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On control of Sobolev norms for some semilinear wave equations with localized data

By

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Abstract

The purpose of this paper is to give an overview of the proof of the result obtained in [15]. Consider the semilinear wave equations

$$\begin{cases} \partial_{tt}u - \triangle u = -|u|^{p-1}u \\ u(t=0) \quad = u_0 \\ \partial_t u(t=0) = u_1 \end{cases} \tag{0.1}$$

on $\mathbb{R}^3$ with $3 \leq p < 5$, data (u_0, u_1) lying in the $H^s \times H^{s-1}$ $(s < 1)$ closure of smooth functions that are compactly supported inside a ball $B(O,R)$. We establish new bounds of the H^s norms of the solution. In order to do that, we perform an analysis in a neighborhood of the cone, using the finite speed of propagation, an almost conservation law, an almost Shatah-Struwe estimate [16], and a low-high frequency decomposition [3, 4]. This allows to establish a decay estimate pointwise-in-time and to estimate the low frequency component of the H^s norm of the solution. Then, in order to estimate the H^s norm of the high frequency component of the position and the H^{s-1} norm of the velocity, we estimate the variation of another almost conservation law.

§ 1. Introduction

The global existence of smooth solutions of (0.1) was solved in [7] for the range $3 \leq p < 5$. The critical power (i.e $p = 5$) was solved in [12] for small data, in [17] for large and radial data and in [6] for large and general data.
The construction of local solutions with rougher data was studied by many authors. It is known (see for example [10]) that (0.1) is locally well-posed in $H^s \times H^{s-1}$ for $s \geq s_c := \frac{3}{2} - \frac{2}{p-1}$. By that we mean that

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- given $(u_0, u_1) \in H^s \times H^{s-1}$ there exist a time of local existence $T_l > 0$ and a unique $(u, \partial_t u)$ lying in a subspace of $\mathcal{C}([0, T_l], H^s) \times \mathcal{C}([0, T_l], H^{s-1})$ such that u satisfies the Duhamel formula for all $t \in [0, T_l]$, i.e

$$
\begin{aligned}
u(t) &= \cos(tD)u_0 + \frac{\sin(tD)}{D}u_1 - \int_0^t \frac{\sin\left((t-t')D\right)}{D}\left[|u|^{p-1}(t')u(t')\right] dt' \\
&=: \Psi_t(u_0, u_1)
\end{aligned}
\tag{1.1}
$$

- $(u_0, u_1) \to \Psi_t(u_0, u_1)$ is uniformly continuous in the $H^s \times H^{s-1}$ topology

Moreover, if $s > s_c$, then the time of local existence depends on the size of the initial data, i.e $T_l := T_l\left(\|(u_0, u_1)\|_{H^s \times H^{s-1}}\right)$ [1]. The next stage is to extend the construction of these solutions for larger times. By iterating the local well-posedness theory, one can define the maximal interval of existence $I_{max} := (-T_{max}, T_{max})$. If $T_{max} = \infty$, then we say that the solution exists globally-in-time. By the local well-posedness theory, the global behavior of H^s solutions of (0.1) is closely related the growth of the Sobolev norms $\|(u(T), \partial_t u(T))\|_{H^s \times H^{s-1}}$. In particular, if one can find a finite bound of $\|(u(T), \partial_t u(T))\|_{H^s \times H^{s-1}}$ for all time T, then one can prove that the H^s solutions of (0.1) exist for all time T. The equation (0.1) satisfies the following energy conservation law

$$
\begin{aligned}
E(u(t)) &:= \frac{1}{2}\int_{\mathbb{R}^3} |\partial_t u(t,x)|^2\, dx + \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u(t,x)|^2\, dx + \frac{1}{p+1}\int_{\mathbb{R}^3} |u(t,x)|^{p+1}\, dx \\
&= E(u(0)).
\end{aligned}
\tag{1.2}
$$

It is straightforward to see from the conservation of (1.2) that H^1 solutions of (0.1) exist for all time. It remains to better understand the global behavior of H^s solutions of (0.1) if $s < 1$. This question is delicate since there is no known conservation law at these levels of regularity. It has been studied in [1, 9, 5, 14, 13] (see [11] for higher dimensions). To our knowledge the best results regarding the optimal index of regularity for which the solution exists globally in time are the following ones:

- $p = 3$: $s > \frac{13}{18}$ for general data ([14]) and $s > \frac{7}{10}$ for radial data ([13])
- $5 > p > 3$: $s > s_p := \frac{26p - 3p^2 - 39}{2(p-1)(7-p)}$ for general data lying in slightly different spaces, i.e $(u_0, u_1) \in \dot{H}^s \cap L^{p+1} \times \dot{H}^{s-1}$ ([9]).

