm{Op}(\chi_\varepsilon)(e^{-itH}\,\mathrm{Op}(\chi_\varepsilon)^* - J(\Psi, a))$, has a smooth, uniformly bounded kernel for sufficiently small t.

The proof of Theorem 2.3 (2) is based on a standard idea by [31], see also [6, 4]. Strichartz estimates with loss of derivatives $\langle H\rangle^{1/(2p)}$ follow from semiclassical Strichartz estimates up to time scales of order h, which can be verified by the standard WKB method. Moreover, under the nontrapping condition, we will prove that the missing $1/p$ derivative loss can be recovered by using the local smoothing effect due to Doi [11].

The proof of Theorem 2.4 is based on a slight modification of that of Theorem 2.3 (2). By virtue of the Strichartz estimates for $\mathrm{Op}(\chi_\varepsilon)e^{-itH}$ and the above Littlewood-Paley estimates, it suffices to show

$$||\,\mathrm{Op}_h(a_h)e^{-itH}\varphi||_{L^p([-T,T];L^q)} \le C_T h^{-\varepsilon}||\varphi||_{L^2}, \quad 0 < h \ll 1.$$

To prove this, we first prove semiclassical Strichartz estimates for $e^{-itH}\,\mathrm{Op}_h(a_h)^*$ up to time scales of order hR, where $R = \pi_x(\inf \operatorname{supp} a_h)$. The proof is based on a refinement of the standard WKB method for the semiclassical propagator $e^{-itH^h/h}\,\mathrm{Op}_h(a_h)^*$. Combining semiclassical Strichartz estimates with a partition of unity argument with respect to x, we will obtain the following Strichartz estimate with an inhomogeneous error term:

$$\begin{aligned}&||\,\mathrm{Op}_h(a_h)e^{-itH}\varphi||_{L^p([-T,T];L^q)}\\ &\le C_T||\varphi||_{L^2} + C||\langle x\rangle^{-1/2-\varepsilon}h^{-1/2-\varepsilon}\,\mathrm{Op}_h(a_h)e^{-itH}\varphi||_{L^2([-T,T];L^2)},\end{aligned}$$

for any $\varepsilon > 0$, which, combined with the local smoothing effect, implies the assertion. □

§ 3. Schrödinger equations on manifolds with ends

In this section we consider the following model. Let (M, g) be a smooth, connected complete Riemannian manifold of dimension $d \ge 2$ such that M is decomposed into two parts $M = M_c \cup M_\infty$, where $M_c \Subset M$ is a d-dimensional relatively compact open submanifold and M_∞ is diffeomorphic to $(0, \infty) \times S$ with a $(d-1)$-dimensional smooth closed manifold S. We suppose that there exists $R_M \ge 1$ such that g takes the form

$$g = dr^2 + r^{2\sigma}g_S(r) \quad \text{on } [R_M, \infty) \times S,$$

where $\sigma \ge 1$ and $g_S(r)$ is a family of smooth Riemannian metrics on S smoothly depending on r. In local coordinates, $g_S(r)$ is of the form $g_S(r) = g_{S,jk}(r, \theta)d\theta^j d\theta^k$ using Einstein's summation convention.

Let $k_S \in C^\infty([R_M,\infty)\times T^*S;\mathbb{R})$ be the classical kinetic energy associated to $g_S(r)$, that is the principal symbol of the free Schrödinger operator $-\frac{1}{2}\Delta_{g_S(r)}$ on S associated to $g_S(r)$, which, in local coordinates, is of the form

$$k_S(r,\theta,\omega) := \frac{1}{2}g_S^{jk}(r,\theta)\omega_j\omega_k, \quad r\in[R_M,\infty),\ (\theta,\omega)\in T^*S,$$

where $(g_S^{jk}) = (g_{S,jk})^{-1}$. For sufficiently large $R \geq R_M$, we then impose that

