RIMS Kôkyûroku Bessatsu
B42 (2013), 137–152

Dispersive effect of the Coriolis force for the Navier-Stokes equations in the rotational framework

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1 Introduction

This note is a survey of our paper [16] on the initial value problems for the Navier-Stokes equations with the Coriolis force in $\mathbb{R}^3$, describing the motion of viscous incompressible fluids in the rotational framework,

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + \Omega e_3 \times u + (u \cdot \nabla)u + \nabla p = 0 & \text{in } \mathbb{R}^3 \times (0,\infty), \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^3 \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^3, \end{cases} \tag{NSC}$$

where the unknown functions $u = u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ and $p = p(x,t)$ denote the velocity field and the pressure of the fluid, respectively, while $u_0 = u_0(x) = (u_{0,1}(x), u_{0,2}(x), u_{0,3}(x))$ denotes the given initial velocity field satisfying the compatibility condition $\operatorname{div} u_0 = 0$. Here $\Omega \in \mathbb{R}$ represents the speed of rotation around the vertical unit vector $e_3 = (0,0,1)$, which is called the Coriolis parameter.

The main purpose of this note is to prove the local existence and the uniqueness of a mild solution to (NSC). In particular, we are interested in the dispersive effect of the Coriolis force and consider how the speed of rotation $|\Omega|$ affects the size of the existence time T of solutions to (NSC). We make it clear the relation between the time interval T of local existence and the size of Ω.

For the local existence of solutions to (NSC), Sawada [22] proved the local existence and uniqueness of the classical solution to (NSC) in the framework of the Besov space $\dot{B}^0_{\infty,1}(\mathbb{R}^3)$. Giga, Inui, Mahalov and Matsui [12] proved the uniform local solvability of (NSC) for large initial velocity in FM_0. Here the uniform local solvability means that the length of the existence time interval of solutions is independent of the Coriolis parameter Ω. Moreover, they [13]

Received September 3, 2012.
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showed the local existence and uniqueness of the mild solution to (NSC) in the framework of $L^\infty(\mathbb{R}^3)$, and obtained the lower estimate for the existence time T as $T(1+\Omega T)^{6+4\delta} \geqslant C/\|u_0\|^2_{L^\infty_{\sigma,a}}$ for arbitrary $\delta > 0$.

For the global existence of solutions to (NSC), Chemin, Desjardins, Gallagher and Grenier [6] [7] proved that for given initial velocity $u_0 \in L^2(\mathbb{R}^2)^3 + \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)^3$ with $\operatorname{div} u_0 = 0$, there exists a positive parameter Ω_0 such that for every $\Omega \in \mathbb{R}$ with $|\Omega| \geqslant \Omega_0$, (NSC) possesses a unique global solution. Babin, Mahalov and Nicolaenko [2] [3] [4] obtained the global existence and regularity of solutions to (NSC) for large $|\Omega|$ under the periodic boundary conditions. On the other hand, Giga, Inui, Mahalov and Saal [14] established the uniform global solvability of (NSC) for small initial velocity in $\mathrm{FM}_0^{-1}(\mathbb{R}^3)$. Here the uniform global solvability means that the smallness condition on the initial velocity is independent of the size of the speed of ratation Ω. Hieber and Shibata [15] and Konieczny and Yoneda [19] obatained the uniform global solvability of (NSC) in the Sobolev space $H^{\frac{1}{2}}(\mathbb{R}^3)$ and the function spaces of Besov type $F\dot{B}^{2-3/p}_{p,\infty}(\mathbb{R}^3)$ with $1 < p \leqslant \infty$, respectively. In the case $\Omega = 0$, (NSC) correspond to the original Navier-Stokes equations. For the global well-posedness for the original Navier-Stokes equations in the scaling invariant spaces, we refer to Fujita and Kato [9], Kato [17], Kozono and Yamazaki [20], Koch and Tataru [18], Germain [10], Bourgain and Pavlović [5] and Yoneda [23].

In order to state our results, we first introduce the notion of mild solutions to (NSC). Let $\{R_i\}_{i=1}^3$ denote the Riesz transforms, and let $\mathbb{P} = (\delta_{ij} + R_i R_j)_{1\leqslant i,j\leqslant 3}$ denotes the Helmholtz projection onto the divergence-free vector fields. Then, let $T_\Omega(\cdot)$ denotes the semigroup generated by the linearized operator $L := -\Delta + \mathbb{P}\Omega e_3 \times$ associated with (NSC), which is given explicitly by

$$T_\Omega(t)f = \mathscr{F}^{-1}\left[\cos\left(\Omega\frac{\xi_3}{|\xi|}t\right)e^{-|\xi|^2 t} I\widehat{f}(\xi) + \sin\left(\Omega\frac{\xi_3}{|\xi|}t\right)e^{-|\xi|^2 t}R(\xi)\widehat{f}(\xi)\right]$$

for $t \geqslant 0$ and divergence-free vector fields f. Here I is the identity matrix in $\mathbb{R}^3$ and $R(\xi)$ is the skew-symmetric matrix related to the symbol of the Riesz transforms, which is defined by

$$R(\xi) := \frac{1}{|\xi|}\begin{pmatrix} 0 & \xi_3 & -\xi_2 \\ -\xi_3 & 0 & \xi_1 \\ \xi_2 & -\xi_1 & 0 \end{pmatrix} \qquad \xi \in \mathbb{R}^3 \setminus \{0\}.$$

For the derivation of the explicit form of $T_\Omega(\cdot)$, we refer to Babin, Mahalov and Nicolaenko [1] [2] [3], Giga, Inui, Mahalov and Matsui [13] and Hieber and Shibata [15]. In this note, we consider the solution to the following integral equations:

$$u(t) = T_\Omega(t)u_0 - \int_0^t T_\Omega(t-\tau)\mathbb{P}\left[(u\cdot\nabla)u\right](\tau)d\tau. \tag{IE}$$

We call that u is a mild solution to (NSC) if u satisfies (IE) in some appropriate function space.

Before stating our result, we impose the following assumptions for our solution spaces.

Assumption (A). Let the exponent s satisfy $1/2 < s < 5/4$.
Assumption (B). Let the exponents p and q satisfy

$$0 < \frac{1}{p} < \frac{3-2s}{6}, \quad \max\left\{\frac{3-2s}{6}, \frac{s}{3}\right\} < \frac{1}{q} < \min\left\{\frac{1}{2}, \frac{5-2s}{6}\right\}, \quad \frac{1}{p} + \frac{1}{q} > \frac{1}{2}.$$

Assumption (C). Let the exponents θ_1 and θ_2 satisfy

$$\frac{3}{4}-\frac{3}{2p}-\frac{s}{2}<\frac{1}{\theta_1}<\min\left\{1-\frac{2}{p}-\frac{s}{2},\frac{1}{2}-\frac{3}{2p}\right\},$$
$$\frac{5}{4}-\frac{3}{2q}-\frac{s}{2}<\frac{1}{\theta_2}<\min\left\{\frac{1}{2},\frac{3}{2}-\frac{2}{q}-\frac{s}{2},1-\frac{3}{2q}\right\}.$$

Note that for every fixed $1/2 < s < 5/4$, the set of $(p,q,\theta_1,\theta_2) \in (2,\infty)^4$ satisfying the Assumptions (B) and (C) is not empty. The pairs (p,θ_1) and (q,θ_2) correspond to H^s admissible pairs of the Strichartz estimates for the free propagator $e^{it\Delta}$ of the Schrödinger equations.

Our result on the local existence of the mild solution now reads :

Theorem 1.1. *Let s,p,q,θ_1 and θ_2 satisfy Assumptions $(A),(B)$ and (C). Then there exists a positive constant $C = C(s,p,q,\theta_1,\theta_2)$ such that for every $\Omega \in \mathbb{R}\setminus\{0\}$ and for every initial velocity field $u_0 \in \dot{H}^s(\mathbb{R}^3)^3$ with $\operatorname{div} u_0 = 0$, there exists a positive time $T = T(s,p,q,\theta_1,\theta_2,|\Omega|,\|u_0\|_{\dot{H}^s})$ such that* (NSC) *possesses a unique mild solution $u \in X_T$. Here*

$$X_T := \left\{u \in C([0,T];\dot{H}^s(\mathbb{R}^3))^3 \;\middle|\; \|u\|_{X_T} \leqslant C\|u_0\|_{\dot{H}^s}, \operatorname{div} u = 0\right\}$$

with

$$\|u\|_{X_T} := \sup_{0<t<T}\|u(t)\|_{\dot{H}^s} + |\Omega|^{\frac{1}{\theta_1}-\left(\frac{3}{4}-\frac{3}{2p}-\frac{s}{2}\right)}\|u\|_{L^{\theta_1}(0,T;L^p(\mathbb{R}^3))}$$
$$+ |\Omega|^{\frac{1}{\theta_2}-\left(\frac{5}{4}-\frac{3}{2q}-\frac{s}{2}\right)}\|\nabla u\|_{L^{\theta_2}(0,T;L^q(\mathbb{R}^3))}.$$