Moreover the H^s norm of the high frequency component of the position u and the H^{s-1} norm of the velocity $\partial_t u$ grow like $T^{\sim(1-s)}$ in a neighborhood of $s = 1$. The main theorem of [15] is the following:

[1] we shall not discuss the case $s = s_c$

Theorem 1.1. *[15] Let $R > 0$ and $B(O,R) := \left\{ x \in \mathbb{R}^3, \, |x| < R \right\}$. Let u be a solution of (0.1) with data (u_0, u_1) in the closure of $\mathcal{C}_c^{\infty}(B(O,R)) \times \mathcal{C}_c^{\infty}(B(O,R))$ with respect to the $H^s \times H^{s-1}$ topology, $s < 1$. If $1 > s > \frac{26p-3p^2-39}{2(p-1)(7-p)}$, there exist $\alpha_1 := \alpha_1(s,p) > 0$ and $\alpha_2 := \alpha_2(s,p) > 0$ such that $\lim_{s \to 1} \alpha_1 < \infty$, $\lim_{s \to 1} \alpha_2 = 0$,*

$$\|(P_{>1} u(T), \partial_t u(T))\|^2_{H^s \times H^{s-1}} \lesssim T^{\alpha_1(s,p)(1-s)^2} \tag{1.3}$$

and

$$\|P_{<1} u(T)\|^2_{H^s} \lesssim T^{\frac{3p-5}{p+1}(1+\alpha_2(s,p))}. \tag{1.4}$$

In particular the H^s norm of the high frequency component of the solution and the H^{s-1} norm of the velocity grow like $T^{\sim(1-s)^2}$, i.e at a slower rate than $T^{\sim(1-s)}$. If we compare our results with [14, 13] regarding the H^s norm of the low frequency component of the solution, it grows more slowly by a factor $T^{-\gamma}$ for some $\gamma := \gamma(p) > 0$ in a neighborhood of $s = 1$.

§ 2. Ideas of Theorem 1.1

First we recall the general framework in which we estimate these Sobolev norms on an interval $[0,T]$: the I method. The I method was designed in [4] and is inspired from the Fourier truncation method, designed in [3]. The steps are the following:

- *First Step*: we introduce a multiplier I_N defined in the Fourier domain by $\widehat{I_N f}(\xi) := m\left(\frac{\xi}{N}\right) \hat{f}(\xi)$ with

$$m(\xi) := \begin{cases} 1, \, |\xi| \leq 1 \\ \frac{1}{|\xi|^{1-s}}, \, |\xi| \geq 2 \end{cases}, \tag{2.1}$$

  and $N \gg 1$ a parameter to be chosen.

- *Second Step*: we insert this multiplier into (1.2): this defines a new functional

$$E(I_N u(t)) := \frac{1}{2} \int_{\mathbb{R}^3} |\partial_t I_N u(t)|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla I_N u(t)| + \frac{1}{p+1} \int_{\mathbb{R}^3} |I_N u(t)|^{p+1} \, dx. \tag{2.2}$$

  The main interest of introducing this functional is that, unlike the energy conservation law, it is finite in H^s. Moreover, as N goes to infinity, the symbol of this multiplier approaches one so we expect the variation of this functional to be slow for $N \gg 1$.

- *Third Step*: we estimate the variation of $E(I_N u)$ on an interval $J \subset [0,T]$ small in some sense by using local estimates such as the Strichartz estimates and an *a priori* bound of $E(I_N u)$ on $[0,T]$. In order to do that, we must first find out how this *a priori* bound looks like. It can be proved (see [13] for example) that

$$E(I_N u(0)) \lesssim N^{2(1-s)} \tag{2.3}$$

Since we aim at proving that $E(I_N u)$ does not vary much, a good candidate for an *a priori* bound of $E(I_N u)$ is the following [2]:

$$\sup_{t\in[0,T]} E(I_N u(t)) \lesssim N^{2(1-s)}. \tag{2.4}$$

Then we introduce on J the following number $Z(J,u)$

$$Z(J,u) := \sup_{m\in[0,1]} Z_{m,s}(J,u) \tag{2.5}$$

with

$$Z_{m,s}(J,u) := \sup_{(q,r)-m-wave\,adm} \|\partial_t D^{-m} I_N u\|_{L_t^q L_x^r(J)} + \|D^{1-m} I_N u\|_{L_t^q L_x^r(J)}$$

