Mean continuity for potentials of functions in Musielak-Orlicz spaces

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Abstract

Our aim in this paper is to show mean continuity in a certain strong sense at points except in a small set for potentials of functions in Musielak-Orlicz spaces.

§1. Introduction

For the Riesz potential

\[ I_\alpha f(x) := \int_{\mathbb{R}^N} |x - y|^{\alpha-N} f(y) \, dy, \]

where \(0 < \alpha < N\) and \(f \in L^{p}_{\text{loc}}(\mathbb{R}^N)\) \((1 \leq p < \infty)\) is assumed to satisfy

\[ \int_{\mathbb{R}^N} (1 + |x|)^{\alpha-N} |f(x)| \, dx < \infty, \]

the following mean continuity is known (see, e.g., [1], [10] and [14]):
If \( p > 1 \), \( \alpha p < N \) and \( 1/p^* = 1/p - \alpha/N \), then
\[
\lim_{r \to 0^+} \frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} |I_\alpha f(x) - I_\alpha f(x_0)|^{p^*} dx = 0
\]
for \( x_0 \in \mathbb{R}^N \setminus E \) with a set \( E \) of \((\alpha, p)\)-capacity zero. (\(|B(x_0, r)|\) denotes the Lebesgue measure of \( B(x_0, r) \).)

Variable exponent Lebesgue spaces and Sobolev spaces were introduced to discuss nonlinear partial differential equations with non-standard growth conditions. Mean continuity of Riesz potentials of functions in variable exponent Lebesgue spaces \( L^{p(\cdot)} \) was investigated in [3] (also, cf. [2] and [4] for mean continuity of functions in variable exponent Sobolev spaces). For Riesz potentials on the two variable exponents spaces \( L^{p(\cdot)}(\log L)^{q(\cdot)} \), see [11]. These spaces are special cases of so-called Musielak-Orlicz spaces ([12]).

Our aim in this paper is to show mean continuity in a certain strong sense at points except in a small set for potentials of functions in Musielak-Orlicz spaces as an extension of the above results. Recently, a capacity defined by potentials of functions in Musielak-Orlicz spaces was introduced in [5]. We discuss the size of the exceptional sets using such capacity.

§ 2. Preliminaries

In this paper, we consider a function
\[
\Phi(x, t) := t\phi(x, t) : \mathbb{R}^N \times [0, \infty) \to [0, \infty)
\]
satisfying the following conditions (\( \Phi 1 \)) - (\( \Phi 4 \)):

\( \Phi 1 \) \( \phi(\cdot, t) \) is measurable on \( \mathbb{R}^N \) for each \( t \geq 0 \) and \( \phi(x, \cdot) \) is continuous on \([0, \infty)\) for each \( x \in \mathbb{R}^N \);

\( \Phi 2 \) there exists a constant \( A_1 \geq 1 \) such that
\[
A_1^{-1} \leq \phi(x, 1) \leq A_1 \quad \text{for all } x \in \mathbb{R}^N;
\]

\( \Phi 3 \) \( \phi(x, \cdot) \) is uniformly almost increasing, namely there exists a constant \( A_2 \geq 1 \) such that
\[
\phi(x, t) \leq A_2 \phi(x, s) \quad \text{for all } x \in \mathbb{R}^N \quad \text{whenever } 0 \leq t < s;
\]

\( \Phi 4 \) there exists a constant \( A_3 \geq 1 \) such that
\[
\phi(x, 2t) \leq A_3 \phi(x, t) \quad \text{for all } x \in \mathbb{R}^N \text{ and } t > 0.
\]
Note that $(\Phi 2)$, $(\Phi 3)$ and $(\Phi 4)$ imply
\[
0 < \inf_{x \in \mathbb{R}^N} \phi(x, t) \leq \sup_{x \in \mathbb{R}^N} \phi(x, t) < \infty
\]
for each $t > 0$.

If $\Phi(x, \cdot)$ is convex for each $x \in \mathbb{R}^N$, then $(\Phi 3)$ holds with $A_2 = 1$; namely $\phi(x, \cdot)$ is non-decreasing for each $x \in \mathbb{R}^N$.

Let $\bar{\phi}(x, t) := \sup_{0 \leq s \leq t} \phi(x, s)$ and
\[
\overline{\Phi}(x, t) := \int_0^t \bar{\phi}(x, r) \, dr
\]
for $x \in \mathbb{R}^N$ and $t \geq 0$. Then $\overline{\Phi}(x, t)$ satisfies $(\Phi 1)-(\Phi 4)$. Furthermore, $\overline{\Phi}(x, \cdot)$ is convex and
\[
\frac{1}{2A_3} \Phi(x, t) \leq \overline{\Phi}(x, t) \leq A_2 \Phi(x, t)
\]
for all $x \in \mathbb{R}^N$ and $t \geq 0$.

By $(\Phi 3)$, we see that
\[
\Phi(x, at) \begin{cases} 
\leq A_2 a \Phi(x, t) & \text{if } 0 \leq a \leq 1 \\
\geq A_2^{-1} a \Phi(x, t) & \text{if } a \geq 1.
\end{cases}
\]

**Example 2.1.** Let $p(\cdot)$ and $q(\cdot)$ be measurable functions on $\mathbb{R}^N$ such that

(P1) $1 \leq p^− := \inf_{x \in \mathbb{R}^N} p(x) \leq \sup_{x \in \mathbb{R}^N} p(x) =: p^+ < \infty$

and

(Q1) $-\infty < q^− := \inf_{x \in \mathbb{R}^N} q(x) \leq \sup_{x \in \mathbb{R}^N} q(x) =: q^+ < \infty$.

Then, $\Phi_{p(\cdot),q(\cdot),a}(x, t) = t^{p(x)}(\log(a + t))^q(x) (a \geq e)$ satisfies $(\Phi 1)$, $(\Phi 2)$ and $(\Phi 4)$. It satisfies $(\Phi 3)$ if $p^− > 1$ or $q^− \geq 0$. As a matter of fact, it satisfies $(\Phi 3)$ if and only if $q(x) \geq 0$ at points $x$ where $p(x) = 1$ and

$$
\sup_{x:p(x)>1,q(x)<0} q(x) \log(p(x) - 1) < \infty
$$

(see section 6: Appendix).