Moreover, there exists a positive constant $C' = C'(s,p,q,\theta_1,\theta_2)$ such that the existence time T can be taken as

$$T \geqslant C'\min\left\{\left(\frac{|\Omega|^{\frac{1}{\theta_1}-\left(\frac{3}{4}-\frac{3}{2p}-\frac{s}{2}\right)}}{\|u_0\|_{\dot{H}^s}}\right)^{\frac{1}{\frac{1}{2}-\frac{3}{2p}-\frac{1}{\theta_1}}}, \left(\frac{|\Omega|^{\frac{1}{\theta_2}-\left(\frac{5}{4}-\frac{3}{2q}-\frac{s}{2}\right)}}{\|u_0\|_{\dot{H}^s}}\right)^{\frac{1}{1-\frac{3}{2q}-\frac{1}{\theta_2}}}\right\}. \tag{1.1}$$

Remark 1.2. Theorem 1.1 states that the mild solution of (NSC) can be constructed locally in time for every $\Omega \in \mathbb{R}\setminus\{0\}$ and for every initial velocity $u_0 \in \dot{H}^s(\mathbb{R}^3)^3$ with $1/2 < s < 5/4$. Moreover, we can characterize the lower bound for a time interval T of its local existence in terms of $|\Omega|$ and $\|u_0\|_{\dot{H}^s}$. In particular, since the power of $|\Omega|$ in (1.1) is positive, the existence time T of the mild solution to (NSC) can be taken arbitrarily large provided the speed of rotation is sufficiently fast.

We remark that Theorem 1.1 holds even in the case $1/\theta_1 = 3/4 - 3/2p - s/2$ and $1/\theta_2 = 5/4 - 3/2q - s/2$. Moreover, in such a case, we can prove the local existence theorem for all $\Omega \in \mathbb{R}$. Our second result on the uniform local solvability of (NSC) reads as follows :

Theorem 1.3. *Let s,p and q satisfy Assumptions (A) and (B), and let $1/\theta_1 = 3/4-3/2p-s/2$ and $1/\theta_2 = 5/4 - 3/2q - s/2$. Then there exists a positive constant $C = C(s,p,q,\theta_1,\theta_2)$ such that for every initial velocity field $u_0 \in \dot{H}^s(\mathbb{R}^3)^3$ with $\operatorname{div} u_0 = 0$, there exists a positive time $T = T(s,p,q,\theta_1,\theta_2,\|u_0\|_{\dot{H}^s})$ independent of $\Omega \in \mathbb{R}$ such that* (NSC) *possesses a unique mild solution $u \in Y_T$ for all $\Omega \in \mathbb{R}$. Here*

$$Y_T := \left\{u \in C([0,T];\dot{H}^s(\mathbb{R}^3))^3 \;\middle|\; \|u\|_{Y_T} \leqslant C\|u_0\|_{\dot{H}^s}, \operatorname{div} u = 0\right\}$$

with

$$\|u\|_{Y_T} := \sup_{0<t<T} \|u(t)\|_{\dot{H}^s} + \|u\|_{L^{\theta_1}(0,T;L^p(\mathbb{R}^3))} + \|\nabla u\|_{L^{\theta_2}(0,T;L^q(\mathbb{R}^3))}.$$

Moreover, there exists a positive constant $C' = C'(s, p, q, \theta_1, \theta_2)$ *such that the existence time* T *can be taken as*

$$T \geqslant \frac{C'}{\|u_0\|_{\dot{H}^s}^{\frac{2}{s-1/2}}}. \tag{1.2}$$

Remark 1.4. In the case $\Omega = 0$, it follows from Kato [17] and Giga [11] that the time interval T for local existence of the strong solution with the initial data u_0 in $H^s(\mathbb{R}^3)$ with $s > 1/2$ is characterized as

$$T \geqslant \frac{C}{\|u_0\|_{H^s}^{\frac{2}{s-1/2}}}, \tag{1.3}$$

which corresponds to our characterization (1.2). Hence Theorem 1.3 covers the local existence results for $\Omega = 0$.

Remark 1.5. The characterization (1.1) of the existence time T in Theorem 1.1 seems to be sharp in the sense that (1.1) coincides with (1.2) and (1.3) in the case $1/\theta_1 = 3/4 - 3/2p - s/2$ and $1/\theta_2 = 5/4 - 3/2q - s/2$. Therefore, our characterization (1.1) and (1.2) can be regarded as a continuous extension of (1.3) with respect to $\Omega \in \mathbb{R}$ from $\{0\}$ to $\mathbb{R}$.

This note is organized as follows. In Section 2, we introduce some notation and function spaces, and show the dispersive estimates for the oscillation part of the semigroup $T_\Omega(t)$. In Section 3, we introduce the admissible pairs (p, θ_1) and (q, θ_2) and establish the estimates of the Strichartz type for the semigroup $T_\Omega(t)$. In Section 4, we prove the nonlinear estimates for (NSC) using the L^p-L^q smoothing properties of the semigroup $T_\Omega(t)$. In Section 5, we present the proofs of Theorem 1.1 and Theorem 1.3.

2 Dispersive Estimates

We first introduce function spaces. Let $\mathscr{S}(\mathbb{R}^3)$ be the Schwartz class of all rapidly decreasing functions, and let $\mathscr{S}'(\mathbb{R}^3)$ be the space of all tempered distributions. We first recall the definition of the homogeneous Littlewood-Paley decompositions. Let φ be a radial function in $\mathscr{S}(\mathbb{R}^3)$ satisfying the following properties:

$$0 \leqslant \widehat{\varphi}(\xi) \leqslant 1 \quad \text{for all } \xi \in \mathbb{R}^3,$$
$$\operatorname{supp} \widehat{\varphi} \subset \left\{ \xi \in \mathbb{R}^3 \;\middle|\; 2^{-1} \leqslant |\xi| \leqslant 2 \right\},$$

and

$$\sum_{j \in \mathbb{Z}} \widehat{\varphi_j}(\xi) = 1 \quad \text{for all } \xi \in \mathbb{R}^3 \setminus \{0\},$$

where $\varphi_j(x) := 2^{3j}\varphi(2^j x)$ and $\widehat{f}$ denotes the Fourier transform of $f \in \mathscr{S}(\mathbb{R}^3)$. Then, we define the Besov space $\dot{B}^s_{p,q}(\mathbb{R}^3)$ by the following definition.

Definition 2.1. For $s \in \mathbb{R}$ and $1 \leqslant p, q \leqslant \infty$, the Besov space $\dot{B}^s_{p,q}(\mathbb{R}^3)$ is defined to be the set of all tempered distributions $f \in \mathscr{S}'(\mathbb{R}^3)$ such that the following semi-norm is finite:

$$\|f\|_{\dot{B}^s_{p,q}} := \left\| \left\{ 2^{sj} \|\varphi_j * f\|_{L^p} \right\}_{j \in \mathbb{Z}} \right\|_{\ell^q}.$$

Next, we shall prove the dispersive estimates for the oscillating parts of the semigroup $T_\Omega(t)$ associated with the linear problem of (NSC). We define the operators $\mathcal{G}_\pm(\tau)$ of oscillatory integral type as

$$\mathcal{G}_\pm(\tau)[f] := \mathscr{F}^{-1}\left[e^{\pm i\tau\frac{\xi_3}{|\xi|}}\mathscr{F}[f]\right]$$

for $\tau \in \mathbb{R}$. Then, we can rewrite the semigroup $T_\Omega(t)$ as

$$T_\Omega(t)f = \frac{1}{2}\mathcal{G}_+(\Omega t)\left[e^{t\Delta}(I+\mathcal{R})f\right] + \frac{1}{2}\mathcal{G}_-(\Omega t)\left[e^{t\Delta}(I-\mathcal{R})f\right] \tag{2.1}$$

for $t \geqslant 0$ and $\Omega \in \mathbb{R}$, where $\mathcal{R}$ denotes the matrix of singular integral operators defined by

$$\mathcal{R} := \begin{pmatrix} 0 & R_3 & -R_2 \\ -R_3 & 0 & R_1 \\ R_2 & -R_1 & 0 \end{pmatrix}.$$

The operators $\mathcal{G}_\pm(\Omega t)$ represent the oscillation parts of $T_\Omega(t)$.

Lemma 2.2. *For any $2 \leqslant p \leqslant \infty$, there exists a positive constant $C = C(p)$ such that*

$$\|\mathcal{G}_\pm(\tau)[f]\|_{\dot{B}^s_{p,q}} \leqslant C\left\{\frac{\log(e+|\tau|)}{1+|\tau|}\right\}^{\frac{1}{2}\left(1-\frac{2}{p}\right)} \|f\|_{\dot{B}^{s+3(1-\frac{2}{p})}_{p',q}}$$

for all $s \in \mathbb{R}, 1 \leqslant q \leqslant \infty, \tau \in \mathbb{R}$ and $f \in \dot{B}^{s+3\left(1-\frac{2}{p}\right)}_{p',q}(\mathbb{R}^3)$ with $1/p + 1/p' = 1$.

