Sobolev’s inequality for Riesz potentials in central Lorentz-Morrey spaces of variable exponent

By

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Abstract

In the present paper we discuss the boundedness of the maximal operator in the central Lorentz-Morrey space of variable exponent defined by the symmetric decreasing rearrangement in the sense of Almut [1]. Further we establish Sobolev’s inequality for Riesz potentials.

§1. Introduction

In this paper we use $B(x, r)$ to denote the open ball centered at $x$ of radius $r > 0$. The volume of a measurable set $E \subset \mathbb{R}^n$ is written as $|E|$. We denote by $\chi_E$ the characteristic function of $E$.

Given a measurable function $f$ on $\mathbb{R}^n$, recall the symmetric decreasing rearrangement of $f$ defined by

$$f^*(x) = \int_0^\infty \chi_{E_f(t^*)}(x)dt,$$

where $E^* = \{x : |B(0, |x|)| < |E|\}$ and $E_f(t) = \{y : |f(y)| > t\}$ (see Almut [1]). Note here that

$$f^*(|B(0, |x|)|) = f^*(x),$$

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where \( f^* \) is the usual decreasing rearrangement of \( f \). The fundamental fact of the symmetric decreasing rearrangement of \( f \) is that
\[
|E_f(t)| = |E_{f^*}(t)|
\]
for all \( t \geq 0 \). This readily gives the rearrangement preserving \( L^p \)-norm property such as
\[
\|f\|_{L^p(\mathbb{R}^n)} = \|f^*\|_{L^p(\mathbb{R}^n)}
\]
for \( 1 \leq p \leq \infty \). For fundamental properties of the symmetric decreasing rearrangement, we refer the reader to the Lecture Notes by Almut [1]; see also his papers [2, 3].

For variable exponents \( p, q \) and a constant \( \mu \geq 0 \), the central Lorentz-Morrey space of variable exponent \( \mathcal{L}^{q,p,\mu}(\mathbb{R}^n) \) is defined as the set of all measurable functions \( f \) on \( \mathbb{R}^n \) with
\[
\|f\|_{\mathcal{L}^{q,p,\mu}(\mathbb{R}^n)} = \inf \left\{ \lambda > 0 : \sup_{r>0} \int_{B(0,r)} r^{-\mu p(y)} |f^*(y)/\lambda|^{p(y)} |y|^{n(\frac{p(y)}{q(y)}-1)} dy \leq 1 \right\} < \infty.
\]
If \( p \) and \( q \) are radial and \( \mu = 0 \), then we refer the reader to the paper by Ephremidze, Kokilashvili and Samko [9].

In central Lorentz-Morrey spaces of variable exponent, we establish the Sobolev inequality for the Riesz potential
\[
I_{\alpha}f(x) = \int_{\mathbb{R}^n} |x-y|^{\alpha-n} f(y) dy
\]
of order \( \alpha \); for fundamental properties of Riesz potentials, see e.g. [12].

\section*{2. Symmetric decreasing rearrangement}

Let us recall the Hardy-Littlewood inequality for the symmetric decreasing rearrangement (see Almut [1, Lemma 1.6]).

\textbf{Lemma 2.1.} For all nonnegative measurable functions \( f \) and \( g \) on \( \mathbb{R}^n \),
\[
\int_{\mathbb{R}^n} f(x)g(x)dx \leq \int_{\mathbb{R}^n} f^*(x)g^*(x)dx.
\]

The (centered) maximal function \( M f \) of a measurable function \( f \) on \( \mathbb{R}^n \) is defined by
\[
M f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy.
\]

\textbf{Lemma 2.2.} For all measurable functions \( f \) on \( \mathbb{R}^n \),
\[
(Mf)^*(x) \leq C \frac{1}{|B(0,|x|)|} \int_{B(0,|x|)} f^*(y)dy \leq CM^*f(x),
\]
where \( C \) is a positive constant independent of \( f \).
Proof. Recall the definition of the symmetric decreasing rearrangement and thus

\[(Mf)^*(x) = \sup\{r > 0 : |B(0, |x|)| < |\{z : Mf(z) \geq r\}|\}.