- (Uniform ellipticity) There exists a constant $c_0 > 0$ such that

$$(g^{jk}(r,\theta))_{j,k} \geq c_0\,\mathrm{Id}, \quad (r,\theta)\in[R,+\infty)\times S. \tag{3.1}$$

- (Symbol-type estimates of order zero) For any $(l,\alpha)\in\mathbb{Z}_+^d := \mathbb{N}^d\cup\{0\}$, g_S^{jk} obeys

$$|\partial_r^l\partial_\theta^\alpha g_S^{jk}(r,\theta)| \leq C_{l\alpha}r^{-l}, \quad (r,\theta)\in[R,+\infty)\times S. \tag{3.2}$$

We also consider the following two conditions:

- (Convex near infinity) There exists $\varepsilon > 0$ such that

$$(2\sigma-\varepsilon)g_S^{jk}(r,\theta) \geq r\frac{\partial g_S^{jk}}{\partial r}(r,\theta), \quad (r,\theta)\in[R,\infty)\times S. \tag{3.3}$$

- (Long-range type condition) There exist a smooth positive $(2,0)$-tensor $(h_S^{jk})_{j,k}$ on S, independent of r, and a constant $\mu > 0$ such that

$$|\partial_r^l\partial_\theta^\alpha(g_S^{jk}(r,\theta) - h_S^{jk}(\theta))| \leq C_{l\alpha}r^{-\mu-l}, \quad (r,\theta)\in[R,+\infty)\times S. \tag{3.4}$$

Remark 3.1. *Let us fix $R \geq R_M$ and set $\tau_0 = \sup_{l=1}||r\partial_r^l g^{jk}||_{L^\infty((R,\infty)\times S)}$. Since $k_S \geq c_0|\omega|^2$ by* (3.1)*, if $\tau_0 < 2\sigma c_0$ then* (3.3) *holds with $\varepsilon = 2\sigma - \tau_0/c_0$. In particular, if $\partial_r g^{jk} = o(r^{-1})$, $r\to+\infty$, then* (3.3) *is satisfied.* (3.3) *hence is strictly weaker than the long-range type condition* (3.4)*.*

Setting $L^p(M) = L^p(M, G(x)dx)$ with $G(x) = \sqrt{\det g(x)}$, we consider the time-dependent Schrödinger equation:

$$i\partial_t u = -\frac{1}{2}\Delta_g u + V(x)u; \quad u|_{t=0} = u_0 \in L^2(M), \tag{3.5}$$

where Δ_g is the Laplace-Beltrami operator associated to g which, in any local coordinates $x = (x^1,...,x^d)\in M$, has the form

$$\Delta_g = \frac{1}{G(x)}\partial_{x^l}g^{lm}(x)G(x)\partial_{x^m},\ (g^{lm}(x)) = (g_{lm}(x))^{-1}.$$

For the potential V we impose the long-range type condition:

Assumption 3.2. *$V \in C^\infty(M;\mathbb{R})$ and there exists $\nu > 0$ such that*

$$|\partial_r^l \partial_\theta^\alpha V(r,\theta)| \le C_{l\alpha} r^{-\nu-l}, \quad (r,\theta) \in [R_M,+\infty) \times S. \tag{3.6}$$

By the completeness of M and Assumption 3.2, it is well-known that $-\frac{1}{2}\Delta_g + V$ is essentially self-adjoint on $C_0^\infty(M)$ and we denote its self-adjoint extension on $L^2(M)$ by H. By the Stone theorem, we have a unique unitary propagator e^{-itH} on $L^2(M)$ generated by H such that the solution to (3.5) is given by $u(t) = e^{-itH}u_0$.