In order to prove Lemma 2.2, we first prove the following lemma.

Lemma 2.3. *There exists a positive constant C such that*

$$\|\mathcal{G}_\pm(\tau)\left[\Phi_j\right]\|_{L^\infty} \leqslant C2^{3j}\left\{\frac{\log(e+|\tau|)}{1+|\tau|}\right\}^{\frac{1}{2}}$$

for all $j \in \mathbb{Z}$ and $\tau \in \mathbb{R}$, where $\Phi_j := \varphi_{j-1} + \varphi_j + \varphi_{j+1}$.

Remark 2.4. We remark that similar dispersive estimates were obtained by Dutrifoy [8] in the context of non-viscous rotating fluids. Our estimates in Lemma 2.3 gives an improvement of [8] in the sense that $\log(e+|\tau|)$ is replaced with $\{\log(e+|\tau|)\}^{\frac{1}{2}}$.

Proof of Lemma 2.3. It suffices to show that there exists a positive constant C such that

$$\left|\int_{\mathbb{R}^3} e^{ix\cdot\xi}e^{\pm i\tau\frac{\xi_3}{|\xi|}}\widehat{\Phi_j}(\xi)d\xi\right| \leqslant C2^{3j}\left\{\frac{\log(e+|\tau|)}{1+|\tau|}\right\}^{\frac{1}{2}} \tag{2.2}$$

for all $x \in \mathbb{R}^3$ with $x_2 = 0$ by the rotational symmetry with respect to (ξ_1, ξ_2). In the case $|\tau| \leqslant e$, it is easy to see that

$$\begin{aligned}\left|\int_{\mathbb{R}^3} e^{ix\cdot\xi}e^{\pm i\tau\frac{\xi_3}{|\xi|}}\widehat{\Phi_j}(\xi)d\xi\right| &\leqslant \int_{\mathbb{R}^3}\left|\widehat{\Phi_j}(\xi)\right|d\xi \\ &\leqslant C2^{3j}\left\{\frac{\log(e+|\tau|)}{1+|\tau|}\right\}^{\frac{1}{2}}.\end{aligned}$$

Therefore it suffices to show (2.2) for $|\tau| > e$. Since $\operatorname{supp}\widehat{\Phi_0} \subset \left\{\xi \in \mathbb{R}^3 \;\middle|\; |\xi| \leqslant 4\right\}$, we decompose the left hand side of (2.2) as

$$\begin{aligned}
\left|\int_{\mathbb{R}^3} e^{ix\cdot\xi} e^{\pm i\tau\frac{\xi_3}{|\xi|}} \widehat{\Phi_j}(\xi)d\xi\right| &= 2^{3j}\left|\int_{\mathbb{R}^3} e^{i2^j x\cdot\xi} e^{\pm i\tau\frac{\xi_3}{|\xi|}} \widehat{\Phi_0}(\xi)d\xi\right| \\
&\leqslant 2^{3j}\left|\int_{|\xi_1|\leqslant 4, |\xi_2|\leqslant 4, |\xi_3|\leqslant \varepsilon} e^{i2^j x\cdot\xi} e^{\pm i\tau\frac{\xi_3}{|\xi|}} \widehat{\Phi_0}(\xi)d\xi\right| \\
&\quad + 2^{3j}\left|\int_{|\xi_1|\leqslant 4, |\xi_2|\leqslant \varepsilon, \varepsilon\leqslant|\xi_3|\leqslant 4} e^{i2^j x\cdot\xi} e^{\pm i\tau\frac{\xi_3}{|\xi|}} \widehat{\Phi_0}(\xi)d\xi\right| \\
&\quad + 2^{3j}\left|\int_{|\xi_1|\leqslant 4, \varepsilon\leqslant|\xi_2|\leqslant 4, \varepsilon\leqslant|\xi_3|\leqslant 4} e^{i2^j x\cdot\xi} e^{\pm i\tau\frac{\xi_3}{|\xi|}} \widehat{\Phi_0}(\xi)d\xi\right| \\
&=: I_1 + I_2 + I_3
\end{aligned} \tag{2.3}$$

for some $\varepsilon \in (0,4)$ to be determined later. For I_1 and I_2, since $|\widehat{\Phi_0}(\xi)| \leqslant 1$, we have

$$I_1 + I_2 \leqslant C2^{3j}\varepsilon. \tag{2.4}$$

For I_3, we have by integration by parts with respect to ξ_2 that

$$\begin{aligned}
I_3 &\leqslant 2\frac{2^{3j}}{\varepsilon|\tau|}\int_{|\xi_1|\leqslant 4, \varepsilon\leqslant|\xi_3|\leqslant 4} \frac{|(\xi_1,\varepsilon,\xi_3)|^3}{|\xi_3|}\left|\widehat{\Phi_0}(\xi_1,\varepsilon,\xi_3)\right| d\xi_1 d\xi_3 \\
&\quad + \frac{2^{3j}}{|\tau|}\left|\int_{|\xi_1|\leqslant 4, \varepsilon\leqslant|\xi_2|\leqslant 4, \varepsilon\leqslant|\xi_3|\leqslant 4} e^{i2^j x\cdot\xi} e^{\pm i\tau\frac{\xi_3}{|\xi|}} \frac{\partial}{\partial\xi_2}\left\{\frac{|\xi|^3}{\xi_2\xi_3}\widehat{\Phi_0}(\xi)\right\} d\xi\right| \\
&=: I_{3,1} + I_{3,2}.
\end{aligned} \tag{2.5}$$

For $I_{3,1}$, we have that

$$\begin{aligned}
I_{3,1} &\leqslant C\frac{2^{3j}}{\varepsilon|\tau|}\int_{\varepsilon\leqslant|\xi_3|\leqslant 4}\frac{1}{|\xi_3|}d\xi_3 \\
&\leqslant C\frac{2^{3j}}{\varepsilon|\tau|}\log\left(\frac{4}{\varepsilon}\right).
\end{aligned} \tag{2.6}$$

For $I_{3,2}$, since

$$\left|\frac{\partial}{\partial\xi_2}\left\{\frac{|\xi|^3}{\xi_2\xi_3}\widehat{\Phi_0}(\xi)\right\}\right| \leqslant \frac{C}{|\xi_2|^2|\xi_3|}$$

for $|\xi| \leqslant 4$, we have that

$$\begin{aligned}
I_{3,2} &\leqslant C\frac{2^{3j}}{|\tau|}\int_{\varepsilon\leqslant|\xi_2|\leqslant 4, \varepsilon\leqslant|\xi_3|\leqslant 4}\frac{1}{|\xi_2|^2|\xi_3|}d\xi_2 d\xi_3 \\
&\leqslant C\frac{2^{3j}}{\varepsilon|\tau|}\log\left(\frac{4}{\varepsilon}\right).
\end{aligned} \tag{2.7}$$

By (2.3), (2.4), (2.5), (2.6) and (2.7), we obtain that

$$\left|\int_{\mathbb{R}^3} e^{ix\cdot\xi} e^{\pm i\tau\frac{\xi_3}{|\xi|}} \widehat{\Phi_j}(\xi)d\xi\right| \leqslant C2^{3j}\left\{\varepsilon + \frac{1}{\varepsilon|\tau|}\log\left(\frac{4}{\varepsilon}\right)\right\}$$

for all $\varepsilon \in (0,4)$. Choosing $\varepsilon = 4|\tau|^{-\frac{1}{2}}\{\log|\tau|\}^{\frac{1}{2}}$, we obtain the desired estimates. □