By using the Strichartz estimates and (2.4) one can show that

$$\begin{aligned} Z(J,u) &\lesssim \sup_{t\in J} E^{\frac{1}{2}}(I_N u(t)) \\ &\lesssim N^{1-s}. \end{aligned} \tag{2.6}$$

We can now estimate the variation of $E(Iu)$ on J through the relation

$$\begin{aligned} \text{Variation}(E(I_N u), J) &\lesssim \frac{Z^{p+1}(J,u)}{N^{\frac{5-p}{2}-}} \\ &\lesssim \frac{N^{(p+1)(1-s)}}{N^{\frac{5-p}{2}-}}. \end{aligned} \tag{2.7}$$

We refer to [13, 14, 15] for more details with regard to the definition of $m-$ wave admissibility pairs and the procedure to estimate the variation of $E(I_N u)$ on J.

- *Fourth Step*: we iterate the procedure described in the last step over subintervals J that make a partition of an arbitrarily long-time interval $[0,T]$. This allows to prove that (2.4) holds *a posteriori* on $[0,T]$.

[2] We shall prove that this bound holds *a posteriori*

- *Fifth Step*: we estimate the H^s norm of the high frequency component of the position and the H^{s-1} norm of the velocity through the following relation (see for example [13])

$$\|(P_{>1}u(T), \partial_t u(T))\|^2_{H^s \times H^{s-1}} \lesssim \sup_{t\in[0,T]} E(I_N u(t)). \tag{2.8}$$

We estimate the H^s norm of the low frequency component of the position through the following relation (see for example [13])

$$\begin{aligned} \|P_{<1}u(T)\|^2_{H^s} &\lesssim T^2 \sup_{t\in[0,T]} \|\partial_t I_N u(t)\|^2_{L^2} \\ &\lesssim T^2 \sup_{t\in[0,T]} E(I_N u(t)). \end{aligned} \tag{2.9}$$

Next we sketch the ideas of the proof of Theorem 1.1.
It is well-known that the long-time behavior of solutions of semilinear wave equations with a defocusing power-type nonlinearity is closely related to the Morawetz-type decay estimate. In the study of the energy-critical wave equation (i.e $p = 5$) a Morawetz-type estimate using the scaling multiplier inside the cone $K_R([0,T])$ defined by

$$K_R([0,T]) := \{(t,x) : t \in [0,T], t > |x| - R\} \tag{2.10}$$

was used. This estimate is of the form (see [2])

$$\int_{|x|\leq T+R} |u|^6(T,x)\, dx \lesssim \tfrac{R}{T+R} E(u) + X$$

with X a boundary term depending on the flux $\mathrm{Flux}\,(u, \partial K_R([0,T]))$ defined by

$$\mathrm{Flux}\,(u, \partial K_R([0,T])) := \tfrac{1}{\sqrt{2}} \int_{\partial K_R([0,T])} \tfrac{1}{2} \left| \tfrac{\nabla u \cdot x}{|x|} + \partial_t u \right|^2 + \tfrac{|u|^6}{6}\, d\sigma.$$

This estimate with general data is a weak decay since it only holds inside the cone and it depends on boundary terms. But, if we work with compactly supported data inside the ball $B(O,R)$, then it is much stronger since, by finite speed of propagation, the flux vanishes. Getting back to (0.1), it is worth trying to establish a decay estimate by using the same multiplier for these data. One finds that for the range of p that we consider (i.e $3 \leq p < 5$), one has

$$\int_{|x|\leq T+R} |u|^{p+1}(T,x)\, dx \lesssim \tfrac{R}{R+T} E. \tag{2.11}$$

The next step is to find the right framework in which we can use this estimate in rougher spaces, i.e $H^s \times H^{s-1}$, $s < 1$. It seems natural to choose data $(u_0, u_1) \in$