To state the main result, we recall the non-trapping condition. Let

$$k(x,\xi) = \frac{1}{2} g^{lm}(x)\xi_l \xi_m, \quad (x,\xi) \in T^*M,$$

be the classical kinetic energy associated to g and let $H_k = \partial_\xi k \cdot \partial_x - \partial_x k \cdot \partial_\xi$ the corresponding Hamilton vector field. By the completeness of M, for any $(x,\xi) \in T^*M$, the Hamilton flow $\exp tH_k(x,\xi)$, generated by H_k, exists for all $t \in \mathbb{R}$. We say that M *is non-trapping* if for any $(x,\xi) \in T^*M \setminus 0$, $\pi(\exp tH_k(x,\xi))$ escapes from any compact set in M as $t \to \pm\infty$, where $\pi : T^*M \to M$ is the projection onto the base space.

§ 3.1. Main results

We now state main results in this section. For the conic case, we obtain Strichartz estimates under the long-range type condition on the angular kinetic energy:

Theorem 3.3 ([24]). *Let $\sigma = 1$. Assume that* (3.1), (3.2) *and* (3.4) *and that Assumption 3.2. Then, there exist a compact set $K \subset M$ and $\chi_K \in C_0^\infty(M)$ satisfying $\chi_K \equiv 1$ on K such that for any $T > 0$ and any admissible pair (p,q),*

$$||(1-\chi_K)e^{-itH}u_0||_{L^p([-T,T];L^q(M))} \le C_T ||u_0||_{L^2(M)}. \tag{3.7}$$

Moreover, if we assume in addition that M is non-trapping then

$$||e^{-itH}u_0||_{L^p([-T,T];L^q(M))} \le C_T ||u_0||_{L^2(M)} \tag{3.8}$$

for any admissible pair (p,q).

When $\sigma > 1$, the same result holds under the convexity condition which is weaker than the long-range condition.

Theorem 3.4 ([27]). *Let $\sigma > 1$. Assume that* (3.1), (3.2) *and* (3.3) *and that Assumption 3.2. Let χ_K be as above. Then, $(1-\chi_K)e^{-itH}$ satisfies local-in-time Strichartz estimates* (3.7) *for any admissible pair (p,q). Under the non-trapping condition, global-in-space estimates* (3.8) *also hold.*

Remark 3.5. *For the asymptotically conic case,* (3.7) *and* (3.8) *have been proved by Hassel-Wunsch-Tao [16] for $p > 2$, however the method of the proof is considerably different. In [3], Bouclet proved* (3.7) *and* (3.8) *for the case on the asymptotically hyperbolic manifold, which is a non-compact manifold M as above equipped with the metric g having the from $g = dr^2 + e^{2r} g_S(r)$, $r \ge R_M$, where $g_S(r)$ satisfies* (3.4)*. The present article is motivated by his work and our proof is based on his idea. Theorem 3.4 may be regarded as an interpolation between [16] and [3].*

§ 3.2. Strategy of the proof

We here give the idea of the proof only and refer to [24, 25] for the details. We only consider the estimate (3.7) for the case when $\sigma > 1$ (The estimates on compact sets are verified by a standard argument due to Staffilani-Tataru [31], see also Bouclet-Tzvetkov [4]). The general strategy is similar to that in the previous section, though the construction of parametrices is slightly different.

First of all, under conditions (3.1) and (3.2), it has been showed by [1] that the Littlewood-Paley estimates of forms (2.4) hold for any $q \in [2, \infty)$. Hence, it suffices to show (3.7) that $(1 - \chi_K) f(h^2 H) e^{-itH}$ satisfies Strichartz estimates uniformly in $h \in (0, 1]$. We next embed the solution into the conic manifold as follows. Let $v(t) = \langle r \rangle^{\sigma(d-1)/2} e^{-itH} u_0$. It is easy to see that $v(t)$ solves $i\partial_t v(t) = \widehat{H} v(t)$ with the initial state $v(0) = \langle r \rangle^{\sigma(d-1)/2} u_0 \in \widehat{L}^2(M)$, where $\widehat{L}^p(M) := L^p(M, \langle r \rangle^{-\sigma(d-1)} G(x) dx)$ and

$$\widehat{H} := \langle r \rangle^{\sigma(d-1)/2} H \langle r \rangle^{-\sigma(d-1)/2},$$

which is self-adjoint on $\widehat{L}^2(M)$. Then, it is sufficient to prove (3.7) that

$$||\langle r \rangle^{\sigma(d-1)/2} (1 - \chi_K) f(h^2 \widehat{H}) e^{-it\widehat{H}} v_0||_{L^p([-T,T]; L^q(M))} \le C_T ||v_0||_{\widehat{L}^2(M)}. \tag{3.9}$$