Given $\Phi(x, t)$ as above and an open set $G$ in $\mathbb{R}^N$, the associated Musielak-Orlicz space on $G$ is defined by
\[
L^\Phi(G) = \left\{ f \in L_{\text{loc}}^1(G); \int_G \Phi(y, |f(y)|) \, dy < \infty \right\},
\]
which is a Banach space with respect to the norm
\[ \|f\|_{L^\Phi(G)} = \inf \left\{ \lambda > 0; \int_G \Phi(y, |f(y)|/\lambda) \, dy \leq 1 \right\} \]
(cf. [12]).

Lemma 2.2.
\[ (2A_3)^{-1} \int_G \Phi(x, |f(x)|) \, dx \leq \|f\|_{L^\Phi(G)} \leq 2 \left( A_2 \int_G \Phi(x, |f(x)|) \, dx \right)^\sigma \]
whenever \( \|f\|_{L^\Phi(G)} \leq 1 \), where \( \sigma = \log 2/\log(2A_3) > 0 \).

Proof. Let \( f \in L^\Phi(G) \) and suppose \( \lambda := \|f\|_{L^\Phi(G)} \leq 1 \). Then by (2.1),
\[ \int_G \Phi(x, |f(x)|) \, dx \leq 2A_3 \int_G \Phi(x, |f(x)|) \, dx \leq 2A_3 \lambda \int_G \Phi(x, |f(x)|/\lambda) \, dx \leq 2A_3 \lambda. \]

On the other hand, suppose \( \lambda^* := \int_G \Phi(x, |f(x)|) \, dx \leq A_2^{-1} \). Choose \( k \in \mathbb{N} \) such that \((2A_3)^{-k} < A_2 \lambda^* \leq (2A_3)^{-k+1}\). Then, by (2.1) and (\( \Phi4 \))
\[ \int_G \Phi(x, 2^{k-1}|f(x)|) \, dx \leq A_2 \int_G \Phi(x, 2^{k-1}|f(x)|) \, dx \leq A_2 (2A_3)^{k-1} \lambda^* \leq 1. \]
Hence \( \|f\|_{L^\Phi(G)} \leq 2^{1-k} \). Since \( 2^{-k} < (A_2 \lambda^*)^{\sigma} \),
\[ \|f\|_{L^\Phi(G)} \leq 2 \left( A_2 \int_G \Phi(x, |f(x)|) \, dx \right)^\sigma. \]
\[ \square \]

We shall also consider the following conditions:

(\( \Phi5 \)) for every \( \gamma > 0 \), there exists a constant \( B_\gamma \geq 1 \) such that
\[ \phi(x, t) \leq B_\gamma \phi(y, t) \]
whenever \( |x - y| \leq \gamma t^{-1/N} \) and \( t \geq 1 \);

(\( \Phi3^* \)) \( t \mapsto t^{-\varepsilon_0} \phi(x, t) \) is uniformly almost increasing on \((0, \infty) \) for some \( \varepsilon_0 > 0 \), namely there exists a constant \( A_{2, \varepsilon_0} \geq 1 \) such that
\[ t^{-\varepsilon_0} \phi(x, t) \leq A_{2, \varepsilon_0} s^{-\varepsilon_0} \phi(x, s) \]
for all \( x \in \mathbb{R}^N \) whenever \( 0 < t < s \).
Example 2.3. Let $\Phi_{p(\cdot),q(\cdot),a}(x,t)$ be as in Example 2.1. It satisfies $(\Phi 5)$ if

(P2) $p(\cdot)$ is log-Hölder continuous, namely

$$|p(x) - p(y)| \leq \frac{C_p}{\log(1/|x-y|)} \quad \text{for } |x-y| \leq \frac{1}{2}$$

with a constant $C_p \geq 0$,

and

(Q2) $q(\cdot)$ is log-log-Hölder continuous, namely

$$|q(x) - q(y)| \leq \frac{C_q}{\log(\log(1/|x-y|))} \quad \text{for } |x-y| \leq e^{-2}$$

with a constant $C_q \geq 0$.

It satisfies $(\Phi 3^*)$ if $p^- > 1$ with $0 < \varepsilon_0 < p^- - 1$.

In this paper, as a kernel function on $\mathbb{R}^N$, we consider $k(x) = k(|x|)$ (with the abuse of notation) with a function $k(r) : (0, \infty) \to (0, \infty)$ satisfying the following conditions:

(k1) $k(r)$ is non-increasing and lower semicontinuous on $(0, \infty)$;

(k2) \( \int_0^1 k(r)r^{N-1}dr < \infty \);

(k3) there exists a constant $K_1 \geq 1$ such that $k(r) \leq K_1k(r+1)$ for all $r \geq 1$.

By (k2), $k(\cdot) \in L^1_{\text{loc}}(\mathbb{R}^N)$. We set $k(0) = \lim_{r \to 0^+} k(r)$.

Let

$$\bar{k}(r) := \frac{N}{r^N} \int_0^r k(\rho)\rho^{N-1}d\rho$$

for $r > 0$. Then $k(r) \leq \bar{k}(r)$, $\bar{k}(r)$ is non-increasing and

$$\lim_{r \to 0^+} r^N \bar{k}(r) = 0.\quad (2.3)$$

For $0 < \alpha < N$, the Riesz kernel $I_\alpha(x) = |x|^{\alpha-N}$ and the Bessel kernel $g_\alpha$ of order $\alpha$ are typical examples of $k(x)$ satisfying above conditions.

We define the $k$-potential of a locally integrable function $f$ on $\mathbb{R}^N$ by

$$k * f(x) = \int_{\mathbb{R}^N} k(x-y)f(y)dy.$$ 

Here it is natural to assume that

$$\int_{\mathbb{R}^N} k(1+|y|)|f(y)|dy < \infty,\quad (2.4)$$
which is equivalent to the condition that \( k * |f| \neq \infty \) by the conditions (k2) and (k3) (see [10, Theorem 1.1, Chapter 2]). Note that \( k * f \in L^1_{\text{loc}}(\mathbb{R}^N) \) under this assumption.

Set
\[
\Gamma(x,s) := s^{-1}k(s^{-1/N})\Phi^{-1}(x,s) \quad (x \in \mathbb{R}^N, \ s > 0),
\]
where \( \Phi^{-1}(x,s) = \sup\{t > 0; \Phi(x,t) < s\} \).

Here we note:
\[
(2.5) \quad \Gamma(x,\Phi(x,t)) \approx t\Phi(x,t)^{-1}k(\Phi(x,t)^{-1/N}),
\]
since \( \Phi^{-1}(x,\Phi(x,t)) \approx t \) (cf. [7, Lemma 5.2 (4)]). (For two functions \( f \) and \( g \), \( f \approx g \) means that there is a constant \( C \geq 1 \) such that \( C^{-1}g \leq f \leq Cg \).)