Proof of Lemma 2.2. Since $\Phi_j * \varphi_j = \varphi_j$ for all $j \in \mathbb{Z}$, we see that

$$\varphi_j * \mathcal{G}_\pm(\tau)\,[f] = \mathcal{G}_\pm(\tau)\,[\Phi_j] * (\varphi_j * f)$$

for all $j \in \mathbb{Z}$. Hence we have by the Hausdorff-Young inequality and Lemma 2.3 that

$$\begin{aligned}\|\Phi_j * \mathcal{G}_\pm(\tau)\,[f]\|_{L^\infty} &\leqslant \sum_{k=-1}^{1} \|\mathcal{G}_\pm(\tau)\,[\Phi_{j+k}]\|_{L^\infty} \|\varphi_{j+k} * f\|_{L^1} \\ &\leqslant C2^{3j}\left\{\frac{\log(e+|\tau|)}{1+|\tau|}\right\}^{\frac{1}{2}} \|f\|_{L^1} \end{aligned} \tag{2.8}$$

for all $j \in \mathbb{Z}$ and $f \in L^1(\mathbb{R}^3)$. In the case $p = 2$, it follows from the Plancherel theorem that

$$\|\Phi_j * \mathcal{G}_\pm(\tau)\,[f]\|_{L^2} \leqslant \|f\|_{L^2} \tag{2.9}$$

for all $j \in \mathbb{Z}$ and $f \in L^2(\mathbb{R}^3)$. From (2.8) and (2.9), we have by the Riesz-Thorin interpolation theorem that

$$\|\Phi_j * \mathcal{G}_\pm(\tau)\,[f]\|_{L^p} \leqslant C2^{3\left(1-\frac{2}{p}\right)j}\left\{\frac{\log(e+|\tau|)}{1+|\tau|}\right\}^{\frac{1}{2}\left(1-\frac{2}{p}\right)} \|f\|_{L^{p'}} \tag{2.10}$$

for all $j \in \mathbb{Z}, 2 \leqslant p \leqslant \infty$ and $f \in L^{p'}(\mathbb{R}^3)$. Since

$$\Phi_j * \mathcal{G}_\pm(\tau)\,[\varphi_j * f] = \varphi_j * \mathcal{G}_\pm(\tau)\,[f]\,,$$

we have by (2.10) that

$$\|\varphi_j * \mathcal{G}_\pm(\tau)\,[f]\|_{L^p} \leqslant C2^{3\left(1-\frac{2}{p}\right)j}\left\{\frac{\log(e+|\tau|)}{1+|\tau|}\right\}^{\frac{1}{2}\left(1-\frac{2}{p}\right)} \|\varphi_j * f\|_{L^{p'}} \tag{2.11}$$

for all $j \in \mathbb{Z}, \tau \in \mathbb{R}$. Multiplying both sides of (2.11) by 2^{sj} and then taking the $\ell^q(\mathbb{Z})$-norm, we complete the proof of Lemma 2.2. $\square$

3 Linear Estimates

In this section, we establish the linear estimates for the semigroup $T_\Omega(t)$. We first recall the behavior of the heat semigroup $e^{t\Delta}$ in the Besov spaces established by Kozono, Ogawa and Taniuchi [21].

Lemma 3.1 (Kozono-Ogawa-Taniuchi [21])**.** *Let $-\infty < s_0 \leqslant s_1 < \infty$. Then there exists a positive constant $C = C(s_0, s_1)$ such that*

$$\left\|e^{t\Delta} f\right\|_{\dot{B}^{s_1}_{p,q}} \leqslant Ct^{-\frac{1}{2}(s_1-s_0)} \|f\|_{\dot{B}^{s_0}_{p,q}}$$

for all $t > 0, 1 \leqslant p, q \leqslant \infty$ and $f \in \dot{B}^{s_0}_{p,q}(\mathbb{R}^3)$.

We prove the uniform boundedness of $T_\Omega(t)$ in $\dot{H}^s(\mathbb{R}^3)$ with respect to $t > 0$ and $\Omega \in \mathbb{R}$.

Lemma 3.2. *There exists a positive constant C such that*

$$\|T_\Omega(t)f\|_{\dot{H}^s} \leqslant C\,\|f\|_{\dot{H}^s}$$

for all $t>0, \Omega\in\mathbb{R}, s\in\mathbb{R}$ and $f\in\dot{H}^s(\mathbb{R}^3)^3$.

Proof. The desired estimate easily follows from the Plancherel theorem and the explicit form of $T_\Omega(t)$. □

Next we shall establish the estimates of the Strichartz type for the semigroup $T_\Omega(t)$. We impose the following assumption for s, p and θ_1.

Assumption (L1). Let the exponent s satisfy $0\leqslant s<3/2$.
Assumption (L2). Let the exponents p and θ_1 satisfy

$$\max\left\{\frac{1-2s}{6},0\right\}<\frac{1}{p}<\frac{3-2s}{6},\quad \frac{3}{4}-\frac{3}{2p}-\frac{s}{2}\leqslant\frac{1}{\theta_1}<\min\left\{\frac{1}{2},1-\frac{2}{p}-\frac{s}{2}\right\}.$$

Note that in the case $1/\theta_1=3/4-3/2p-s/2$, the pair (p,θ_1) corresponds to the H^s admissible pair of the Strichartz estimates for the free propagator of the Schrödinger equations.

Lemma 3.3. *Let s,p and θ_1 satisfy Assumptions $(L1)$ and $(L2)$. Then there exists a positive constant $C=C(s,p,\theta_1)$ such that*

$$\|T_\Omega(\cdot)f\|_{L^{\theta_1}(0,\infty;L^p(\mathbb{R}^3))}\leqslant C|\Omega|^{-\left\{\frac{1}{\theta_1}-\left(\frac{3}{4}-\frac{3}{2p}-\frac{s}{2}\right)\right\}}\|f\|_{\dot{H}^s} \tag{3.1}$$

for all $\Omega\in\mathbb{R}\setminus\{0\}$ and $f\in\dot{H}^s(\mathbb{R}^3)^3$. In particular, in the case $1/\theta_1=3/4-3/2p-s/2$, (3.1) *holds for all $\Omega\in\mathbb{R}$.*

Proof. The proof is based on the duality argument. Since the relation (2.1) holds and since $\mathcal{R}$ is a bounded operator in $\dot{H}^s(\mathbb{R}^3)$, it suffices to prove that

$$\begin{aligned}&\left|\int_0^\infty\int_{\mathbb{R}^3}\mathcal{G}_\pm(\Omega t)[e^{t\Delta}(-\Delta)^{-\frac{s}{2}}f](x)\overline{\phi(x,t)}dxdt\right|\\ &\qquad\leqslant C|\Omega|^{-\left\{\frac{1}{\theta_1}-\left(\frac{3}{4}-\frac{3}{2p}-\frac{s}{2}\right)\right\}}\|f\|_{L^2}\|\phi\|_{L^{\theta_1'}(0,\infty;L^{p'}(\mathbb{R}^3))}\end{aligned}$$

for all $\phi\in C_0^\infty(\mathbb{R}^3\times(0,\infty))$, where $1/p+1/p'=1$ and $1/\theta_1+1/\theta_1'=1$. By the Perseval formula and the Hölder inequality, we have

$$\begin{aligned}&\left|\int_0^\infty\int_{\mathbb{R}^3}\mathcal{G}_\pm(\Omega t)[e^{t\Delta}(-\Delta)^{-\frac{s}{2}}f](x)\overline{\phi(x,t)}dxdt\right|\\ &\quad=\left|\int_0^\infty\int_{\mathbb{R}^3}f(x)\overline{\mathcal{G}_\mp(\Omega t)[e^{t\Delta}(-\Delta)^{-\frac{s}{2}}\phi(t)](x)}dxdt\right|\\ &\quad\leqslant\|f\|_{L^2}\left\|\int_0^\infty\mathcal{G}_\mp(\Omega t)[e^{t\Delta}(-\Delta)^{-\frac{s}{2}}\phi(t)]dt\right\|_{L^2}.\end{aligned} \tag{3.2}$$

Moreover, since the continuous embedding relation $\dot{B}^0_{p,2}(\mathbb{R}^3)\hookrightarrow L^p(\mathbb{R}^3)$ holds for $2\leqslant p<\infty$, we have by the Perseval formula, the Hölder inequality and Lemma 2.2 that

$$\left\|\int_0^\infty\mathcal{G}_\mp(\Omega t)[e^{t\Delta}(-\Delta)^{-\frac{s}{2}}\phi(t)]dt\right\|_{L^2}^2$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^3}\int_0^\infty\int_0^\infty \mathcal{G}_\mp(\Omega t)[e^{t\Delta}(-\Delta)^{-\frac{s}{2}}\phi(t)](x)\overline{\mathcal{G}_\mp(\Omega\tau)[e^{\tau\Delta}(-\Delta)^{-\frac{s}{2}}\phi(\tau)](x)}dtd\tau dx \\
&= \int_0^\infty\int_0^\infty\int_{\mathbb{R}^3} \phi(x,t)\overline{\mathcal{G}_\pm(\Omega(t-\tau))[e^{(t+\tau)\Delta}(-\Delta)^{-s}\phi(\tau)](x)}dxdtd\tau \\
&\leqslant \int_0^\infty\int_0^\infty \|\phi(t)\|_{L^{p'}}\left\|\mathcal{G}_\pm(\Omega(t-\tau))[e^{(t+\tau)\Delta}(-\Delta)^{-s}\phi(\tau)]\right\|_{L^p} dtd\tau \\
&\leqslant C\int_0^\infty\int_0^\infty \|\phi(t)\|_{L^{p'}}\left\|\mathcal{G}_\pm(\Omega(t-\tau))[e^{(t+\tau)\Delta}(-\Delta)^{-s}\phi(\tau)]\right\|_{\dot{B}^0_{p,2}} dtd\tau \\
&\leqslant C\int_0^\infty\int_0^\infty \|\phi(t)\|_{L^{p'}}\left\{\frac{\log(e+|\Omega||t-\tau|)}{1+|\Omega||t-\tau|}\right\}^{\frac{1}{2}\left(1-\frac{2}{p}\right)}\left\|e^{(t+\tau)\Delta}(-\Delta)^{-s}\phi(\tau)\right\|_{\dot{B}^{3\left(1-\frac{2}{p}\right)}_{p',2}} dtd\tau \\
&\leqslant C\int_0^\infty\int_0^\infty \|\phi(t)\|_{L^{p'}}\left\{\frac{\log(e+|\Omega||t-\tau|)}{1+|\Omega||t-\tau|}\right\}^{\frac{1}{2}(1-\frac{2}{p})}\left\|e^{(t+\tau)\Delta}\phi(\tau)\right\|_{\dot{B}^{3(1-\frac{2}{p})-2s}_{p',2}} dtd\tau. \qquad (3.3)
\end{aligned}
$$