Assuming for simplicity that $S = \mathbb{S}^{d-1}$ and $V \equiv 0$, we set $H = -\Delta_g/2$. The corresponding kinetic energy is written in the form

$$k(r, \rho, \omega) = \frac{1}{2}\rho^2 + \frac{1}{2} r^{-2\sigma} k_S(r, \theta, \omega) \quad \text{on } T^*M_\infty \cong T^*\mathbb{R} \times T^*S,$$

where ρ (reap. ω) is the dual variable of r (reap. θ). Since $\langle r \rangle^{-\sigma(d-1)} G(r, \theta) \approx 1$ in M_∞, we can use the standard h-PDO calculus and obtain that the spectral multiplier (near infinity) $(1-\chi_K) f(h^2 \widehat{H})$ can be approximated by a h-PDO, $\mathrm{Op}_h(a) := a(r, \theta, hD_r, hD_\theta)$, modulo some smoothing term, where $a \in C^\infty(T^*M_\infty)$ satisfies

$$|\partial_r^j \partial_\theta^\alpha \partial_\rho^l \partial_\omega^\beta a(r, \theta, \rho, \omega)| \le C_{j\alpha l\beta} \langle r \rangle^{-j-\sigma|\beta|} \quad \text{in } M_\infty,$$

and is supported in $\Gamma(R) = \{(r, \theta, \rho, \omega);\ r > R,\ \theta \in S,\ k(r, \rho, \omega) \in I\}$ with some $R \gg 1$ and $I \Subset (0, \infty)$. We then split Γ into outgoing ("+") and incoming ("−") regions

$$\Gamma^\pm(R) = \{r > R,\ \theta \in S,\ k \in I,\ \rho > \pm(1/2)\sqrt{2k}\}.$$

In what follows, we consider the outgoing case only since the proof for the incoming case is analogous. In the asymptotically Euclidean case, one can construct a long-time parametrix of the propagator $e^{-it\widehat{H}^h/h}\operatorname{Op}_h(a^+)^*$ as in the previous section, where $\widehat{H}^h = h^2\widehat{H}$, $0 \le t \le h^{-1}$ and a^+ is supported in Γ^+. However, this is not the present case since $\partial_\theta k_s$ does not decay at spatial infinity. To overcome this difficulty, following the idea by Bouclet [3] we decompose $a^+ = a_s^+ + a_i^+$, where a_s^+ and a_i^+ are supported in the *strongly outgoing* and *intermediate* regions:

$$\Gamma^+_{\mathrm{stg}}(R) = \Gamma^+(R) \cap \{r^{-2\sigma}k_S \le \varepsilon\},\ \Gamma^+_{\mathrm{int}}(R) = \Gamma^+(R) \setminus \Gamma^+_{\mathrm{stg}}(R), \quad 0 < \varepsilon \ll 1,$$

respectively. In $\Gamma^+_{\mathrm{stg}}(R)$, we obtain a long-time behavior of the classical system:

$$\begin{aligned} &|\partial_{r_0}^j\partial_{\theta_0}^\alpha\partial_{\rho^0}^k\partial_{\omega^0}^\beta(r^t(X_0) - r_0 - t\rho^0)| \lesssim \varepsilon^2|t|,\ |\partial_{r_0}^j\partial_{\theta_0}^\alpha\partial_{\rho^0}^k\partial_{\omega^0}^\beta(\theta^t(X_0) - \theta_0)| \lesssim \varepsilon,\\ &\quad |\partial_{r_0}^j\partial_{\theta_0}^\alpha\partial_{\rho^0}^k\partial_{\omega^0}^\beta(\rho^t(X_0) - \rho^0)| + \langle r_0\rangle^{-1}|\partial_{r_0}^j\partial_{\theta_0}^\alpha\partial_{\rho^0}^k\partial_{\omega^0}^\beta(\omega^t(X_0) - \omega^0)| \lesssim \varepsilon^2, \end{aligned}$$