Set \(r_0 = (Mf)^*(x)\). Then, using the covering property (see [12, Theorem 1.10.1]) and Lemma 2.1, we have for \(0 < r < r_0\)

\[|\{z : Mf(z) \geq r\}| \leq Cr^{-1}\int_{\{z : f(z) > r/2\}} f(y)dy \leq Cr^{-1}\int_{\{z : f^*(z) > r/2\}} f^*(y)dy.
\]

If \(\{z : f^*(z) > r/2\} \subset B(0, |x|)\), then

\[r \leq C \frac{1}{|B(0, |x|)|} \int_{B(0, |x|)} f^*(y)dy \leq C M(f^*)(x).
\]

If \(\{z : f^*(z) > r/2\} \supset B(0, |x|)\), then

\[|\{z : f^*(z) > r/2\}| \leq \frac{2}{r} \int_{\{z : f^*(z) > r/2\}} f^*(y)dy.
\]

Noting that

\[\frac{1}{|B(0, t)|} \int_{B(0, t)} f^*(y)dy \leq \frac{1}{|B(0, s)|} \int_{B(0, s)} f^*(y)dy
\]

when \(0 < s < t\), we obtain

\[\frac{r}{2} \leq \frac{1}{|\{z : f^*(z) > r/2\}|} \int_{\{z : f^*(z) > r/2\}} f^*(y)dy \leq \frac{1}{|B(0, |x|)|} \int_{B(0, |x|)} f^*(y)dy \leq CMf^*(x),
\]

as required. \(\square\)

The following is known as Riesz’ inequality (see Almut [1, §1.3]).

**Lemma 2.3.** For all nonnegative measurable functions \(f, g\) and \(h\) on \(\mathbb{R}^n\),

\[\int_{\mathbb{R}^n} f(x)(g * h)(x)dx \leq \int_{\mathbb{R}^n} f^*(x)(g^* * h^*)(x)dx,
\]

where

\[g * h(x) = \int_{\mathbb{R}^n} g(x - y)h(y)dy.
\]

**Lemma 2.4.** For all nonnegative measurable functions \(f\) on \(\mathbb{R}^n\),

\[(I_\alpha f)^*(x) \leq C \int_{\mathbb{R}^n} (|x| + |y|)^{\alpha-n} f^*(y)dy \leq C(I_\alpha f^*)(x),
\]

where \(C\) is a positive constant independent of \(f\).
Proof. Set $r_0 = (I_{\alpha}f)^*(x)$. For $0 < r < r_0$, write
\[ |\{z : I_{\alpha}f(z) > r\}| = |B(0,t)|.\]
We have
\[
|B(0,t)| = |\{z : I_{\alpha}f(z) > r\}| \\
\leq r^{-1} \int_{\{z : I_{\alpha}f(z) > r\}} I_{\alpha}f(\zeta)d\zeta \\
\leq r^{-1} \int_{\{z : (I_{\alpha}f)^*(z) > r\}} I_{\alpha}f^*(\zeta)d\zeta \quad \text{(by Riesz' inequality)} \\
= r^{-1} \int_{B(0,t)} I_{\alpha}f^*(\zeta)d\zeta \\
= r^{-1} \int_{\mathbb{R}^n} \left( \int_{B(0,t)} |\zeta - y|^{\alpha-n}d\zeta \right) f^*(y)dy \\
\leq C r^{-1} t^n \int_{\mathbb{R}^n} (t + |y|)^{\alpha-n} f^*(y)dy,
\]
so that
\[
r \leq C \int_{\mathbb{R}^n} (t + |y|)^{\alpha-n} f^*(y)dy.
\]
Since $t \geq |x|$, we have
\[
r \leq C \int_{\mathbb{R}^n} (|x| + |y|)^{\alpha-n} f^*(y)dy,
\]
which gives the required inequality. \qed

Remark 2.5. In case $\alpha = 0$, $I_{\alpha}$ might be replaced by the singular integral operator (see [4] and [9, Theorem 3.14]).

§ 3. Central Lorentz-Morrey spaces of variable exponent

A function $p$ on $\mathbb{R}^n$ is said to be log-Hölder continuous if

(P1) $p$ is locally log-Hölder continuous, namely
\[
|p(x) - p(y)| \leq \frac{C_0}{\log(1/|x - y|)} \quad \text{for} \quad |x - y| \leq \frac{1}{e}
\]
with a constant $C_0 \geq 0$, and

(P2) $p$ is log-Hölder continuous at infinity, namely
\[
|p(x) - p(\infty)| \leq \frac{C_{\infty}}{\log(e + |x|)}
\]
with constants $C_\infty \geq 0$ and $p(\infty)$. Let $\mathcal{P}(\mathbb{R}^n)$ be the class of all log-Hölder continuous functions $p$ on $\mathbb{R}^n$. If in addition $p$ satisfies

(P3) $1 < p^- := \inf_{x \in \mathbb{R}^n} p(x) \leq \sup_{x \in \mathbb{R}^n} p(x) =: p^+ < \infty,$

then we write $p \in \mathcal{P}_1(\mathbb{R}^n)$.