We shall consider the following condition (\( \Phi k \)):
\[
(\Phi k) \quad s \mapsto s^{-\varepsilon_1} \Gamma(x,s) \text{ is uniformly almost increasing on } (0, \infty) \text{ for some } \varepsilon_1 > 0, \text{ namely there exists a constant } A_{\Gamma} \geq 1 \text{ such that}
\]
\[
s_1^{-\varepsilon_1} \Gamma(x,s_1) \leq A_{\Gamma}s_2^{-\varepsilon_1} \Gamma(x,s_2)
\]
for all \( x \in \mathbb{R}^N \) whenever \( 0 < s_1 < s_2 \).

**Example 2.4.** If \( k \) is the Riesz kernel \( I_\alpha \), then \( \Phi_{p(\cdot),q(\cdot),a}(x,t) \) in Example 2.1 satisfies (\( \Phi k \)) if \( \alpha p^+ < N \).

We consider a function \( \Psi(x,t) : \mathbb{R}^N \times [0, \infty) \rightarrow [0, \infty) \) satisfying the following conditions:
\[
(\Psi 1) \quad \Psi(\cdot,t) \text{ is measurable on } \mathbb{R}^N \text{ for each } t \geq 0 \text{ and } \Psi(x,\cdot) \text{ is continuous on } [0, \infty) \text{ for each } x \in \mathbb{R}^N;
\]
\[
(\Psi 2) \quad \text{there is a constant } A_4 \geq 1 \text{ such that}
\]
\[
\Psi(x,at) \leq A_4a\Psi(x,t)
\]
for all \( x \in \mathbb{R}^N, t > 0 \) and \( 0 \leq a \leq 1 \);
\[
(\Psi \Phi k) \quad \text{there exists a constant } A_5 \geq 1 \text{ such that}
\]
\[
\Psi(x,\Gamma(x,s)) \leq A_5s
\]
for all \( x \in \mathbb{R}^N \) and \( s > 0 \).

Note: (\( \Psi 2 \)) implies that \( \Psi(x,\cdot) \) is uniformly almost increasing on \( [0, \infty) \); if we assume (\( \Phi k \)), then \( \Gamma(x,t) \rightarrow \infty \) uniformly as \( t \rightarrow \infty \), and hence (\( \Psi \Phi k \)) implies that \( \Psi(\cdot,t) \) is bounded on \( \mathbb{R}^N \) for every \( t > 0 \).
Example 2.5. For \(\Phi_{p(\cdot),q(\cdot),a}(x, t)\) in Example 2.1 and the Riesz kernel \(I_{\alpha}(0 < \alpha < N)\), if \(\alpha p^{+} < N\), then
\[
\Gamma(x, s) \approx s^{1/p_{\#}(x)} [\log(e + s)]^{-q(x)/p(x)}
\]
with
\[
\frac{1}{p_{\#}(x)} := \frac{1}{p(x)} - \frac{\alpha}{N},
\]
so that we may take
\[
\Psi(x, t) = t^{p_{\#}(x)} (\log(e + t))^{p_{\#}(x)q(x)/p(x)}.
\]

We know the following result (see [6, Corollary 6.3]; also cf. [7, Corollary 6.5]; note that condition \((\Psi \Phi k)\) given there is essentially the same as the above one, in view of (2.5)).

Lemma 2.6. Suppose \(\Phi(x, t)\) satisfies \((\Phi 3^*)\), \((\Phi 5)\) and \((\Phi k)\); \(\Psi(x, t)\) satisfies \((\Psi 1)\), \((\Psi 2)\) and \((\Psi \Phi k)\). Then there exists a constant \(C^{*} > 0\), such that
\[
\int_{B(0,1)} \Psi(x, k f(x)/C^{*}) dx \leq 1
\]
for all \(f \geq 0\) satisfying \(\|f\|_{L^{\#}(B(0,1))} \leq 1\).

§ 3. Mean continuity

In this section, we prove our main theorem, which gives an extension of Meyers [9], Harjulehto-Hästö [4] and the authors [3, Theorem 4.5], [11, Theorem 3.4].

For a measurable function \(u\) on \(\mathbb{R}^{N}\), we define the integral mean over a measurable set \(E \subset \mathbb{R}^{N}\) of positive measure by
\[
\int_{E} u(x) \, dx := \frac{1}{|E|} \int_{E} u(x) \, dx.
\]

Theorem 3.1. Let \(f\) be a nonnegative measurable function on \(\mathbb{R}^{N}\) satisfying (2.4) and set
\[
E_{1} := \{ x \in \mathbb{R}^{N} : k * f(x) = \infty \},
\]
\[
E_{2} := \left\{ x \in \mathbb{R}^{N} : \limsup_{r \to 0^{+}} \int_{B(x,r)} \Phi \left( z, r^{N} k(r) f(z) \right) dz > 0 \right\}.
\]

(1) Suppose \(k(r)\) satisfies
(k4) there is a constant $K_2 > 0$ such that

$$k(r/2) \leq K_2 k(r) \quad \text{for all } 0 < r \leq 1.$$  

Then

$$\lim_{r \to 0+} \int_{B(x_0, r)} |k * f(x) - k * f(x_0)| \, dx = 0$$

for all $x_0 \in \mathbb{R}^N \setminus (E_1 \cup E_2)$.

(2) Besides the assumptions on $k(r), \Phi(x, t)$ and $\Psi(x, t)$ given in Lemma 2.6, assume further that $k(r)$ satisfies

(k5) there is a constant $K_3 > 0$ such that

$$k(rs) \leq K_3 \bar{k}(r)k(s) \quad \text{for all } 0 < r \leq 1, 0 < s \leq 1.$$  

Then

$$\lim_{r \to 0+} \int_{B(x_0, r)} \Psi(x, |k * f(x) - k * f(x_0)|) \, dx = 0$$

for all $x_0 \in \mathbb{R}^N \setminus (E_1 \cup E_2)$.

Note that (k5) implies (k4) with $K_2 = K_3 \bar{k}(1/2)$. The Riesz kernel $I_{\alpha} (0 < \alpha < N)$ satisfies (k5).