$\overline{C_c^\infty}(B(0,R)) \times \overline{C_c^\infty}(B(0,R))$, where the closure is taken with respect to the $H^s \times H^{s-1}$ topology. Then one would like to use this estimate and the I-method in order to estimate the H^s norms of the solution. By introducing the multiplier I_N, one aims at proving an estimate of the form

$$\int_{|x|\leq R+T} |I_N u(T,x)|^{p+1}\,dx \lesssim \tfrac{R}{R+T} E(I_N u(0)) + Error\,terms, \tag{2.12}$$

the error terms coming from the fact that the multiplier I_N does not commute with the nonlinearity. On then aims at estimating $E(I_N u)$ by using this decay estimate. More precisely, one would like to prove on larger subintervals J that (2.7) holds, which would reduce the number of the Js making a partition of $[0,T]$ and eventually yield a better estimate of $E(I_N u)$ on $[0,T]$.
But before starting the procedure, one must be careful. Indeed, recall that the decay estimate (2.11) is useful if we work with data supported in $B(O,R)$. The introduction of the multiplier I_N kills the localization of the data and consequently the localization of the solution inside the cone. But, although we cannot perform an analysis inside the cone, we manage to perform an analysis in a neighborhood of it [3] and outside it by using finite speed of propagation and a more or less localization of smoothness result: see Proposition 3.2.

§ 3. Overview of the proof of Theorem 1.1

For convenience, we shall only discuss the case $p = 3$. The other cases (i.e $3 < p < 5$) can be treated in a similar fashion. The proof of Theorem 1.1 relies upon some propositions that we state now.

§ 3.1. Propositions

The first proposition shows that if u is a solution of (0.1), then we have a partial decay of the potential term of the mollified energy inside the cone. This decay is partial since only the first term of the right-hand side of (3.1) shows that there is decay

Proposition 3.1. *Let $(a,b) \in \mathbb{R}^+ \times \mathbb{R}^+$. Let u be a solution of (0.1) on $[a,b]$. Then*

$$\begin{aligned}\int_{|x|\leq b+R'} |I_N u(b,x)|^4\,dx &\lesssim \tfrac{a+R'}{b+R'} E(I_N u(a)) + \tfrac{1}{\sqrt{2}(b+R')} \int_{\partial K_{R'}([a,b])} \tfrac{|\nabla I_N u\cdot x+(t+R')\partial_t I_N u+I_N u|^2}{t+R'}\,d\sigma \\ &+ \tfrac{1}{b+R'} \int_{K_{R'}([a,b])} \Re\left[\begin{array}{c}\Big((t+R')\partial_t I_N u + x\cdot\nabla I_N u + I_N u\Big) \\ \Big(|I_N u|^2 I_N u - I_N(|u|^2 u)\Big)\end{array}\right] dz.\end{aligned} \tag{3.1}$$

[3] More precisely the neighborhood of the cone we consider is $K_{R'}([0,T])$ with $R' := R+1$

Proof. (Sketch)
By defining $\tilde{u}$ in the following fashion

$$\tilde{u}(t+R',x) := u(t,x),$$

and by finite speed of propagation, we may assume, without loss of generality, that $R'=0$ in (3.1). Next

- we introduce the scaling multiplier $S(f) := t\partial_t f + x\cdot\nabla f + f$ introduced by Struwe [17].
- we apply this scaling multiplier to $I_N u$; we integrate $\Re\left(\overline{S(I_N u)}(\partial_{tt} I_N u - \triangle I_N u + I_N(|u|^{p-1}u))\right) = 0$ inside the cone $K_{R'=0}([a,b])$. Two terms appear: X_1 and X_2 defined by

$$X_1 := \int_{K_{R'=0}([a,b])} \Re\left(\overline{S(I_N u)}\left(\partial_{tt} I_N u - \triangle I_N u + |I_N u|^2 I_N u\right)\right)$$

and

$$X_2 := \int_{K_{R'=0}([a,b])} \Re\left(\overline{S(I_N u)}\left(I_N(|u|^2 u) - |I_N u|^2 I_N u\right)\right).$$

Notice that $-\frac{X_2}{b}$ is the second term on the right hand side of (3.1). So we just need to modify the form of X_1. We use an argument of Shatah-Struwe [16]. More precisely we integrate by part X_1 to get

$$H(b) = H(a) + \frac{1}{\sqrt{2}} \int_{\partial K_{R'=0}([a,b])} \left(P + Q\cdot\frac{x}{|x|}\right) d\sigma \tag{3.2}$$

with $H(t) := \int_{|x|\leq t} P(u(t,x))\,dx$, $P := P(u)$ and $Q := Q(u)$ two functions of u: we refer to [15] for more details. It can be proved that for $t\in[a,b]$,

$$H(t) + \int_{|x|=t} \frac{|I_N u|^2(t,x)}{2}\,d\sigma \lesssim tE(I_N u(t)), \tag{3.3}$$