Here, since $s < 3/2$ and $1/p < (3-2s)/6$, we see that $3(1-\frac{2}{p})-2s > 0$. Therefore, it follows from Lemma 3.1 and the continuous embedding relation $L^{p'}(\mathbb{R}^3) \hookrightarrow \dot{B}^0_{p',2}(\mathbb{R}^3)$ that

$$
\begin{aligned}
\left\|e^{(t+\tau)\Delta}\phi(\tau)\right\|_{\dot{B}^{3(1-\frac{2}{p})-2s}_{p',2}} &\leqslant \frac{C}{(t+\tau)^{\frac{3}{2}(1-\frac{2}{p})-s}}\|\phi(\tau)\|_{\dot{B}^0_{p',2}} \\
&\leqslant \frac{C}{|t-\tau|^{\frac{3}{2}(1-\frac{2}{p})-s}}\|\phi(\tau)\|_{L^{p'}}. \qquad (3.4)
\end{aligned}
$$

Combining (3.3) and (3.4), we have by the Hölder inequality that

$$
\begin{aligned}
&\left\|\int_0^\infty \mathcal{G}_\mp(\Omega t)[e^{t\Delta}(-\Delta)^{-\frac{s}{2}}\phi(t)]dt\right\|_{L^2}^2 \\
&\quad\leqslant C\int_0^\infty\int_0^\infty \|\phi(t)\|_{L^{p'}}\left\{\frac{\log(e+|\Omega||t-\tau|)}{1+|\Omega||t-\tau|}\right\}^{\frac{1}{2}(1-\frac{2}{p})}\frac{1}{|t-\tau|^{\frac{3}{2}(1-\frac{2}{p})-s}}\|\phi(\tau)\|_{L^{p'}}dtd\tau \\
&\quad\leqslant C\|\phi\|_{L^{\theta_1'}(0,\infty;L^{p'}(\mathbb{R}^3))}\left\|\int_0^\infty h(\cdot-\tau)\|\phi(\tau)\|_{L^{p'}}d\tau\right\|_{L^{\theta_1}(0,\infty)}, \qquad (3.5)
\end{aligned}
$$

where we put

$$
h(t) := \left\{\frac{\log(e+|\Omega||t|)}{1+|\Omega||t|}\right\}^{\frac{1}{2}(1-\frac{2}{p})}\frac{1}{|t|^{\frac{3}{2}(1-\frac{2}{p})-s}}.
$$

In the case $1/\theta_1 > 3/4 - 3/2p - s/2$, since $h \in L^{\frac{\theta_1}{2}}(\mathbb{R})$ and

$$
\|h\|_{L^{\frac{\theta_1}{2}}(\mathbb{R})} = C|\Omega|^{-\left\{\frac{2}{\theta_1}-\left(\frac{3}{2}-\frac{3}{p}-s\right)\right\}},
$$

we have by (3.5) and the Hausdorff-Young inequality that

$$
\left\|\int_0^\infty \mathcal{G}_\mp(\Omega t)[e^{t\Delta}(-\Delta)^{-\frac{s}{2}}\phi(t)]dt\right\|_{L^2}^2 \leqslant C|\Omega|^{-\left\{\frac{2}{\theta_1}-\left(\frac{3}{2}-\frac{3}{p}-s\right)\right\}}\|\phi\|^2_{L^{\theta_1'}(0,\infty;L^{p'}(\mathbb{R}^3))}. \qquad (3.6)
$$

In the case $1/\theta_1 = 3/4 - 3/2p - s/2$, since $h(t) \leqslant |t|^{-\frac{3}{2}(1-\frac{2}{p})+s}$, we have by (3.5) and the Hardy-Littlewood-Sobolev inequality that

$$
\left\|\int_0^\infty \mathcal{G}_\mp(\Omega t)[e^{t\Delta}(-\Delta)^{-\frac{s}{2}}\phi(t)]dt\right\|_{L^2}^2 \leqslant C\|\phi\|^2_{L^{\theta_1'}(0,\infty;L^{p'}(\mathbb{R}^3))}. \qquad (3.7)
$$

Combining (3.2), (3.6) and (3.7), we complete the proof of Lemma 3.3. □

Next we prove the estimates of the Strichartz type for the derivative of the semigroup $T_\Omega(t)$. To this end, we impose the following assumption on s, q and θ_2.

Assumption (L3). Let the exponent s satisfy $0 < s < 3/2$.
Assumption (L4). Let the exponents q and θ_2 satisfy

$$\frac{3-2s}{6} < \frac{1}{q} < \min\left\{\frac{1}{2}, \frac{5-2s}{6}\right\}, \quad \frac{5}{4} - \frac{3}{2q} - \frac{s}{2} \leqslant \frac{1}{\theta_2} < \min\left\{\frac{1}{2}, \frac{3}{2} - \frac{2}{q} - \frac{s}{2}\right\}.$$

Lemma 3.4. *Let s, q and θ_2 satisfy Assumptions $(L3)$ and $(L4)$. Then there exists a positive constant $C = C(s, q, \theta_2)$ such that*

$$\|\nabla T_\Omega(\cdot) f\|_{L^{\theta_2}(0,\infty;L^q(\mathbb{R}^3))} \leqslant C|\Omega|^{-\left\{\frac{1}{\theta_2} - \left(\frac{5}{4} - \frac{3}{2q} - \frac{s}{2}\right)\right\}} \|f\|_{\dot{H}^s} \tag{3.8}$$

for all $\Omega \in \mathbb{R} \setminus \{0\}$ and $f \in \dot{H}^s(\mathbb{R}^3)^3$. In particular, in the case $1/\theta_2 = 5/4 - 3/2q - s/2$, (3.8) holds for all $\Omega \in \mathbb{R}$.

Proof. The proof is based on the duality argument and similar to that of Lemma 3.3. Since the relation (2.1) holds and since $\mathcal{R}$ is a bounded operator in $\dot{H}^s(\mathbb{R}^3)$, it suffices to prove that

$$\begin{aligned}
&\left| \int_0^\infty \int_{\mathbb{R}^3} \partial_{x_j} \mathcal{G}_\pm(\Omega t)[e^{t\Delta}(-\Delta)^{-\frac{s}{2}} f](x) \overline{\phi(x,t)} dx dt \right| \\
&\qquad \leqslant C|\Omega|^{-\left\{\frac{1}{\theta_2} - \left(\frac{5}{4} - \frac{3}{2q} - \frac{s}{2}\right)\right\}} \|f\|_{L^2} \|\phi\|_{L^{\theta_2'}(0,\infty;L^{q'}(\mathbb{R}^3))}
\end{aligned}$$

for all $\phi \in C_0^\infty(\mathbb{R}^3 \times (0,\infty))$ and $j = 1, 2, 3$, where $1/q + 1/q' = 1$ and $1/\theta_2 + 1/\theta_2' = 1$. By the Perseval formula and the Hölder inequality, we have

$$\begin{aligned}
&\left| \int_0^\infty \int_{\mathbb{R}^3} \partial_{x_j} \mathcal{G}_\pm(\Omega t)[e^{t\Delta}(-\Delta)^{-\frac{s}{2}} f](x) \overline{\phi(x,t)} dx dt \right| \\
&= \left| \int_0^\infty \int_{\mathbb{R}^3} f(x) \overline{\mathcal{G}_\mp(\Omega t)[e^{t\Delta}(-\Delta)^{-\frac{s}{2}} \partial_{x_j} \phi(t)](x)} dx dt \right| \\
&\leqslant \|f\|_{L^2} \left\| \int_0^\infty \mathcal{G}_\mp(\Omega t)[e^{t\Delta}(-\Delta)^{-\frac{s}{2}} \partial_{x_j} \phi(t)] dt \right\|_{L^2}.
\end{aligned} \tag{3.9}$$