**Lemma 3.2.** Let $x_0 \in \mathbb{R}^N$ and let $f$ be a nonnegative measurable function on $\mathbb{R}^N$ satisfying

$$\lim_{r \to 0+} \int_{B(x_0, r)} \Phi(z, r^N \bar{k}(r)f(z)) \, dz = 0.$$  

Then

$$\lim_{r \to 0+} \bar{k}(r) \int_{B(x_0, r)} f(y) \, dy = 0.$$  

**Proof.** For $\epsilon > 0$ ($\epsilon \leq 1$), we see from (P3), (P2) and (P4) that

$$\int_{B(x_0, r)} f(y) \, dy \leq \int_{B(x_0, r)} \epsilon r^{-N} \bar{k}(r)^{-1} \, dy + A_2 \int_{B(x_0, r)} f(y) \frac{\phi(y, \epsilon^{-1}r^N \bar{k}(r)f(y))}{\phi(y, 1)} \, dy$$

$$\leq \nu_N \epsilon \bar{k}(r)^{-1} + A_1 A_2 \epsilon r^{-N} \bar{k}(r)^{-1} \int_{B(x_0, r)} \Phi(y, \epsilon^{-1}r^N \bar{k}(r)f(y)) \, dy$$

$$\leq \nu_N \epsilon \bar{k}(r)^{-1} + A(\epsilon) r^{-N} \bar{k}(r)^{-1} \int_{B(x_0, r)} \Phi(y, r^N \bar{k}(r)f(y)) \, dy,$$
where \( \nu_N = |B(0,1)| \), so that
\[
\limsup_{r \to 0^+} \int_{B(x_0,r)} f(y) \, dy \leq \nu_N \varepsilon.
\]

Hence, we have the required result. \( \square \)

**Proof of Theorem 3.1.** Let \( x_0 \in \mathbb{R}^N \setminus (E_1 \cup E_2) \) and write
\[
k*f(x) - k*f(x_0) = \int_{B(x_0,2|x-x_0|)} k(x-y)f(y) \, dy \\
+ \int_{\mathbb{R}^N \setminus B(x_0,2|x-x_0|)} k(x-y)f(y) \, dy - k*f(x_0)
= I_1(x) + I_2(x).
\]

(1) If \( y \in \mathbb{R}^N \setminus B(x_0,2|x-x_0|) \), then \( |x_0-y| \leq 2|x-y| \). Hence, if \( |x_0-y| \leq 1 \), then \( k(x-y) \leq k(|x_0-y|/2) \leq K_2 k(x_0-y) \) by (k1) and (k4); if \( 1 < |x_0-y| \leq 2 \), then \( |x-y| \geq |x_0-y|/2 > 1/2 \), so that \( k(x-y) \leq k(1/2) \leq k(1/2)k(2)^{-1}k(x_0-y) \) by (k1); if \( |x_0-y| > 2 \) and \( |x-x_0| \leq 1 \), then \( k(x-y) \leq k(|x_0-y| - 1) \leq K_1 k(x_0-y) \) by (k1) and (k3). Thus,
\[
k(x-y) \leq K'k(x_0-y)
\]
with \( K' = \max\{K_2, k(1/2)/k(2), K_1\} \), whenever \( y \in \mathbb{R}^N \setminus B(x_0,2|x-x_0|) \) and \( |x-x_0| \leq 1 \).

By (k1), \( k(r) \) is continuous a.e. on \((0, \infty)\), so that \( k(x-y) \to k(x_0-y) \) as \( x \to x_0 \) for almost every \( y \in \mathbb{R}^N \). Since \( k*f(x_0) < \infty \), noting (3.3) we can apply Lebesgue’s dominated convergence theorem to obtain
\[
\lim_{x \to x_0} I_2(x) = 0.
\]

Hence
\[
\lim_{r \to 0^+} \int_{B(x_0,r)} |I_2(x)| \, dx = 0.
\]

For \( I_1 \), note that
\[
0 \leq I_1(x) \leq \int_{B(x_0,r)} k(x-y)f(y) \, dy = k*f_r(x)
\]
for \( x \in B(x_0,r/2) \), where \( f_r := f \chi_{B(x_0,r)} \) and \( \chi_E \) is the characteristic function of \( E \). Hence,
\[
\int_{B(x_0,r/2)} I_1(x) \, dx \leq \int_{B(x_0,r/2)} k*f_r(x) \, dx \\
= \int_{B(x_0,r)} \left( \int_{B(x_0,r/2)} k(x-y) \, dx \right) f(y) \, dy.
\]
Since
\[ \int_{B(x_{0}, r/2)} k(x - y) \, dy \leq \int_{B(x_{0}, r/2)} k(x_{0} - y) \, dy = \tilde{k}(r/2) \leq 2^{N} \tilde{k}(r), \]
we have
\[ \lim_{r \to 0^+} \int_{B(x_{0}, r)} I_1(x) \, dx = 0 \]
by Lemma 3.2. Thus, together with (3.5), we obtain (3.1).

(2) Since (k5) implies (k4), (3.4) holds under our assumptions. Hence

\[ \lim_{r \to 0^+} \int_{B(x_{0}, r)} \Psi(x, 2|I_2(x)|) \, dx = 0 \]
by (Ψ2) and the boundedness of Ψ(x, 1).

We will show that

\[ \lim_{r \to 0^+} \int_{B(x_{0}, r)} \Psi(x, 2k \ast f_r(x)) \, dx = 0. \]

Let 0 < r ≤ 1, x = x_0 + rz with |z| < 1. For y ∈ B(x_0, r), write y = x_0 + rw with |w| < 1. If |z - w| ≤ 1, then by (k5) k(x - y) ≤ K_3 \tilde{k}(r)k(z - w). If 1 < |z - w| < 2, then r < |x - y| < 2r, so that by (k1), (k5) and (k3)

\[ k(x - y) \leq k(r) \leq K_3 \tilde{k}(r)k(1) \leq K_1 K_3 \tilde{k}(r)k(z - w). \]
Hence

\[ k \ast f_r(x) = \int_{B(x_0, r)} k(x - y) f(y) \, dy \leq K_1 K_3 \int_{B(0, 1)} r^{N} \tilde{k}(r)k(z - w)f(x_0 + rw) \, dw \]
if 0 < r ≤ 1. Thus, to prove (3.7) it is enough to show

\[ \lim_{r \to 0^+} \int_{B(0, 1)} \Psi(x_0 + rz, 2k \ast g_r(z)) \, dz = 0, \]
where \( g_r(w) = r^{N} \tilde{k}(r) f_r(x_0 + rw). \)

Let

\[ \Phi_{x_0, r}(x, t) = \Phi(x_0 + rx, t) \quad \text{and} \quad \Psi_{x_0, r}(x, t) = \Psi(x_0 + rx, t). \]

Then, \( \Phi_{x_0, r} \) satisfies (Φ1), (Φ2), (Φ3*), (Φ4) and (Φk) with the same constants \( A_1, \varepsilon_0, A_2, \varepsilon_0, A_3, \varepsilon_1 \) and \( A_1 \). Further, it satisfies (Φ5) with the same \( B_1 \) whenever \( 0 < r \leq 1 \).