**Definition 3.1.** Let $\nu \in \mathcal{P}(\mathbb{R}^n)$, $p \in \mathcal{P}_1(\mathbb{R}^n)$ and $\mu \geq 0$. For an open set $G \subset \mathbb{R}^n$, we define the $L^{p(\cdot)}$-norm by

$$\|f\|_{L^{p(\cdot)}(G)} = \inf \left\{ \lambda > 0 : \int_G |f(y)/\lambda|^{p(y)}dy \leq 1 \right\}$$

for measurable functions $f$ on $G$. By $L^{p(\cdot),\nu}(G)$ we denote the weighted $L^{p(\cdot)}$-Morrey space of all functions $f$ on $G$ with

$$\|f\|_{L^{p(\cdot),\nu}(G)} = \sup_{r>0} r^{-\mu} \|f(x)|x|^{\nu(x)}\|_{L^{p(\cdot)}(G \cap B(0,r))} < \infty.$$ We write $L^{0,p,0}(G) = L^{p(\cdot)}(G)$ and $L^{0,p,\mu}(G) = L^{p,\mu}(G)$ for simplicity; set $\|f\|_{L^{0,p,0}(G)} = \|f\|_{L^{p(\cdot)}(G)}$ and $\|f\|_{L^{0,p,\mu}(G)} = \|f\|_{L^{p,\mu}(G)}$.

**Definition 3.2.** For $p, q \in \mathcal{P}_1(\mathbb{R}^n)$, set $\nu(x) = n \left( \frac{1}{q(x)} - \frac{1}{p(x)} \right)$. We denote by $L^{q,p,\mu}(\mathbb{R}^n)$ the family of measurable functions $f$ on $\mathbb{R}^n$ such that

$$\|f\|_{L^{q,p,\mu}(\mathbb{R}^n)} = \|f^*\|_{L^{q,p,\mu}(\mathbb{R}^n)} < \infty.$$

Throughout this paper, let $C$ denote various constants independent of the variables in question. For functions $f, g$, we write $f \sim g$ if there is a constant $C > 1$ such that $C^{-1}g \leq f \leq Cg$.

**Lemma 3.3.** Let $p \in \mathcal{P}_1(\mathbb{R}^n), \mu \geq 0$ and $\nu \in \mathcal{P}(\mathbb{R}^n)$. Set

$$\omega(r) = \begin{cases} r^{\mu p(0)} & \text{for } 0 < r \leq 1, \\ r^{\mu p(\infty)} & \text{for } r > 1. \end{cases}$$

Then there exists a constant $C > 0$ such that

$$\int_{B(0,r)} |f(y)|^{p(y)}|y|^{\nu(y)p(y)}dy \leq C\omega(r)$$

for all $r > 0$, whenever $\|f\|_{L^{p(\cdot),\nu}(\mathbb{R}^n)} \leq 1$. Conversely, there exists a constant $C' > 0$ such that

$$\|f\|_{L^{p(\cdot),\nu}(\mathbb{R}^n)} \leq C'$$

whenever $\int_{B(0,r)} |f(y)|^{p(y)}|y|^{\nu(y)p(y)}dy \leq \omega(r)$ for all $r > 0$. 
Proof. We only show the case $\mu > 0$. If $0 < r < 1$, then, noting from (P1) that $r^{p(y)} \sim r^{p(0)}$ for $y \in B(0, r)$, we find

$$\int_{B(0,r)} r^{-\mu p(y)} |f(y)|^{p(y)} |y|^{\nu(y)p(y)} dy \sim r^{-\mu p(0)} \int_{B(0,r)} |f(y)|^{p(y)} |y|^{\nu(y)p(y)} dy.$$  

This proves the case when $0 < r < 1$.