Moreover, since the continuous embedding relation $\dot{B}^0_{q,2}(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$ holds for $2 \leqslant q < \infty$, similarly to (3.3), we have by the Perseval formula, the Hölder inequality and Lemma 2.2 that

$$\begin{aligned}
&\left\| \int_0^\infty \mathcal{G}_\mp(\Omega t)[e^{t\Delta}(-\Delta)^{-\frac{s}{2}} \partial_{x_j} \phi(t)] dt \right\|_{L^2}^2 \\
&\leqslant C \int_0^\infty \int_0^\infty \|\phi(t)\|_{L^{q'}} \left\| \mathcal{G}_\pm(\Omega(t-\tau))[e^{(t+\tau)\Delta}(-\Delta)^{-s} \partial_{x_j}^2 \phi(\tau)] \right\|_{\dot{B}^0_{q,2}} dt d\tau \\
&\leqslant C \int_0^\infty \int_0^\infty \|\phi(t)\|_{L^{q'}} \left\{ \frac{\log(e + |\Omega||t-\tau|)}{1 + |\Omega||t-\tau|} \right\}^{\frac{1}{2}\left(1-\frac{2}{q}\right)} \left\| e^{(t+\tau)\Delta} \phi(\tau) \right\|_{\dot{B}^{5-\frac{6}{q}-2s}_{q',2}} dt d\tau.
\end{aligned} \tag{3.10}$$

Here, since $s < 3/2$ and $1/q < (5-2s)/6$, we see that $5 - \frac{6}{q} - 2s > 0$. Therefore, it follows from Lemma 3.1 and the continuous embedding relation $L^{q'}(\mathbb{R}^3) \hookrightarrow \dot{B}^0_{q',2}(\mathbb{R}^3)$ that

$$\left\| e^{(t+\tau)\Delta}\phi(\tau) \right\|_{\dot{B}^{5-\frac{6}{q}-2s}_{q',2}} \leqslant \frac{C}{|t-\tau|^{\frac{5}{2}-\frac{3}{q}-s}} \|\phi(\tau)\|_{L^{q'}}. \tag{3.11}$$

Combining (3.10) and (3.11), we have by the Hölder inequality that

$$\begin{aligned} &\left\| \int_0^\infty \mathcal{G}_{\mp}(\Omega t)[e^{t\Delta}(-\Delta)^{-\frac{s}{2}}\partial_{x_j}\phi(t)]dt \right\|_{L^2}^2 \\ &\leqslant C \int_0^\infty \int_0^\infty \|\phi(t)\|_{L^{q'}} \left\{ \frac{\log(e+|\Omega||t-\tau|)}{1+|\Omega||t-\tau|} \right\}^{\frac{1}{2}(1-\frac{2}{q})} \frac{1}{|t-\tau|^{\frac{5}{2}-\frac{3}{q}-s}} \|\phi(\tau)\|_{L^{q'}} dt d\tau \\ &\leqslant C\|\phi\|_{L^{\theta_2'}(0,\infty;L^{q'}(\mathbb{R}^3))} \left\| \int_0^\infty \widetilde{h}(\cdot - \tau)\|\phi(\tau)\|_{L^{q'}} d\tau \right\|_{L^{\theta_2}(0,\infty)}, \end{aligned} \tag{3.12}$$

where we put

$$\widetilde{h}(t) := \left\{ \frac{\log(e+|\Omega||t|)}{1+|\Omega||t|} \right\}^{\frac{1}{2}(1-\frac{2}{q})} \frac{1}{|t|^{\frac{5}{2}-\frac{3}{q}-s}}.$$

In the case $1/\theta_2 > 5/4 - 3/2q - s/2$, since $\widetilde{h} \in L^{\frac{\theta_2}{2}}(\mathbb{R})$ and

$$\|\widetilde{h}\|_{L^{\frac{\theta_2}{2}}(\mathbb{R})} = C|\Omega|^{-\left\{\frac{2}{\theta_2} - \left(\frac{5}{2}-\frac{3}{q}-s\right)\right\}},$$

we have by (3.12) and the Hausdorff-Young inequality that

$$\left\| \int_0^\infty \mathcal{G}_{\mp}(\Omega t)[e^{t\Delta}(-\Delta)^{-\frac{s}{2}}\partial_{x_j}\phi(t)]dt \right\|_{L^2}^2 \leqslant C|\Omega|^{-\left\{\frac{2}{\theta_2} - \left(\frac{5}{2}-\frac{3}{q}-s\right)\right\}} \|\phi\|^2_{L^{\theta_2'}(0,\infty;L^{q'}(\mathbb{R}^3))}. \tag{3.13}$$

In the case $1/\theta_2 = 5/4 - 3/2q - s/2$, since $\widetilde{h}(t) \leqslant |t|^{-\frac{5}{2}+\frac{3}{q}+s}$, we have by (3.13) and the Hardy-Littlewood-Sobolev inequality that

$$\left\| \int_0^\infty \mathcal{G}_{\mp}(\Omega t)[e^{t\Delta}(-\Delta)^{-\frac{s}{2}}\partial_{x_j}\phi(t)]dt \right\|_{L^2}^2 \leqslant \|\phi\|^2_{L^{\theta_2'}(0,\infty;L^{q'}(\mathbb{R}^3))}. \tag{3.14}$$

Combining (3.9), (3.13) and (3.14), we complete the proof of Lemma 3.4. □

4 Nonlinear Estimates

In this section, we consider the estimates for the Duhamel term of (NSC). Put

$$N(u,v)(t) := \int_0^t T_\Omega(t-\tau)\mathbb{P}[(u\cdot\nabla)v](\tau)d\tau$$

for $t \geqslant 0$. We first recall the L^p-L^q smoothing properties for the semigroup T_Ω obtained by Hieber and Shibata [15].

Lemma 4.1 (Hieber-Shibata [15])**.** *Let* $1 \leqslant p \leqslant 2 \leqslant q \leqslant \infty$. *Then for any* $\alpha \in (\mathbb{N} \cup \{0\})^3$ *there exists a positive constant* $C = C(p, q, \alpha)$ *such that*

$$\|\partial^\alpha T_\Omega(t) f\|_{L^q} \leqslant C t^{-\frac{|\alpha|}{2} - \frac{3}{2}(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p}$$

for all $t > 0, \Omega \in \mathbb{R}$ *and* $f \in L^p(\mathbb{R}^3)^3$ *with* $\operatorname{div} f = 0$.

Next we prove the bilinear estimates for $N(\cdot, \cdot)$ in our solution spaces associated with the linear estimates. We impose the following assumption on the exponents of our function spaces.

Assumption (NL1). Let the exponent s satisfy $0 \leqslant s < 3/2$.
Assumption (NL2). Let the exponents p, q, θ_1 and θ_2 satisfy

$$0 < \frac{1}{p} < \frac{1}{3}, \quad \frac{s}{3} < \frac{1}{q} < \frac{1}{2}, \quad \frac{1}{p} + \frac{1}{q} > \frac{1}{2},$$
$$0 < \frac{1}{\theta_1} < \frac{1}{2} - \frac{3}{2p}, \quad 0 < \frac{1}{\theta_2} < 1 - \frac{3}{2q},$$

Lemma 4.2. *Let* s, p, q, θ_1 *and* θ_2 *satisfy Assumptions* $(NL1)$ *and* $(NL2)$. *Then there exists a positive constant* $C = C(s, p, q, \theta_1, \theta_2)$ *such that*

$$\sup_{0<t<T} \|N(u, v)(t)\|_{\dot{H}^s} \leqslant C T^{1 - \frac{3}{2q} - \frac{1}{\theta_2}} \sup_{0<t<T} \|u(t)\|_{\dot{H}^s} \|\nabla v\|_{L^{\theta_2}(0,T;L^q(\mathbb{R}^3))} \tag{4.1}$$

for all $T > 0, \Omega \in \mathbb{R}, u \in L^\infty(0, T; \dot{H}^s(\mathbb{R}^3))^3$ *and* $v \in L^{\theta_2}(0, T; \dot{W}^{1,q}(\mathbb{R}^3))^3$, *and*

$$\|N(u, v)\|_{L^{\theta_1}(0,T;L^p(\mathbb{R}^3))} \leqslant C T^{1 - \frac{3}{2q} - \frac{1}{\theta_2}} \|u\|_{L^{\theta_1}(0,T;L^p(\mathbb{R}^3))} \|\nabla v\|_{L^{\theta_2}(0,T;L^q(\mathbb{R}^3))}, \tag{4.2}$$
$$\|\nabla N(u, v)\|_{L^{\theta_2}(0,T;L^q(\mathbb{R}^3))} \leqslant C T^{\frac{1}{2} - \frac{3}{2p} - \frac{1}{\theta_1}} \|u\|_{L^{\theta_1}(0,T;L^p(\mathbb{R}^3))} \|\nabla v\|_{L^{\theta_2}(0,T;L^q(\mathbb{R}^3))} \tag{4.3}$$

for all $T > 0, \Omega \in \mathbb{R}, u \in L^{\theta_1}(0, T; L^p(\mathbb{R}^3))^3$ *and* $v \in L^{\theta_2}(0, T; \dot{W}^{1,q}(\mathbb{R}^3))^3$.