As to \( \Psi_{x_0, r} \), it satisfies (Ψ1) and (Ψ2) with the same constant \( A_4 \). The pair \( (\Phi_{x_0, r}, \Psi_{x_0, r}) \) satisfies (ΨΦk) with the same constant \( A_5 \).

Therefore, by Lemma 2.6, there exists a constant \( C^* > 0 \) independent of \( x_0 \) and 0 < r ≤ 1 such that

\[ \int_{B(0, 1)} \Psi_{x_0, r} \left( z, \frac{k \ast g_r(z)}{C^* \lambda_r} \right) \, dz \leq 1, \]
or
\[ \int_{B(0,1)} \Psi \left( x_0 + rz, \frac{k \cdot g_r(z)}{C^* \lambda_r} \right) \, dz \leq 1, \]
where \( \lambda_r = \| g_r \|_{L^{\Phi_{x_0,r}}(B(0,1))} \). Then, by (\( \Psi 2 \)), we have
\[ \int_{B(0,1)} \Psi \left( x_0 + rz, 2k \cdot g_r(z) \right) \, dz \leq 2A_4 C^* \lambda_r \]
whenever \( 2C^* \lambda_r \leq 1 \). Now, \( x_0 \notin E_2 \) implies
\[
\int_{B(0,1)} \Phi \left( z, g_r(z) \right) \, dz = \int_{B(0,1)} \Phi \left( x_0 + rz, r^N \overline{k}(r)f_r(x_0 + rz) \right) \, dz
= |B(0,1)| \int_{B(x_0,r)} \Phi \left( x, r^N \overline{k}(r)f(x) \right) \, dx \rightarrow 0 \quad \text{as} \ r \to 0+.\]
Hence, by Lemma 2.2, \( \lambda_r \to 0 \) as \( r \to 0+ \). Thus (3.8), and hence (3.7) holds. Since
\[
\Psi \left( x, |k \cdot f(x) - k \cdot f(x_0)| \right) \leq A_4 \Psi \left( x, I_1(x) \right) + |I(x))
\leq A_4^2 \left( \Psi \left( x, 2I_1(x) \right) + \Psi \left( x, 2|I_2(x)| \right) \right)
\]
by (\( \Psi 2 \)), and
\[
\int_{B(x_0,r/2)} \Psi \left( x, 2I_1(x) \right) \, dx \leq A_4 \int_{B(x_0,r/2)} \Psi \left( x, 2k \cdot f_r(x) \right) \, dx
\leq 2^N A_4 \int_{B(x_0,r)} \Psi \left( x, 2k \cdot f_r(x) \right) \, dx,
\]
(3.2) follows from (3.6) and (3.7).

\[ \square \]

\section*{§ 4. Mean continuity (II)}

Set
\[ u_{B(x_0,r)} := \int_{B(x_0,r)} u(y) \, dy \]
for \( u \in L^1_{\text{loc}}(\mathbb{R}^N) \).

Combining (3.1) and (3.2) in Theorem 3.1, we see that
\[ \lim_{r \to 0+} \int_{B(x_0,r)} \Psi \left( x, |k \cdot f(x) - (k \cdot f)_B(x_0,r)| \right) \, dx = 0 \]
holds for \( x_0 \in \mathbb{R}^N \setminus (E_1 \cup E_2) \). In this section, we shall show that this holds also for \( x_0 \in E_1 \setminus E_2 \) under the following additional condition for \( k \):
there exists a constant $K_4 > 0$ such that
\[ k(r) - k(s) \leq K_4(s - r)r^{-1}k(r) \]
whenever $0 < r < s$.

The Riesz kernel $I_\alpha(x) = |x|^\alpha |N|$ (0 $< \alpha < N$) satisfies this condition.

Note that if $k$ satisfies (k6), then $k$ is continuous and
\[
d(-r^{-1}k(r)) \leq (1 + K_4)r^{-1}k(r) \frac{dr}{r}.
\]

**Theorem 4.1.** Besides the assumptions on $k(r)$, $\Phi(x, t)$ and $\Psi(x, t)$ given in Lemma 2.6, assume further that $k(r)$ satisfies (k5) and (k6). Let $f$ be a nonnegative measurable function on $\mathbb{R}^N$ satisfying (2.4). Then (4.1) holds for all $x_0 \in \mathbb{R}^N \setminus E_2$, where
\[
E_2 = \left\{ x \in \mathbb{R}^N : \limsup_{r \to 0+} \int_{B(x, r)} \Phi(z, r^N \overline{k}(r)f(z)) \, dz > 0 \right\}.
\]

**Lemma 4.2.** Let $x_0 \in \mathbb{R}^N$ and let $f$ be a nonnegative measurable function on $\mathbb{R}^N$ satisfying (2.4). Then
\[
g(t) := k(t) \int_{B(x_0, t)} f(y) \, dy
\]
is bounded on $[\delta, \infty)$ for $\delta > 0$.

**Proof.** It is enough to show that $g(t)$ is bounded on $[1, \infty)$, since $\int_{B(x_0, 1)} f(y) \, dy < \infty$ by (2.4).

If $1 \leq |x_0 - y| < t$, then $1 + |y| \leq m + t$ for an integer $m$ such that $m \geq 1 + |x_0|$. Hence, by (k3), $k(t) \leq K_1^m k(m + t) \leq K_1^m k(1 + |y|)$. Therefore
\[
g(t) \leq k(1) \int_{B(x_0, 1)} f(y) \, dy + K_1^m \int_{B(x_0, t) \setminus B(x_0, 1)} k(1 + |y|) f(y) \, dy
\leq k(1) \int_{B(x_0, 1)} f(y) \, dy + K_1^m \int_{\mathbb{R}^N} k(1 + |y|) f(y) \, dy < \infty
\]
for $t \geq 1$.