for $t \ge 0$ and $X_0 = (r_0, \theta_0, \rho^0, \omega^0) \in \Gamma^+_{\mathrm{stg}}(R)$, where $(r^t, \theta^t, \rho^t, \omega^t) = \exp tH_k$ is the Hamilton flow in T^*M_∞. We here have used the assumption (3.3) and the fact that $\sigma > 1$. When $\sigma = 1$, (3.3) is not sufficient to obtain these estimates and we need to assume (3.4). These estimates tell us that the strong outgoing region is invariant under the Hamilton flow for any $t \ge 0$ if $\varepsilon > 0$ is sufficiently small. Taking $\varepsilon > 0$ small enough and using a same argument as that in [3], we then can construct the semiclassical IK parametrix of the form

$$e^{-it\widehat{H}^h/h}\operatorname{Op}_h(a_s^+)^* = J_h(S^+, b_h^+)e^{ith\partial_r^2/2}J_h(S^+, c_h^+)^* + O_{\widehat{L}^2\to\widehat{L}^2}(h^N), \quad 0 \le t \le h^{-1}.$$

Here $J_h(S^+, b_h^+)$ and $J_h(S^+, c_h^+)$ are h-FIOs with symbols $b_h^+, c_h^+ \in C_b^\infty(T^*M_\infty)$ supported in a strongly outgoing region and the phase S^+ solves the Eikonal equation:

$$k(r, \theta, \partial_r S^+, \partial_\theta S^+) = \rho^2/2 \text{ on a neighborhood of } \operatorname{supp} a_s^+.$$

Moreover, S^+ is essentially of the form

$$S^+(r, \theta, \rho, \omega) = r\rho + \theta\cdot\omega + \frac{1}{\rho}\int_0^\infty \frac{k_S(r+\lambda, \theta, \omega)}{(r+\lambda)^{2\sigma}}d\lambda. \tag{3.10}$$

and satisfies $\partial_r^j\partial_\theta^\alpha\partial_\rho^l\partial_\omega^\beta(S^+ - r\rho - \theta\cdot\omega) = O(r^{1-j-\sigma|\beta|}(r^{-2\sigma}k_S)^{1-|\beta|/2})$. We here note that these estimates are even worse than that of the both of asymptotically Euclidean and asymptotically hyperbolic cases. Indeed, $\partial_{\rho,\omega}\otimes\partial_{r,\theta}S^+$ is not bounded in general since $\partial_\theta^\alpha\partial_\rho^m S^+$ can be grow linearly as $r \to +\infty$, while, in the above two cases, we see that $\partial_{\rho,\omega}\otimes\partial_{r,\theta}S^+ \approx \mathrm{Id}$. We, however, see that $\det\partial_{\rho,\omega}\otimes\partial_{r,\theta}S^+ \approx 1$ if $\langle r\rangle^{-2\sigma}k_S$ is small enough and $\sigma \ge 1$. Using this non-degeneracy, we can make a change of variables $(\rho, \omega) \mapsto (\rho_+, \omega_+)$, where $(\rho_+, \omega_+) = (\rho_+, \omega_+)(r, \theta, r_0, \theta_0, \rho, \omega)$ is the inverse of