Suppose $\|f\|_{L^{\nu,p,\mu}(\mathbb{R}^n)} \leq 1$. If $r > 1$, then, since $|x|^{p(x)} \sim |x|^{p(\infty)}$ when $x \in \mathbb{R}^n \setminus B(0,1/2)$ by (P2), we have

$$r^{-\mu p(\infty)} \int_{B(0,r) \setminus B(0,1)} |f(y)|^{p(y)} |y|^{\nu(y)p(y)} dy \leq r^{-\mu p(\infty)} \sum_{j=1}^{j_0} \int_{B(0,2^{-j+1}r) \setminus B(0,2^{-j}r)} (2^{-j}r)^{-\mu p(\infty)} |f(y)|^{p(y)} |y|^{\nu(y)p(y)} dy$$

$$= \sum_{j=1}^{j_0} 2^{-j\mu p(\infty)} \int_{B(0,2^{-j+1}r) \setminus B(0,2^{-j}r)} (2^{-j}r)^{-\mu p(\infty)} |f(y)|^{p(y)} |y|^{\nu(y)p(y)} dy$$

$$\leq C \sum_{j=1}^{j_0} 2^{-j\mu p(\infty)} \int_{B(0,2^{-j+1}r) \setminus B(0,2^{-j}r)} (2^{-j}r)^{-\mu p(y)} |f(y)|^{p(y)} |y|^{\nu(y)p(y)} dy$$

$$\leq C \sum_{j=1}^{\infty} 2^{-j\mu} < \infty,$$

where $j_0$ is the positive integer such that $2^{-j_0} r \leq 1 < 2^{-j_0+1} r$. Conversely, suppose

$$\sup_{r>1} r^{-\mu p(\infty)} \int_{B(0,r) \setminus B(0,1)} |f(y)|^{p(y)} |y|^{\nu(y)p(y)} dy \leq 1.$$  

For $r > 1$, taking $j_0$ as above, we obtain by (P2)

$$r^{-\mu p(y)} \int_{B(0,r) \setminus B(0,1)} |f(y)|^{p(y)} |y|^{\nu(y)p(y)} dy$$

$$\leq \sum_{j=1}^{j_0} 2^{-j\mu} \int_{B(0,2^{-j+1}r) \setminus B(0,2^{-j}r)} (2^{-j}r)^{-\mu p(y)} |f(y)|^{p(y)} |y|^{\nu(y)p(y)} dy$$

$$\leq C \sum_{j=1}^{\infty} 2^{-j\mu} < \infty.$$  

The case $r > 1$ is now established, with the aid of the first case.
§ 4. The boundedness of maximal operator in central Lorentz-Morrey spaces of variable exponent

**Theorem 4.1.** Let $\nu \in \mathcal{P}(\mathbb{R}^n)$, $p \in \mathcal{P}_1(\mathbb{R}^n)$ and $\mu \geq 0$. Suppose

(T1) $\mu p(0) - n < \nu(0)p(0) < n(p(0) - 1)$ and $\mu p(\infty) - n < \nu(\infty)p(\infty) < n(p(\infty) - 1)$.

Then the maximal operator $\mathcal{M} : f \mapsto Mf$ is bounded from $L^{\nu,p,\mu}(\mathbb{R}^n)$ into itself, namely, there is a constant $C > 0$ such that

$$\|Mf\|_{L^{\nu,p,\mu}(\mathbb{R}^n)} \leq C\|f\|_{L^{\nu,p,\mu}(\mathbb{R}^n)}$$

for all $f \in L^{\nu,p,\mu}(\mathbb{R}^n)$.

Our Theorem 4.1 includes the boundedness of the maximal operator in the weighted $L^{p(\cdot)}$-Morrey space as in [13], which is an extension of Diening [8] and Cruz-Uribe, Fiorenza and Neugebauer [6]. When $\mu = 0$, Theorem 4.1 is a special case of Theorem 1.1 in Hästö and Diening [10] (see also Cruz-Uribe, Diening and Hästö [5] and Cruz-Uribe, Fiorenza and Neugebauer [7]).

With the aid of Lemma 2.2, we find the following result.

**Corollary 4.2.** Let $p, q \in \mathcal{P}_1(\mathbb{R}^n)$. Then the maximal operator $\mathcal{M} : f \mapsto Mf$ is bounded in $\mathcal{L}^{q,p,\mu}(\mathbb{R}^n)$.