$$t\int_{|x|\leq t} \frac{|I_N u|^4}{4}\,dx - \int_{|x|=t} \frac{|I_N u|^2}{2}\,d\sigma \leq H(t), \tag{3.4}$$

and

(3.5)
$$\frac{1}{\sqrt{2}}\int_{\partial K_{R'=0}([a,b])} P + Q \cdot \frac{x}{|x|}\, d\sigma$$
$$= \frac{1}{\sqrt{2}}\int_{\partial K_{R'=0}([a,b])} \frac{|\nabla I_N u \cdot x + t\partial_t I_N u + I_N u|^2}{t}\, d\sigma - \left(\int_{|y|=b} \frac{|I_N u|^2(b,y)}{2}\, d\sigma - \int_{|y|=a} \frac{|I_N u|^2(a,y)}{2}\, d\sigma \right).$$

By (3.2), (3.3), (3.4) and (3.5), we see that (3.1) holds.

□

The next proposition shows that if the support of function is localized inside a ball $B(0,R_0)$, then its smoothness (measured by the multiplier I_N) is more or less localized:

Proposition 3.2. *Let $(R_0, L, R_0') \in \mathbb{R}^3$ such that $R_0 > \frac{1}{N^{1-}}$, $0 < L \lesssim N$ and $R_0' - R_0 \geq \frac{L}{N}$. Let $q \geq 1$. Let f be a smooth function supported on the ball $B(O,R_0)$. Then* [4]

$$\|I_N f\|_{L^q(|x|\geq R_0')} \lesssim_{\infty-} \frac{1}{L^{\infty-}} \|I_N f\|_{L^q} \tag{3.6}$$

and

$$\|\nabla I_N f\|_{L^2(|x|\geq R_0')} \lesssim_{\infty-} \frac{1}{L^{\infty-}} \left(\|I_N f\|_{L^2(|x|\leq R_0)} + \|\nabla I_N f\|_{L^2} \right). \tag{3.7}$$

In particular, if $R_0' := R_0 + 1$, then

$$\|I_N f\|_{L^q} \sim \|I_N f\|_{L^q(|x|\leq R_0')}. \tag{3.8}$$

Proof. (Sketch)
First we decompose f into its low frequency part and its high frequency part, i.e

$$f = P_{\lesssim N} f + P_{\gg N} f. \tag{3.9}$$

This seems natural to proceed like this, since

- the left hand side of the estimates (3.6), (3.7) and (3.8) involves the multiplier I_N
- the symbol of the multiplier I_N behaves differently for amplitudes $|\xi| << N$ and amplitudes $|\xi| \gtrsim N$

[4] the notation $x \lesssim_{\infty-} \frac{1}{L^{\infty-}} y$ means that for all $m \geq 0$ there exists $C := C\left(m, \|(u_0,u_1)\|_{H^s \times H^{s-1}}, R\right) > 0$ such that $x \leq C y^m$

Let us say a few words about the proof of (3.6). The proof of (3.7) and (3.8) is an easy modification of that of (3.6). Plugging (3.9), one has to deal with two terms. The first involves the low frequency component, i.e $X_1 := \|P_{\lesssim N} f\|_{L^q(|x| \geq R_0')}$ and the second involves the high frequency component, i.e $X_2 := \|P_{\gg N} f\|_{L^q(|x| \geq R_0')}$. We shall only discuss how we deal with X_1. In order to take into account the fact that f is localized, we write $f = \chi_{R_0} f$. Moreover, since the right-hand side of (3.6) also involve the multiplier I_N we use again the decomposition (3.9). So we have to estimate $X_{1,1} := \|P_{\lesssim N}(\chi_{R_0} P_{\lesssim N} f)\|_{L^q(|x| \geq R_0')}$ and $X_{1,2} := \|P_{\lesssim N}(\chi_{R_0} P_{\gg N} f)\|_{L^q(|x| \leq R_0')}$. Since $P_{<<N}$ is a an average operator at scale $\frac{1}{N}$ we expect X_1 to be small: this can be rigorously proved by writing $P_{\lesssim N}$ as a convolution. In order to deal with X_2, we perform a Paley-Littlewood decomposition $P_{\gg N} = \sum_{M \gg N} P_M$, in order to use to its full extent the quantitative value of the symbol $m\left(\frac{\xi}{N}\right)$ at frequency $|\xi| \sim M$. We also use the fact that the terms that we get after this decomposition are mostly supported in the Fourier domain on $|\xi| \sim M$ (since $M \gg N$), which yields very good decays.