Proof. We first prove (4.1). Choose r_1 such that

$$\frac{1}{r_1} = \frac{3 - 2s}{6} + \frac{1}{q}.$$

Since $s/3 < 1/q < 1/2$, it is easy to see that $1 < r_1 < 2$. Hence by Lemma 3.1, Lemma 4.1, the Hölder inequality and the Sobolev embedding theorem, we have

$$\begin{aligned} \|N(u, v)(t)\|_{\dot{H}^s} &\leqslant C \int_0^t (t - \tau)^{-\frac{s}{2} - \frac{3}{2}(\frac{1}{r_1} - \frac{1}{2})} \|\mathbb{P}[(u \cdot \nabla) v](\tau)\|_{L^{r_1}} \, d\tau \\ &\leqslant C \int_0^t (t - \tau)^{-\frac{3}{2q}} \|\nabla v(\tau)\|_{L^q} d\tau \sup_{0<t<T} \|u(t)\|_{\dot{H}^s} \end{aligned} \tag{4.4}$$

for all $0 < t < T$. Here since $\frac{1}{\theta_2} < 1 - \frac{3}{2q}$, we see that $\frac{3}{2q}\theta_2' < 1$, where $1/\theta_2 + 1/\theta_2' = 1$. Hence by the Hölder inequality we have

$$\int_0^t (t - \tau)^{-\frac{3}{2q}} \|\nabla v(\tau)\|_{L^q} d\tau \leqslant C T^{1 - \frac{3}{2q} - \frac{1}{\theta_2}} \|\nabla v\|_{L^{\theta_2}(0,T;L^q(\mathbb{R}^3))} \tag{4.5}$$

for all $0 < t < T$. Substituting (4.5) into (4.4), we obtain the estimate (4.1).

Next we shall prove (4.2) and (4.3). Choose r_2 such that

$$\frac{1}{r_2} = \frac{1}{p} + \frac{1}{q}.$$

Note that $1 < r_2 < 2$ since $1/p < 1/3, 1/q < 1/2$ and $1/p + 1/q > 1/2$. It follows from Lemma 4.1 and the Hölder inequality that

$$\|N(u,v)(t)\|_{L^p} \leqslant C \int_0^t (t-\tau)^{-\frac{3}{2q}} \|u(\tau)\|_{L^p} \|\nabla v(\tau)\|_{L^q} d\tau \tag{4.6}$$

for all $0 < t < T$. Putting

$$\frac{1}{\theta_3} := \frac{1}{\theta_1} + \frac{2q-3}{2q},$$

we see that

$$\frac{1}{\theta_1} = \frac{1}{\theta_3} - \left(1 - \frac{3}{2q}\right).$$

Hence (4.6), the Hardy-Littlewood-Sobolev inequality and the Hölder inequality yield that

$$\begin{aligned} \|N(u,v)\|_{L^{\theta_1}(0,T;L^p(\mathbb{R}^3))} &\leqslant C \big\| \|u(t)\|_{L^p} \|\nabla v(t)\|_{L^q} \big\|_{L^{\theta_3}(0,T)} \\ &\leqslant C \|u\|_{L^{\theta_1}(0,T;L^p(\mathbb{R}^3))} \|\nabla v\|_{L^{\frac{2q}{2q-3}}(0,T;L^q(\mathbb{R}^3))}. \end{aligned} \tag{4.7}$$

Here we remark that the assumption $\frac{1}{\theta_2} < 1 - \frac{3}{2q}$ implies that $\frac{2q}{2q-3} < \theta_2$. Hence by the Hölder inequality we have that

$$\|\nabla v\|_{L^{\frac{2q}{2q-3}}(0,T;L^q(\mathbb{R}^3))} \leqslant T^{1-\frac{3}{2q}-\frac{1}{\theta_2}} \|\nabla v\|_{L^{\theta_2}(0,T;L^q(\mathbb{R}^3))}. \tag{4.8}$$

Substituting (4.8) into (4.7), we obtain (4.2).

Similarly to (4.6), it follows from Lemma 4.1 and the Hölder inequality that

$$\|\nabla N(u,v)(t)\|_{L^q} \leqslant C \int_0^t (t-\tau)^{-\frac{1}{2}-\frac{3}{2p}} \|u(\tau)\|_{L^p} \|\nabla v(\tau)\|_{L^q} d\tau \tag{4.9}$$

for all $0 < t < T$. Putting

$$\frac{1}{\theta_4} := \frac{1}{\theta_2} + \frac{p-3}{2p},$$

we see that

$$\frac{1}{\theta_2} = \frac{1}{\theta_4} - \left(\frac{1}{2} - \frac{3}{2p}\right).$$

Hence (4.9), the Hardy-Littlewood-Sobolev inequality and the Hölder inequality yield that

$$\begin{aligned} \|\nabla N(u,v)\|_{L^{\theta_2}(0,T;L^p(\mathbb{R}^3))} &\leqslant C \big\| \|u(t)\|_{L^p} \|\nabla v(t)\|_{L^q} \big\|_{L^{\theta_4}(0,T)} \\ &\leqslant C \|u\|_{L^{\frac{2p}{p-3}}(0,T;L^p(\mathbb{R}^3))} \|\nabla v\|_{L^{\theta_2}(0,T;L^q(\mathbb{R}^3))}. \end{aligned} \tag{4.10}$$

Here we remark that the assumption $\frac{1}{\theta_1} < \frac{1}{2} - \frac{3}{2p}$ implies that $\frac{2p}{p-3} < \theta_1$. Hence by the Hölder inequality we have that

$$\|u\|_{L^{\frac{2p}{p-3}}(0,T;L^p(\mathbb{R}^3))} \leqslant T^{\frac{1}{2}-\frac{3}{2p}-\frac{1}{\theta_1}} \|u\|_{L^{\theta_1}(0,T;L^q(\mathbb{R}^3))}. \tag{4.11}$$

Substituting (4.11) into (4.10), we obtain (4.3). This completes the proof of Lemma 4.2. ☐

5 Proof of Theorems

In this section, we shall give the proof of Theorem 1.1. One can prove Theorem 1.3 in the similar way to that of Theorem 1.1.

Proof of Theorem 1.1. Let s, p, q, θ_1 and θ_2 satisfy Assumptions (A), (B) and (C). Note that these exponents satisfy Assumptions (L1), (L2), (L3), (L4), (NL1) and (NL2). Let $\Omega \in \mathbb{R}\setminus\{0\}$, and suppose that $u_0 \in \dot{H}^s(\mathbb{R}^3)^3$ satisfying $\operatorname{div} u_0 = 0$. Lemma 3.2, Lemma 3.3 and Lemma 3.4 yield that there exists a positive constant $C_1 = C_1(s, p, q, \theta_1, \theta_2)$ such that

$$\begin{aligned}\sup_{0<t<T} \|T_\Omega(t)u_0\|_{\dot{H}^s} + |\Omega|^{\frac{1}{\theta_1}-\left(\frac{3}{4}-\frac{3}{2p}-\frac{s}{2}\right)} \|T_\Omega(\cdot)u_0\|_{L^{\theta_1}(0,T;L^p(\mathbb{R}^3))} \\ + |\Omega|^{\frac{1}{\theta_2}-\left(\frac{5}{4}-\frac{3}{2q}-\frac{s}{2}\right)} \|\nabla T_\Omega(\cdot)u_0\|_{L^{\theta_2}(0,T;L^q(\mathbb{R}^3))} \leqslant C_1\|u_0\|_{\dot{H}^s}\end{aligned} \tag{5.1}$$

for all $T > 0$. Then, we define the map Ψ and the solution space X_T by

$$\begin{gathered}\Psi(u)(t) := T_\Omega(t)u_0 - N(u,u)(t), \\ X_T := \left\{ u \in C([0,T];\dot{H}^s(\mathbb{R}^3))^3 \;\middle|\; \|u\|_{X_T} \leqslant 2C_1\|u_0\|_{\dot{H}^s}, \operatorname{div} u = 0 \right\}\end{gathered}$$

with

$$\begin{aligned}\|u\|_{X_T} := \sup_{0<t<T} \|u(t)\|_{\dot{H}^s} + |\Omega|^{\frac{1}{\theta_1}-\left(\frac{3}{4}-\frac{3}{2p}-\frac{s}{2}\right)} \|u\|_{L^{\theta_1}(0,T;L^p(\mathbb{R}^3))} \\ + |\Omega|^{\frac{1}{\theta_2}-\left(\frac{5}{4}-\frac{3}{2q}-\frac{s}{2}\right)} \|\nabla u\|_{L^{\theta_2}(0,T;L^q(\mathbb{R}^3))}\end{aligned}$$

for some $T > 0$ to be chosen later, where $N(\cdot,\cdot)$ is defined in Section 4. From (5.1) and Lemma 4.2, there exists a positive constant $C_2 = C_2(s, p, q, \theta_1, \theta_2)$ such that