In what follows, we prepare lemmas required for a proof of Theorem 4.1. For a measurable function $f$ on an open set $G \subset \mathbb{R}^n$, we define

$$M_G f(x) = \sup_{r > 0} \frac{1}{|B(x,r)|} \int_{B(x,r) \cap G} |f(y)| dy.$$ 

Let us begin with the following result due to Cruz-Uribe, Fiorenza and Neugebauer [6].

**Lemma 4.3.** Let $p \in \mathcal{P}(\mathbb{R}^n)$ and $G$ be an open set in $\mathbb{R}^n$. If

$$p^{-}(G) = \inf_{x \in G} p(x) > 1,$$

then there exists a constant $C > 0$ such that

$$\|M_G f\|_{L^{p(\cdot)}(G)} \leq C\|f\|_{L^{p(\cdot)}(G)}.$$ 

**Lemma 4.4.** Let $p \in \mathcal{P}_1(\mathbb{R}^n), \mu \geq 0, \nu \in \mathcal{P}(\mathbb{R}^n)$ and $0 < r_0 < 1 < R_0 < \infty$. If $\mu p(0) - n < \nu(0)p(0)$ and $\nu(\infty)p(\infty) < n(p(\infty) - 1)$, then there exists a constant $C > 0$ such that

$$\|Mf_{r_0,R_0}\|_{L^{\nu,p,\mu}(\mathbb{R}^n)} \leq C$$

for all $f \geq 0$ satisfying $\|f\|_{L^{\nu,p,\mu}(\mathbb{R}^n)} \leq 1$, where $f_{r_0,R_0}(y) = \chi_{B(0,R_0) \setminus B(0,r_0)}(y)$ with $\chi_E$ denoting the characteristic function of a measurable set $E \subset \mathbb{R}^n$. 
Proof. Let $f$ be a nonnegative measurable function on $\mathbb{R}^n$ satisfying

$$\|f\|_{L^{\nu, p, \nu}(\mathbb{R}^n)} \leq 1,$$

and $0 < r_0 < 1 < R_0 < \infty$. Then note that $\|f_{r_0, R_0}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C$.

For $x \in B(0, r_0/2)$, we see that

$$Mf_{r_0, R_0}(x) \leq \frac{1}{|B(0, r_0/2)|} \int_{B(0, r_0/2)} f_{r_0, R_0}(y) \, dy$$

$$= \frac{1}{|B(0, r_0/2)|} \left( |B(0, R_0)| + \int_{B(0, r_0/2)} \{f_{r_0, R_0}(y)\}^{p(y)} \, dy \right) \leq C,$$

so that

$$\int_{B(0,t)} \{Mf_{r_0, R_0}(x)\}^{p(x)}|x|^{|v(x)p(x)|} \, dx \leq C \int_{B(0,t)} |x|^{v(x)p(x)} \, dx \leq C t^{v(0)p(0)+n} \leq C t^\mu p(0)$$

for all $0 < t < r_0/2$, since $\mu p(0) - n < v(0)p(0)$.

Moreover, for $r_0/2 < t < 2R_0$ we have by Lemma 4.3

$$\int_{B(0,t) \setminus B(0,r_0/2)} \{Mf_{r_0, R_0}(x)\}^{p(x)}|x|^{|v(x)p(x)|} \, dx \leq C \int_{B(0,t) \setminus B(0,r_0/2)} \{Mf_{r_0, R_0}(x)\}^{p(x)} \, dx$$

$$\leq C \leq C \min\{t^{\mu p(0)}, t^{\mu p(\infty)}\},$$

since $\|f_{r_0, R_0}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C$.

Finally, we find for $x \in \mathbb{R}^n \setminus B(0, 2R_0)$

$$Mf_{r_0, R_0}(x) \leq \frac{1}{|B(0, |x|)|} \int_{B(0, |x|)} f_{r_0, R_0}(y) \, dy \leq C|x|^{-n}.$$  

Hence we obtain for $t > 2R_0$

$$\int_{B(0,t) \setminus B(0,2R_0)} \{Mf_{r_0, R_0}(x)\}^{p(x)}|x|^{|v(x)p(x)|} \, dx \leq C \int_{B(0,t) \setminus B(0,2R_0)} |x|^{v(x)p(x)-np(x)} \, dx$$

$$\leq C t^{\mu p(\infty)},$$

since $v(\infty)p(\infty) < n(p(\infty) - 1)$, which completes the proof with the aid of Lemma 3.3. \qed