$\square$

The second proposition shows that if we have an a priori bound of the mollified energy $E(I_N u)$ on an interval J(see (3.10)), then we can control $Z(J, u)$ assuming that J is small in some sense (see (3.11) and (3.12)):

Proposition 3.3. *Let u be a solution of (0.1) on $[0, T]$. Let $J := [a, b] \subset [0, T]$. Let $R' := R + 1$. Assume that*

$$\sup_{t \in J} E(I_N u(t)) \lesssim N^{2(1-s)}. \tag{3.10}$$

There exists $\epsilon > 0$ small enough such that if

$$\|I_N u\|^2_{L_t^\infty L_x^4(K_{R'}(J))} |J|^{\frac{1}{2}+} \leq \epsilon \tag{3.11}$$

and

$$|J|^+ \leq \epsilon N^{(2s-1)+}, \tag{3.12}$$

then (2.6) holds.

Proof. (Sketch)

In order to prove that (2.6) holds, we use

- local estimates or, more precisely, the Strichartz estimates: see for example [8] for the statement of these estimates and their proof

- the *a priori* estimate (3.10).
- the estimate (3.8)

The kind of estimates we get is roughly speaking

(3.13)
$$
\begin{aligned}
Z_{\frac{1}{2},s}(J,u) &\lesssim \|(\partial_t I_N u(a), DI_N u(a))\|_{L^2} + \|D^{1-\frac{1}{2}} I_N(|u|^2 u)\|_{L_t^{\frac{4}{3}} L_x^{\frac{4}{3}}(J)} \\
&\lesssim E^{\frac{1}{2}}(I_N u(a)) + \|D^{1-\frac{1}{2}} I_N u\|_{L_t^4 L_x^4(J)} \|u\|_{L_t^4 L_x^4(J)} \\
&\lesssim E^{\frac{1}{2}}(I_N u(a)) + \|D^{1-\frac{1}{2}} I_N u\|_{L_t^4 L_x^4(J)} \left(\|P_{<<N} u\|^2_{L_t^4 L_x^4(J)} + \|P_{\gtrsim N} u\|^2_{L_t^4 L_x^4(J)} \right) \\
&\lesssim E^{\frac{1}{2}}(I_N u(a)) + \|D^{1-\frac{1}{2}} I_N u\|_{L_t^4 L_x^4(J)} \left(\|I_N u\|^2_{L_t^4 L_x^4(J)} + \|P_{\gtrsim N} u\|^2_{L_t^4 L_x^4(J)} \right) \\
&\lesssim E^{\frac{1}{2}}(I_N u(a)) + \|D^{1-\frac{1}{2}} I_N u\|_{L_t^4 L_x^4(J)} \left(|J|^{\frac{1}{2}} \|I_N u\|^2_{L_t^\infty L_x^4(K_{R'}(J))} + \|P_{\gtrsim N} u\|^2_{L_t^4 L_x^4(J)} \right) \\
&\lesssim N^{1-s} + o\left(Z_{\frac{1}{2},s}(J,u) \right) + \frac{Z^3_{\frac{1}{2},s}(J,u)}{N^{1-}},
\end{aligned}
$$

where we used (3.10) and (3.11) in the last line [5]. Therefore (2.6) holds by a continuity argument.

□

The last proposition shows that for a large number of mollified energies $E(I_{N_0} u)$, the decay of the potential term is total:

Proposition 3.4. *Let u be a solution of (0.1) on $[0,T]$. Assume that*

$$N_0^{4s-3} \gg \langle T \rangle^{1+}. \tag{3.14}$$

Let $t \in [0,T]$. Then we have

$$\int_{|x| \leq R'+t} |I_{N_0} u(t,x)|^4 \, dx \lesssim \frac{R'}{R'+t} N_0^{2(1-s)}. \tag{3.15}$$

Proof. (Sketch)

The proof of Proposition 3.4 relies upon Proposition 3.1, Proposition 3.2 and finite speed of propagation. More precisely, we would like to use (3.1) with $a := 0$, $b := T$ and $N := N_0$ satisfying (3.14). But, in order to get an estimate that looks like (3.15) one must make X_1 and X_2 small with X_1 and X_2 defined by