$(\rho,\omega) \to \int_0^1 (\partial_{r,\theta}S^+)(\lambda r + (1-\lambda)r_0, \lambda\theta + (1-\lambda)\theta_0, \rho, \omega)d\lambda$. The distribution kernel of the IK parametrix $J_h(S^+, b_h^+)e^{ith\partial_r^2/2}J_h(S^+, c_h^+)^*$ then reads

$$\frac{1}{(2\pi h)^d}\int e^{ith^{-1}\Phi^+(t,r,\theta,r_0,\theta_0,\rho,\omega)}A^+(r,\theta,r_0,\theta_0,\rho,\omega)d\rho d\omega,$$

where A^+ and all of its derivatives are uniformly bounded and

$$\Phi^+(t,r,\theta,r_0,\theta_0,\rho,\omega) = \frac{r-r_0}{t}\rho + \frac{\theta-\theta_0}{t}\cdot\omega - \frac{1}{2}\rho_+(r,\theta,r_0,\theta_0,\rho,\omega)^2.$$

Using the expression (3.10) and estimates (3.1), (3.2) and (3.3), we learn that ρ_+^2 is essentially of the form $\rho_+^2 = \rho^2 + q_+(r,\theta,r_0,\theta_0,\omega)$, where

$$q_+(r,\theta,r_0,\theta_0,\omega) = q_+^{jk}(r,\theta,r_0,\theta_0)\omega_j\omega_k,\ q_+^{jk}(r,\theta,r_0,\theta_0) \gtrsim \begin{cases} r^{-2\sigma+1}r_0^{-1}\,\mathrm{Id}_{\mathbb{R}^{d-1}} & \text{if } t \le 0, \\ r^{-1}r_0^{-2\sigma+1}\,\mathrm{Id}_{\mathbb{R}^{d-1}} & \text{if } t \ge 0. \end{cases}$$

Then, the stationary phase method shows that $J_h(S^+, b^+)e^{ith\partial_r^2/2}J_h(S^+, c^+)^*$ satisfies a weighted $L^1 \to L^\infty$ estimate

$$||\langle r\rangle^{-\sigma(d-1)/2}J_h(S^+,b^+)e^{ith\partial_r^2/2}J_h(S^+,c^+)^*\langle r\rangle^{\sigma(d-1)/2}||_{L^1\to L^\infty} \lesssim \min(|th|^{-d/2}, h^{-d}),$$

from which, combining with the $\widehat{L}^2 \to L^2$ boundedness of $\langle r\rangle^{-\sigma(d-1)/2}e^{-it\widehat{H}}$, we obtain

$$||\langle r\rangle^{-\sigma(d-1)/2}\,\mathrm{Op}_h(a_s^+)e^{-it\widehat{H}}v_0||_{L^p([-T,T];L^q(M))} \le C_T||v_0||_{\widehat{L}^2(M)}. \tag{3.11}$$

For the intermediate case, choosing $\delta > 0$ small enough and splitting the interval $(\varepsilon/2, 1]$ into small intervals $I_{\delta,l}$ of size δ, $l \lesssim 1/\delta$, we decompose $\Gamma_{\mathrm{int}}^+(R)$ as follows:

$$\Gamma_{\mathrm{int}}^+(R) \subset \bigcup_{l\lesssim 1/\delta}\Gamma_{\mathrm{int}}^+(R)\cap\{(2k)^{-1}r^{-2\sigma}k_S \in I_{\delta,l}\} = \bigcup_{l\lesssim 1/\delta}\Gamma_{\mathrm{int},\delta,l}^+(R).$$

We also set $\Omega_{\mathrm{int},\delta,l}^+(R) = \Gamma_{\mathrm{int},\delta,l}^+(R)\cap\{R<r<4R\}$. Then, we obtain a behavior of the corresponding classical system:

$$|\partial^\alpha(\exp tH_k(X_0) - X_0)| \lesssim \langle r_0\rangle^{-1}|t| \tag{3.12}$$

if $X_0 = (r_0,\theta_0,\rho^0,\omega^0) \in \Gamma^+(R)$ and $0 \le t \lesssim \langle r_0\rangle$. Although we cannot obtain the precise long-time behavior as in the strongly outgoing case, the following support property holds: for all $0 < \varepsilon \ll 1$ and $\varepsilon_1 > 0$, we can find $\delta = \delta(\varepsilon,\varepsilon_1) > 0$ such that, for sufficiently large $R_1 \ge R > 0$,

$$\Gamma_{\mathrm{int},\delta,l}^+(R)\cap\exp tH_k\left(\Omega_{\mathrm{int},\delta,l}^+(R_1)\right) = \emptyset \quad \text{if } t \ge R_1\varepsilon_1. \tag{3.13}$$