**Lemma 4.3.** Let $x_0 \in \mathbb{R}^N$ and let $f$ be a nonnegative measurable function on $\mathbb{R}^N$ satisfying (2.4) and
\[
\lim_{r \to 0+} \int_{B(x_0, r)} \Phi(z, r^N \overline{k}(r)f(z)) \, dz = 0.
\]
Then
\[
\lim_{r \to 0+} r \int_{2r}^{\infty} t^{-1}k(t) \left( \int_{B(x_0, t)} f(y) \, dy \right) \frac{dt}{t} = 0.
\]
Proof. Let $\varepsilon > 0$. Then, by Lemma 3.2 and $k(t) \leq \bar{k}(t)$, there exists a constant $0 < \delta \leq 1$ such that
\[
k(t) \int_{B(x_{0}, t)} f(y) \, dy \leq \varepsilon
\]
for all $t \in (0, \delta)$. By the previous lemma, there exists $M > 0$ such that
\[
k(t) \int_{B(x_{0}, t)} f(y) \, dy \leq M < \infty
\]
for all $t \in [\delta, \infty)$. Hence, for $0 < r \leq \delta/2$, we have
\[
\int_{2r}^{\infty} t^{-1} k(t) \left( \int_{B(x_{0}, t)} f(y) \, dy \right) \frac{dt}{t} \leq \varepsilon \int_{2r}^{\delta} t^{-1} \frac{dt}{t} + M \int_{\delta}^{\infty} t^{-1} \frac{dt}{t} \leq \varepsilon r^{-1} + M \delta^{-1},
\]
so that
\[
\limsup_{r \to 0+} r \int_{2r}^{\infty} t^{-1} k(t) \left( \int_{B(x_{0}, t)} f(y) \, dy \right) \frac{dt}{t} \leq \varepsilon.
\]
Hence, we have the required result. \qed

Proof of Theorem 4.1. Let $x_{0} \in \mathbf{R}^{N} \setminus E_{2}$ and let $x \in B(x_{0}, r)$. Also, let $0 < r \leq 1$. Write
\[
k \ast f(x) - (k \ast f)_{B(x_{0}, r)} = \int_{B(x_{0}, 2r)} k(x-y) f(y) \, dy
\]
\[+ \int_{\mathbf{R}^{N} \setminus B(x_{0}, 2r)} k(x-y) f(y) \, dy - (k \ast f)_{B(x_{0}, r)}
\]
\[= \int_{B(x_{0}, 2r)} k(x-y) f(y) \, dy
\]
\[+ \int_{\mathbf{R}^{N} \setminus B(x_{0}, 2r)} \left( \int_{B(x_{0}, r)} (k(x-y) - k(y-z)) \, dz \right) f(y) \, dy
\]
\[= I_{1}(x) + I_{2}(x) - I_{3}.
\]
For $I_{2}$, let $|x_{0} - x| < r$, $|x_{0} - z| < r$ and $|x_{0} - y| \geq 2r$. Then, by (k6)
\[
|k(x-y) - k(z-y)| \leq 2K_{4} |x-z| |x_{0} - y|^{-1} \max\{k(x-y), k(z-y)\}.
\]
As in the proof of Theorem 3.1, we see that
\[
k(x-y) \leq K'k(x_{0} - y) \quad \text{and} \quad k(z-y) \leq K'k(x_{0} - y)
\]
with $K' = \max\{K_3 \overline{k}(1/2), k(1/2)/k(2), K_1\}$. Hence

$$|I_2(x)| \leq 2K_4 K' \left( \int_{B(x_0,r)} |x - z| \, dz \right) \int_{\mathbb{R}^N \setminus B(x_0,2r)} |x_0 - y|^{-1} k(x_0 - y) f(y) \, dy$$

$$\leq C r \int_{2r}^{\infty} t^{-1} k(t) dF_{x_0}(t),$$

where $F_{x_0}(t) = \int_{B(x_0,t)} f(y) \, dy$. In view of (4.2) and Lemma 4.2, integration by parts yields

$$\int_{2r}^{\infty} t^{-1} k(t) dF_{x_0}(t) \leq C \int_{2r}^{\infty} t^{-1} k(t) F_{x_0}(t) \frac{dt}{t}.$$

Therefore by Lemma 4.3,

$$\lim_{r \to 0^+} \sup_{x \in B(x_0,r)} |I_2(x)| = 0.$$

As to $I_3$, we have by Lemma 3.2

$$0 \leq I_3 \leq \overline{k}(r) \int_{B(x_0,2r)} f(y) \, dy \leq 2^N \overline{k}(2r) \int_{B(x_0,2r)} f(y) \, dy \to 0$$

as $r \to 0^+$.

Hence, by \(\Psi 2\)

$$\lim_{r \to 0^+} \int_{B(x_0,r)} \Psi(x, 2|I_2(x) - I_3|) \, dx = 0.$$

On the other hand, the arguments to obtain (3.7) in the proof of Theorem 3.1 show that

$$\lim_{r \to 0^+} \int_{B(x_0,r)} \Psi(x, 2I_1(x)) \, dx = 0.$$

Hence again using \(\Psi 2\) we see that

$$\lim_{r \to 0^+} \int_{B(x_0,r)} \Psi(x, |k \ast f(x) - (k \ast f)_{B(x_0,r)}|) \, dx = 0.$$

\(\square\)

§ 5. Size of exceptional sets

First, we introduce a notion of capacity (cf. [5]). For a set $E \subset \mathbb{R}^N$ and an open set $G \subset \mathbb{R}^N$, we define the $(k, \Phi)$-capacity of $E$ relative to $G$ by

$$C_{k,\Phi}(E;G) = \inf_{f \in S_k(E;G)} \int_G \Phi(y, f(y)) \, dy,$$

where $S_k(E;G)$ is the family of all nonnegative measurable functions $f$ on $\mathbb{R}^N$ such that $f$ vanishes outside $G$ and $k \ast f(x) \geq 1$ for every $x \in E$. Here, note that $E \subset G$ is not required.
Lemma 5.1 ([5, Proposition 3.1]). The set function $C_{k,\Phi}(\cdot;G)$ is countably subadditive and nondecreasing.

We say that $E$ is of $(k, \Phi)$-capacity zero, written as $C_{k,\Phi}(E) = 0$, if

$$C_{k,\Phi}(E \cap G; G) = 0$$

for every bounded open set $G$.