[5] Here we ignore the + sign in (3.11) for convenience. The estimates we get are in fact more complicated than (3.13). We refer to [15] for more details

$$X_1 := \frac{1}{\sqrt{2}(T+R')} \int_{\partial K_{R'}([0,T])} \frac{|\nabla I_{N_0} u \cdot x + (t+R')\partial_t I_{N_0} u + I_{N_0} u|^2}{t+R'} \, d\sigma$$

and

$$X_2 := \frac{1}{T+R'} \int_{K_{R'}([0,T])} \Re \left[\begin{array}{l} \left(\overline{(t+R')\partial_t I_{N_0} u + x \cdot \nabla I_{N_0} u + I_{N_0} u} \right) \\ \left(|I_{N_0} u|^2 I_{N_0} u - I_{N_0}(|u|^2 u) \right) \end{array} \right] dz.$$

In order to do that, we assume that we have the *a priori* estimates (3.15) and (2.4) [6]. In order to use the full power of the *a priori* estimate (3.15) we apply the following procedure

- divide $[0,T]$ into subintervals $J_j = [j-1, j+1]_{1 \leq j \leq J}$ so that we are localized in time on each J_j
- divide each J_j into subintervals $J_{j,k}$ of size $\sim \left(\frac{R'+j}{R'} \right)^{1-} N_0^{2(s-1)-}$ so that we can apply Proposition 3.3 on each $J_{j,k}$ and estimate the number $Z(J_{j,k}, u)$

Then we iterate over j and k to cover $[0,T]$: this allows to make X_2 small compare with the natural upper bound of the decay term of (3.1) [7], that is $\frac{R'}{R'+T} N_0^{2(1-s)}$. It is much easier to make X_1 smaller than this upper bound. Indeed, by integrating the mollified energy identity [8] outside the cone $K_{R'}([0,T])$, we can bound this integral over the surface $\partial K_{R'}([0,T])$ by the sum of two terms

- the mollified energy outside the ball $|x| > R'$ at time 0, i.e

$$\begin{array}{l} E_{R',ext}\left(I_{N_0} u_0\right) := \frac{1}{2} \int_{|x| \geq R'} |\partial_t I_{N_0} u(0)|^2 \, dx + \frac{1}{2} \int_{|x| \geq R'} |\nabla I_{N_0} u(0)|^2 \, dx \\ \qquad + \frac{1}{4} \int_{|x| \geq R'} |I_{N_0} u(0)|^4 \, dx \end{array}$$

- an error term that appears because I does not commute with the nonlinearity

$$Error\,Term = \left| \int_{K_{R'}^c([0,T])} \Re \left(\overline{\partial_t I_{N_0 u}} \left(I_{N_0}(|u|^2 u) - |I_{N_0} u|^2 I_{N_0} u \right) \right) dz \right|$$

By our assumptions regarding the data, by finite speed of propagation and by Proposition 3.2, both terms can be made very small since they involve integrals outside the region where there is localization of smoothness. We refer to [15] for more details.

□

[6] Again we shall prove that these estimates hold *a posteriori*

[7] Here we use (2.3)

[8] i.e the identity that we get after plugging the multiplier I_{N_0} into the energy identity

§ 3.2. Sketch of the proof of Theorem 1.1

First we estimate $\|P_{<1}u(T)\|_{H^s}$. By finite speed of propagation, Proposition 3.2 and Proposition 3.4, we see that [9]

$$
\begin{aligned}
\|P_{<1}u(T)\|^2_{H^s} &\lesssim \|I_{N_0}u(T)\|^2_{L^2} \\
&\sim \|I_{N_0}u(T)\|^2_{L^2(B(O,R'+T))} \\
&\lesssim T^{1+\alpha_2(s,3)},
\end{aligned} \tag{3.16}
$$

the last inequality coming from the optimization of (3.15), in view of the constraint (3.14). Hence we proved (1.4).
Next we estimate $\|(P_{>1}u(T), \partial_t u(T))\|_{H^s \times H^{s-1}}$. Notice that we cannot use (2.8) with $N := N_0$. Indeed, recall that $\sup_{t\in[0,T]} E(I_{N_0}u(t))$ and $\|I_{N_0}u\|^4_{L^\infty_t L^4_x(|x|\leq R'+t)}$ were estimated at the same time in the proof of Proposition 3.4. Since the error appearing in the proof of (3.15) is more difficult to control than that appearing in the proof of (2.4), one has to choose a parameter N_0 very large, which yields a bad estimate of $E(I_{N_0}u)$. The idea is to introduce a new almost conservation law