$$\begin{aligned}\|\Psi(u)\|_{X_T} &\leqslant C_1\|u_0\|_{\dot{H}^s} + C_2\left\{ \frac{T^{\frac{1}{2}-\frac{3}{2p}-\frac{1}{\theta_1}}}{|\Omega|^{\frac{1}{\theta_1}-(\frac{3}{4}-\frac{3}{2p}-\frac{s}{2})}} + \frac{T^{1-\frac{3}{2q}-\frac{1}{\theta_2}}}{|\Omega|^{\frac{1}{\theta_2}-(\frac{5}{4}-\frac{3}{2q}-\frac{s}{2})}} \right\} \|u\|_{X_T}^2 \\ &\leqslant C_1\|u_0\|_{\dot{H}^s} + 4C_1^2C_2\|u_0\|_{\dot{H}^s}^2 \left\{ \frac{T^{\frac{1}{2}-\frac{3}{2p}-\frac{1}{\theta_1}}}{|\Omega|^{\frac{1}{\theta_1}-(\frac{3}{4}-\frac{3}{2p}-\frac{s}{2})}} + \frac{T^{1-\frac{3}{2q}-\frac{1}{\theta_2}}}{|\Omega|^{\frac{1}{\theta_2}-(\frac{5}{4}-\frac{3}{2q}-\frac{s}{2})}} \right\}\end{aligned} \tag{5.2}$$

for all $T > 0$ and $u \in X_T$. Moreover, it follows from Lemma 4.2 that there exists a positive constant $C_3 = C_3(s, p, q, \theta_1, \theta_2)$ such that

$$\begin{aligned}\|\Psi(u) - \Psi(v)\|_{X_T} &\leqslant C_3\left\{ \frac{T^{\frac{1}{2}-\frac{3}{2p}-\frac{1}{\theta_1}}}{|\Omega|^{\frac{1}{\theta_1}-(\frac{3}{4}-\frac{3}{2p}-\frac{s}{2})}} + \frac{T^{1-\frac{3}{2q}-\frac{1}{\theta_2}}}{|\Omega|^{\frac{1}{\theta_2}-(\frac{5}{4}-\frac{3}{2q}-\frac{s}{2})}} \right\} \left(\|u\|_{X_T} + \|v\|_{X_T}\right) \|u-v\|_{X_T} \\ &\leqslant 4C_1C_3\|u_0\|_{\dot{H}^s} \left\{ \frac{T^{\frac{1}{2}-\frac{3}{2p}-\frac{1}{\theta_1}}}{|\Omega|^{\frac{1}{\theta_1}-(\frac{3}{4}-\frac{3}{2p}-\frac{s}{2})}} + \frac{T^{1-\frac{3}{2q}-\frac{1}{\theta_2}}}{|\Omega|^{\frac{1}{\theta_2}-(\frac{5}{4}-\frac{3}{2q}-\frac{s}{2})}} \right\} \|u-v\|_{X_T}\end{aligned} \tag{5.3}$$

for all $T > 0$ and $u, v \in X_T$. Choose $T > 0$ such that

$$\frac{T^{\frac{1}{2}-\frac{3}{2p}-\frac{1}{\theta_1}}}{|\Omega|^{\frac{1}{\theta_1}-(\frac{3}{4}-\frac{3}{2p}-\frac{s}{2})}} \leqslant \min\left\{ \frac{1}{8C_1C_2\|u_0\|_{\dot{H}^s}}, \frac{1}{16C_1C_3\|u_0\|_{\dot{H}^s}} \right\}$$

and

$$\frac{T^{1-\frac{3}{2q}-\frac{1}{\theta_2}}}{|\Omega|^{\frac{1}{\theta_2}-(\frac{5}{4}-\frac{3}{2q}-\frac{s}{2})}} \leqslant \min\left\{\frac{1}{8C_1C_2\|u_0\|_{\dot{H}^s}}, \frac{1}{16C_1C_3\|u_0\|_{\dot{H}^s}}\right\}.$$

Then we obtain from (5.2) and (5.3) that

$$\|\Psi(u)\|_{X_T} \leqslant 2C_1\|u_0\|_{\dot{H}^s}, \quad \|\Psi(u)-\Psi(v)\|_{X_T} \leqslant \frac{1}{2}\|u-v\|_{X_T}$$

for all u and v in X_T. Therefore, by the contraction mapping principle, we complete the proof of Theorem 1.1. □

Acknowledgement The authors would like to express their sincere gratitude to Professor Hideo Kozono for his valuable suggestions and continuous encouragement. The second author is partly supported by Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists.

References

[1] A. Babin, A. Mahalov, and B. Nicolaenko, *Long-time averaged Euler and Navier-Stokes equations for rotating fluids*, Nonlinear Waves in Fluids (Hannover, 1994), Adv. Ser. Nonlinear Dynam., vol. 7, World Sci. Publ., River Edge, NJ, 1995, pp. 145–157.

[2] ______, *Regularity and integrability of 3D Euler and Navier-Stokes equations for rotating fluids*, Asymptot. Anal. **15** (1997), 103–150.

[3] ______, *Global regularity of 3D rotating Navier-Stokes equations for resonant domains*, Indiana Univ. Math. J. **48** (1999), 1133–1176.

[4] ______, *3D Navier-Stokes and Euler equations with initial data characterized by uniformly large vorticity*, Indiana Univ. Math. J. **50** (2001), 1–35.

[5] J. Bourgain and N. Pavlović, *Ill-posedness of the Navier-Stokes equations in a critical space in 3D*, J. Funct. Anal. **255** (2008), 2233–2247.

[6] J.-Y. Chemin, B. Desjardins, I. Gallagher, and E. Grenier, *Anisotropy and dispersion in rotating fluids*, Stud. Math. Appl., vol. 31, North-Holland, Amsterdam, 2002, pp. 171–192.

[7] ______, *Mathematical geophysics*, Oxford Lecture Series in Mathematics and its Applications, vol. 32, The Clarendon Press Oxford University Press, Oxford, 2006.

[8] A. Dutrifoy, *Examples of dispersive effects in non-viscous rotating fluids*, J. Math. Pures Appl. **84** (2005), 331-356.

[9] H. Fujita and T. Kato, *On the Navier-Stokes initial value problem. I*, Arch. Rational Mech. Anal. **16** (1964), 269–315.

[10] P. Germain, *The second iterate for the Navier-Stokes equation*, J. Funct. Anal. **255** (2008), 2248–2264.

[11] Y. Giga, *Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system*, J. Differential Equations **62** (1986), 186–212.

[12] Y. Giga, K. Inui, A. Mahalov, and S. Matsui, *Uniform local solvability for the Navier-Stokes equations with the Coriolis force*, Methods Appl. Anal. **12** (2005), 381–393.

[13] ______, *Navier-Stokes equations in a rotating frame in $\mathbb{R}^3$ with initial data nondecreasing at infinity*, Hokkaido Math. J. **35** (2006), 321–364.

[14] Y. Giga, K. Inui, A. Mahalov, and J. Saal, *Uniform global solvability of the rotating Navier-Stokes equations for nondecaying initial data*, Indiana Univ. Math. J. **57** (2008), 2775–2791.

[15] M. Hieber and Y. Shibata, *The Fujita-Kato approach to the Navier-Stokes equations in the rotational framework*, Math. Z. **265** (2010), 481–491.

[16] T. Iwabuchi and R. Takada, *Dispersive effect of the Coriolis force and the local well-posedness for the Navier-Stokes equations in the rotational framework*, submitted.

[17] T. Kato, *Strong L^p-solutions of the Navier-Stokes equation in $\mathbf{R}^m$, with applications to weak solutions*, Math. Z. **187** (1984), 471–480.

[18] H. Koch and D. Tataru, *Well-posedness for the Navier-Stokes equations*, Adv. Math. **157** (2001), 22–35.

[19] P. Konieczny and T. Yoneda, *On dispersive effect of the Coriolis force for the stationary Navier-Stokes equations*, J. Differential Equations **250** (2011), 3859–3873.

[20] H. Kozono and M. Yamazaki, *Semilinear heat equations and the Navier-Stokes equation with distributions in new function spaces as initial data*, Comm. Partial Differential Equations **19** (1994), 959–1014.

[21] H. Kozono, T. Ogawa, and Y. Taniuchi, *Navier-Stokes equations in the Besov space near L^∞ and BMO*, Kyushu J. Math. **57** (2003), 303–324.

[22] O. Sawada, *The Navier-Stokes flow with linearly growing initial velocity in the whole space*, Bol. Soc. Parana. Mat. (3) **22** (2004), 75–96.

[23] T. Yoneda, *Ill-posedness of the 3D-Navier-Stokes equations in a generalized Besov space near* BMO^{-1}, J. Funct. Anal. **258** (2010), 3376–3387.