Lemma 5.2 ([5, Proposition 3.3]). For $E \subset \mathbb{R}^N$, $C_{k,\Phi}(E) = 0$ if and only if there exists a nonnegative function $f \in L^\Phi(\mathbb{R}^N)$ such that $k*f \not\equiv \infty$ and

$$k*f(x) = \infty \quad \text{whenever } x \in E.$$

By Lemma 5.2 we have

Proposition 5.3. If $f \in L^\Phi(\mathbb{R}^N)$, then $E_1$ in Theorem 3.1 has $(k, \Phi)$-capacity zero.

To estimate the size of $E_2$ in Theorem 3.1, we introduce a Hausdorff measure defined by the (variable) measure function

$$h(r;x) = r^N \Phi(x, r^{-N}\overline{k}(r)^{-1})$$

for $x \in \mathbb{R}^N$ and $r > 0$.

We define the Hausdorff $h$-measure of $E \subset \mathbb{R}^N$ by

$$H_h(E) = \inf \left\{ \sum_j h(r_j;x_j) : \bigcup_j B(x_j, r_j) \supset E, \ 0 < r_j < 1 \right\}.$$

Here we note that

(h1) there exists a constant $A > 0$ such that $h(5r;x) \leq Ah(r;x)$ for all $x \in \mathbb{R}^N$ and $r > 0$;

(h2) $\lim_{r \to 0} r^{-N}(\inf_x h(r;x)) = \infty$.

We show the following result (cf. Meyers [8, 9]; also cf. [10, Chapter 5, Lemma 8.2]).

Lemma 5.4. If $f \in L^\Phi(\mathbb{R}^N)$, then $H_h(E_{h,f}) = 0$, where

$$E_{h,f} := \left\{ x \in \mathbb{R}^N : \limsup_{r \to 0^+} \frac{1}{h(r;x)} \int_{B(x,r)} \Phi(y, |f(y)|) \, dy > 0 \right\}.$$
Proof. It suffices to show that $H_h(E(a)) = 0$ for each $a > 0$, where

$$E(a) := \left\{ x \in \mathbb{R}^N : \lim_{r \to 0^+} \sup_{0+} \frac{1}{h(r; x)} \int_{B(x, r)} \Phi(y, |f(y)|) \, dy > a \right\}.$$ 

For $\varepsilon > 0$, by (h2) we can find $\delta > 0$ ($\delta \leq 1$) such that

$$h(r; x) > \varepsilon^{-1} r^N$$

for all $x \in \mathbb{R}^N$ and $0 < r < \delta$. For each $x \in E(a)$, take $B(x, r(x))$ such that $0 < r(x) < \delta$ and

$$\frac{1}{h(r(x); x)} \int_{B(x, r)} \Phi(y, |f(y)|) \, dy > a.$$ 

By a covering lemma (see, e.g., [1, Theorem 1.4.1]), we can take a disjoint subfamily \{\{B(x_j, r(x_j))\}\} such that $E(a) \subset \bigcup_j B(x_j, 5r(x_j))$. Then

$$H_h(E(a)) \leq \sum_j h(5r(x_j); x_j) \leq A \sum_j h(r(x_j); x_j) \leq Aa^{-1} \int_{\bigcup_j B(x_j, r(x_j))} \Phi(y, |f(y)|) \, dy.$$ 

Note here that

$$\varepsilon^{-1} \sum_j r(x_j)^N \leq \sum_j h(r(x_j); x_j) \leq a^{-1} \int_{\bigcup_j B(x_j, r(x_j))} \Phi(y, |f(y)|) \, dy,$$

so that

$$\left| \bigcup_j B(x_j, r(x_j)) \right| \leq Ca^{-1} \varepsilon \int_{\mathbb{R}^N} \Phi(y, |f(y)|) \, dy.$$ 

Since $f \in L^\Phi(\mathbb{R}^N)$, by the absolute continuity of integrals we see that $H_h(E(a)) = 0$, as required. \qed

On the other hand, by [5, Corollary 4.8], we have the following result.

**Lemma 5.5.** Suppose $\Phi(x, t)$ satisfies $(\Phi 5)$. If $f \in L^\Phi(\mathbb{R}^N)$, then $C_{k, \Phi}(E_{h, f}) = 0$. 

Here note that the condition
\begin{equation}
\limsup_{r \to 0+} \sup_{y \in B(x, r)} \frac{\Phi(y, r^{-N}k(r)^{-1})}{\inf_{y \in B(x, r)} \Phi(y, r^{-N}k(r)^{-1})} < \infty
\end{equation}
in [5, Corollary 4.8] is satisfied by (\Phi_5), since \( r^Nk(r) \leq 1 \) for small \( r > 0 \) by (2.3).

Now, we consider a further condition on \( \Phi(x, t) \):
\begin{itemize}
\item[(\Phi 6)] there exists a constant \( A_6 > 0 \) such that
\[ \Phi(x, s) \Phi(x, t) \leq A_6 \Phi(x, st) \]
for all \( x \in \mathbb{R}^N, \ s \geq 1 \) and \( t > 0 \).
\end{itemize}

**Example 5.6.** Let \( \Phi_{p(\cdot), q(\cdot), a}(x, t) \) be as in Example 2.1. It satisfies (\Phi 6) if and only if \( q^+ \leq 0 \); cf. [11, Proposition 3.7].

**Lemma 5.7.** Suppose \( \Phi(x, t) \) satisfies (\Phi_5) and (\Phi_6). Let \( f \) be a nonnegative measurable function on \( \mathbb{R}^N \) and let \( E_2 \) be as in Theorem 3.1. Then \( E_2 \subset E_{h,f} \).

**Proof.** Let \( f \) be a nonnegative measurable function on \( \mathbb{R}^N \) and let \( x \in \mathbb{R}^N \). By (2.3), there is \( 0 < r_1 \leq 1 \) such that \( r_1^Nk(r_1) \leq 1 \). If \( 0 < r \leq r_1 \) and \( y \in B(x, r) \), then by (\Phi 6) and (\Phi 5),
\[ \Phi(y, r^Nk(r)f(y)) \leq A_6B_\gamma \frac{\Phi(y, f(y))}{\Phi(x, r^{-N}k(r)^{-1})}, \]
where \( \gamma = k(r_1)^{-1/N} \). Hence \( E_2 \subset E_{h,f} \). \( \square \)

Combining this lemma with Lemmas 5.4 and 5.5, we obtain

**Proposition 5.8.** Assume that \( \Phi \) satisfies (\Phi_5) and (\Phi_6). If \( f \in L^{\Phi}(\mathbb{R}^N) \), then \( E_2 \) in Theorem 3.1 has Hausdorff \( h \)-measure zero, that is, \( H_h(E_2) = 0 \), and it has \((k, \Phi)\)-capacity zero.