(3.17)
$$
E(I_{N_1}u(t)) := \frac{1}{2}\int_{\mathbb{R}^3} |\partial_t I_{N_1}u(t)|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^3} |\nabla I_{N_1}u(t)|^2\,dx + \frac{1}{p+1}\int_{\mathbb{R}^3} |I_{N_1}u(t)|^4\,dx
$$

defined by a new parameter $N_1 \gg 1$ (to be chosen) and to estimate the variation of this new almost conservation law through the decay estimate (3.15) on an arbitrarily long time interval $[0,T]$. Since we do no longer need to establish again a decay estimate, we expect to choose a parameter $N_1 << N_0$ in order to control the error term appearing in the proof of this new almost conservation law. In order to use the full power of this decay estimate, we apply again the procedure explained in Subsection 3.1, starting from " divide $[0,T]$ " and finishing by " $Z(J_{j,k},u)$ " [10]. Consequently we can estimate the variation of $E(I_{N_1}u)$ on $J_{j,k}$. By iterating over j and k, one can control the variation of $E(I_{N_1}u)$ on $[0,T]$ if one chooses

$$
N_1 \sim N_0^{\frac{2(1-s)}{2s-1}} \langle T\rangle^{+}.
$$

As it is expected, we find $N_1 << N_0$, which justifies all the computations above. Now using (2.8) with $N := N_1$, we get (1.3).

References

[9] Recall that $R' := R+1$

[10] it is easy to see that, since $N_1 << N_0$, it is enough to prove (3.11) with $I := I_{N_0}$: see [15] for more details

[1] H.Bahouri and Jean-Yves Chemin, On global well-posedness for defocusing cubic wave equation, *Int. Math. Res. Not.* 2006, Art. ID 54873, 12 pp

[2] H. Bahouri and P. Gerard, *High frequency approximation of solutions to critical nonlinear wave equations*, AJM, Vol 121, Number 1, Feb 1999, pp 131-175

[3] J. Bourgain, New global well-posedness results for non-linear Schrödinger equations, *AMS Publications*, 1999

[4] J.Colliander, M.Keel, G.Staffilani, H.Takaoka, T.Tao, Almost conservation laws and global rough solutions to a nonlinear Schrödinger equation, *Math. Res. Letters* 9 (2002), pp. 659-682

[5] I. Gallagher and F. Planchon, On global solutions to a dofocusing semi-linear wave equation, *Revista Mathematicá Iberoamericana*, 19, 2003, pp. 161-177

[6] M. Grillakis, Regularity and asymptotic behavior of the wave equation with a critical nonlinearity, *Ann of Math* 132 (1990), 485-509

[7] Jörgens, K, Das Anfangswertproblem im Grossen für ein Klasse nichtlinearer Wellengleichungen, *Math. Z.*, 77 (1961), 295-308

[8] M. Keel and T. Tao, Endpoint Strichartz estimates, *Amer. J. Math*, 120 (1998), 955-980

[9] C. E. Kenig, G. Ponce and L. Vega, Global well-posedness for semi-linear wave equations, *Communications In Partial Differential Equations*, 25 (2000), pp. 1741-1752

[10] H. Lindblad, C. D Sogge, On existence and scattering with minimal regularity for semilinear wave equations, *J.Func.Anal*, 219 (1995), 227-252

[11] C. Miao and B. Zhang, H^s- global well-posedness for semilinear wave equations, *J.Math. Anal. Appl.*, 283 (2003), 645-666

[12] J. Rauch, The u^5 Klein-Gordon equation, nonlinear partial differential equations and their applications, *(H. Brezis and J. L Lions, eds), Pitman*, Boston, 1982, pp 335-364

[13] T. Roy, Global well-posedness for the radial defocusing cubic wave equation on $\mathbb{R}^3$ and for rough data, *EJDE*, 166, 2007, 1-22

[14] T.Roy, Adapted linear-nonlinear decomposition and global well-posedness for solutions to the defocusing cubic wave equation on $\mathbb{R}^3$, *Disc. Cont. Dynam. Systems A*, Vol 24, Number 4, 2009, 1307-1323

[15] T. Roy, On control of Sobolev norms for some semilinear wave equations with localized data, to appear, Journal of Functional Analysis

[16] J. Shatah, M. Struwe, Regularity results for nonlinear wave equations, *Annals of Mathematics*, 138 (1993), 503-518

[17] M. Struwe, Globally regular solutions to the u^5 Klein-Gordon equations, *Ann. Sci. Norm. Sup. Pisa*, 15 (1988), 495-513