Let us fix $\varepsilon > 0$ such that (3.11) holds. Using the dyadic partition of unity $\{\chi_j\}$ with respect to r-variable, we split $a_i^+ = a_{i,1}^+ + a_{i,2}^+ + a_{i,3}^+ + \cdots$, where $a_{i,j}^+ = \chi_j a_i^+$. Then, we learn by (3.12) that there exists $\varepsilon_1 > 0$ such that we can construct the standard WKB type parametrix of $e^{-it\widehat{H}^h/h}\operatorname{Op}_h(a_{i,j}^+)^*$ for $|t| \le \varepsilon_1 2^j$ (see [24]) and hence obtain

$$||\langle r\rangle^{-\sigma(d-1)/2}\operatorname{Op}_h(a_i^+)e^{it\widehat{H}^h/h}\operatorname{Op}_h(a_{i,j}^+)^*\langle r\rangle^{\sigma(d-1)/2}||_{L^1\to L^\infty} \lesssim |th|^{-d/2}, \tag{3.14}$$

for $0 < |t| \le \varepsilon_1 2^j$, uniformly with respect to h and j. On the other hand, splitting

$$a_i^+ = \sum_l a_i^{+,l},\ a_{i,j}^+ = \sum_l a_{i,j}^{+,l}$$

with $\operatorname{supp} a_i^{+,l} \subset \Gamma^+_{\mathrm{int},\delta,l}(R)$, $\operatorname{supp} a_{i,j}^{+,l} \subset \Omega^+_{\mathrm{int},\delta,l}(2^j)$ and $\delta > 0$ depending on $\varepsilon, \varepsilon_1$, using the support property (3.13) and the Egorov type lemma, we see that

$$||\operatorname{Op}_h(a_i^{+,l})e^{it\widehat{H}^h/h}\operatorname{Op}_h(a_{i,j}^{+,l})^*||_{\widehat{L}^2\to\widehat{L}^2} = O(h^\infty), \quad \varepsilon_1 2^j \le t \le h^{-1},\ l \lesssim \delta^{-1}, \tag{3.15}$$

uniformly in h and j. The estimates (3.14), (3.15), the Sobolev embedding imply

$$||\langle r\rangle^{-\sigma(d-1)/2}\operatorname{Op}_h(a_i^{+,l})e^{it\widehat{H}^h/h}\operatorname{Op}_h(a_i^{+,l})^*\langle r\rangle^{\sigma(d-1)/2}||_{L^1\to L^\infty} \lesssim |th|^{-d/2}, \tag{3.16}$$

for $0 < t \le h^{-1}$, uniformly in h. We here have used the fact that

$$\operatorname{Op}_h(a_{i,j}^{+,l}) = \operatorname{Op}_h(a_{i,j}^{+,l})\widetilde{\chi}_j + O((2^{-j}h)^\infty)$$

and that $\sum_j ||\widetilde{\chi}_j f||_{L^1} \lesssim ||f||_{L^1}$, where $\widetilde{\chi}_j\chi_j \equiv \chi_j$ and $\operatorname{supp}\widetilde{\chi}_j \subset \{r \approx 2^j\}$. The former follows from the standard off-diagonal decay of h-PDOs. By the TT^*-argument, we then conclude

$$||\langle r\rangle^{-\sigma(d-1)/2}\operatorname{Op}_h(a_i^+)e^{-it\widehat{H}}v_0||_{L^p([-T,T];L^q(M)} \le C_T||v_0||_{\widehat{L}^2(M)},$$

which, combined with (3.11), implies (3.9). □

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