**Remark 1.** The above definition of the Hausdorff measure is slightly different from the one in [5]. However, noting (5.1), we see that the proof of [5, Theorem 4.10] is valid for \( H_h \) and we have the following result:

Suppose \( \Phi(x, t) \) satisfies (\Phi_5). If \( H_h(E) = 0 \), then \( C_{k, \Phi}(E) = 0 \).

Applying Theorem 3.1, Proposition 5.3 and Proposition 5.8 to \( k = I_\alpha \), we can state:

**Corollary 5.9.** Let \( 0 < \alpha < N \) and let \( f \in L^{\Phi}(\mathbb{R}^N) \) satisfy (2.4) with \( k = I_\alpha \). Suppose \( \Phi(x, t) \) satisfies (\Phi3^*), (\Phi_5), (\Phi_6) and
\((\Phi I_{\alpha}) \ s \mapsto s^{-\varepsilon_{1}/N} \Phi^{-1}(x, s)\) is uniformly almost increasing on \((0, \infty)\) for some \(\varepsilon_{1} > 0\);

\(\Psi(x, t)\) satisfies \((\Psi 1), (\Psi 2)\) and

\((\Psi \Phi I_{\alpha})\) there exists a constant \(A'_{5} \geq 1\) such that

\[
\Psi \left( x, s^{-\alpha/N} \Phi^{-1}(x, s) \right) \leq A'_{5}s
\]

for all \(x \in \mathbb{R}^{N}\) and \(s > 0\).

Then

\[
\lim_{r \to 0+} \int_{B(x_{0}, r)} \Psi(x, |I_{\alpha} \ast f(x) - I_{\alpha} \ast f(x_{0})|) \, dx = 0
\]

holds for all \(x_{0} \in \mathbb{R}^{N} \setminus E\) for a set \(E\) of \((I_{\alpha}, \Phi)\)-capacity zero.

§ 6. Appendix: uniform almost-increasingness of \(t^{p(\xi)}(\log(e + t))^{q(\xi)}\)

In this section, we give an outline of a proof of the equivalence stated in the last part of Example 2.1.

For a positive function \(f(t)\) on \((0, \infty)\), set

\[
A[f] := \sup_{t>0, \lambda>1} \frac{f(t)}{f(\lambda t)}.
\]

\(f\) is almost increasing on \((0, \infty)\) if and only if \(A[f] < \infty\). Note that \(f\) is non-decreasing on \((0, \infty)\) iff \(A[f] = 1\).

A family \(\{f_{\xi}(t)\}_{\xi \in X}\) of positive functions on \((0, \infty)\) is uniformly almost increasing if and only if

\[
\sup_{\xi \in X} A[f_{\xi}] < \infty.
\]

For \(p \geq 0\) and \(q \in \mathbb{R}\), we consider the function

\[
F_{p,q}(t) = t^{p} (\log(e + t))^{q}, \quad t \in [0, \infty).
\]

Obviously, if \(q \geq 0\), then \(F_{p,q}(t)\) is non-decreasing on \((0, \infty)\). If \(p = 0\) and \(q < 0\), then \(F_{0,q}(t)\) is not almost increasing. In case \(p > 0\) and \(q < 0\), it is easy to see that \(F_{p,q}(t)\) is almost increasing. We are interested in the evaluation of \(A[F_{p,q}]\) in this case. Since

\[
A[F_{p,q}] = A[F_{p/(-q),-1}]^{-q},
\]

we will evaluate \(A[F_{r,-1}]\) for \(r > 0\).
Let $c_0 := \log(e + 1)$. We see that
\[
\frac{1}{c_0} \log(e + \lambda) \leq \sup_{t > 0} \frac{\log(e + \lambda t)}{\log(e + t)} \leq 1 + \log \lambda \leq 2 \log(e + \lambda)
\]
for $\lambda \geq 1$. Hence, letting
\[
L(r) := \sup_{\lambda \geq 1} \lambda^{-r} \log(e + \lambda),
\]
we have
\[
(6.1) \quad \frac{1}{c_0} L(r) \leq A[F_{r,-1}] \leq 2L(r) \quad (r > 0).
\]
Here note that $\sup_{1 \leq \lambda \leq e} \lambda^{-r} \log(e + \lambda) \leq 2$,
\[
\sup_{\lambda > e} \lambda^{-r} \log(e + \lambda) \leq 2 \sup_{\lambda > e} \lambda^{-r} \log \lambda \leq \frac{2}{er},
\]
$L(r) \geq \log(e + 1) = c_0$ and
\[
L(r) \geq \frac{1}{e} \log(e + e^{1/r}) > \frac{1}{er},
\]
so that
\[
\max \left( \frac{1}{er}, c_0 \right) \leq L(r) \leq 2 \max \left( \frac{1}{er}, 1 \right) \quad (r > 0).
\]
Hence, by (6.1),
\[
\max \left( \frac{1}{c_0 er}, 1 \right) \leq A[F_{r,-1}] \leq 4 \max \left( \frac{1}{er}, 1 \right) \quad (r > 0).
\]
Thus, for $p > 0$ and $q < 0$,
\[
\left[ \max \left( \frac{-q}{c_0 ep}, 1 \right) \right]^{-q} \leq A[F_{p,q}] \leq 4 \max \left( \frac{-q}{p}, 1 \right)^{-q}.
\]
Note that $e^{-1/e} \leq (-q)^{-q} \leq \max(1, (-q_0)^{-q_0})$ if $q_0 \leq q < 0$. Then from the above inequalities we have:

**Proposition 6.1.** Let $X$ be a nonempty set and let $p(\cdot)$ and $q(\cdot)$ be real valued functions on $X$ such that $p(\xi) \geq 0$ for all $\xi \in X$ and $\inf_{\xi \in X} q(\xi) > -\infty$. Then, the following (1) and (2) are equivalent to each other:

1. The family $\{F_{p(\xi),q(\xi)}(t)\}_{\xi \in X}$ is uniformly almost increasing on $(0, \infty)$;
2. $q(\xi) \geq 0$ at points $\xi \in X$ where $p(\xi) = 0$, and
\[
\sup_{\xi \in X, \ p(\xi) > 0, \ q(\xi) < 0} \frac{q(\xi) \log p(\xi)}{p(\xi)} < \infty.
\]
References