Lemma 4.5. Let $p \in \mathcal{P}_1(\mathbb{R}^n)$ and $\nu \in \mathcal{P}(\mathbb{R}^n)$. Let $p_0 > 1$. Suppose

$$v(0) < n \left( 1 - \frac{1}{p_0} \right).$$

For $0 < t \leq 2$ and a measurable function $f$ on $\mathbb{R}^n$, set

$$I = \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| \, dy.$$
and

\[ J_t = \left( \frac{1}{|B(x, r)|} \int_{B(x, r)} g_{0,t}(y) dy \right)^{1/p_0}, \]

where \( g_{0,t}(y) = \{|f(y)||y|^\nu(y)^{p_0} \chi_{B(0,3t)}(y) \}. \) Then there exists a constant \( C > 0 \) such that

\[ I \leq C \left\{ |x|^{-\nu(x)} J_t + H(x) \right\} \]

for all \( x \in B(0, t), r > 0 \) and \( 0 < t \leq 2 \), where

\[ H(x) = Hf(x) = \int_{\mathbb{R}^n \backslash B(0, |x|)} |f(y)||y|^{-n} dy. \]

**Proof.** Let \( f \) be a nonnegative measurable function on \( \mathbb{R}^n \) and let \( x \in B(0, t) \) be fixed.

First suppose \( r < 2|x| \). Then \( B(x, r) \subset B(0,3t) \), and we have by Hölder’s inequality

\[ I \leq J_t \left( \frac{1}{|B(x, r)|} \int_{B(x, r)} |y|^{-\nu(y)} dy \right)^{1/p_0}. \]

If \( r \leq |x|/2 < t/2 \), then \( |y| \sim |x| \) and

\[ |y|^\nu(y) \sim |y|^\nu(0) \sim |x|^\nu(0) \sim |x|^\nu(x) \]

by (P1). Hence, in this case,

\[ \frac{1}{|B(x, r)|} \int_{B(x, r)} |y|^{-\nu(y)} dy \leq C \frac{1}{|B(x, r)|} \int_{B(x, r)} |x|^{-\nu(x)} dy \leq C|x|^{-\nu(x)}p_0. \]

If \( |x|/2 < r \leq 2|x| < 2t \), then, since \( |y|^\nu(y) \sim |y|^\nu(0) \) for \( y \in B(0, 6) \), we find

\[ \frac{1}{|B(x, r)|} \int_{B(x, r)} |y|^{-\nu(y)} dy \leq C \frac{1}{|B(0, 3|x|)|} \int_{B(0, 3|x|)} |y|^{-\nu(0)} dy \leq C|x|^{-\nu(0)}p_0 \leq C|x|^{-\nu(x)}p_0 \]

since \( \nu(0) < n(1 - 1/p_0) \). Hence

\[ I \leq C|x|^{-\nu(x)} J_t \]

when \( r \leq 2|x| < 2t \).

Finally, if \( r > 2|x| \), then

\[ I \leq \frac{1}{|B(x, r)|} \int_{B(x, r) \cap B(0, |x|)} |f(y)| dy + \frac{1}{|B(x, r)|} \int_{B(x, r) \backslash B(0, |x|)} |f(y)| dy \]

\[ \leq \frac{1}{|B(x, 2|x|)|} \int_{B(x, 2|x|)} |f(y)| dy + CH(x) \]

\[ \leq C|x|^{-\nu(x)} J_t + CH(x) \]
by the above discussions in the case $r = 2|x|$.

Now the present lemma is obtained. \hfill \square

Next we treat the Hardy type operator $H$ along the same manner as in [13].

**Lemma 4.6.** Let $p \in P_1(\mathbb{R}^n)$, $\mu \geq 0$, $\nu \in P(\mathbb{R}^n)$ and $\varepsilon > 0$. For a measurable function $f$ on $\mathbb{R}^n$ and $\beta \geq 0$, set

$$H_{\beta,1} = H_{\beta,1}(x) = H_{\beta,1}f(x) = \int_{B(0,1) \setminus B(0,|x|)} |f(y)||y|^{-\beta - n} dy$$

and

$$K_{\varepsilon,1} = K_{\varepsilon,1}(x) = \left( |x|^{\varepsilon - \mu p(x)} \int_{B(0,1) \setminus B(0,|x|)} g(y)|y|^{-\varepsilon} dy \right)^{1/p(x)},$$

where $g(y) = |f(y)|^{p(y)}|y|^{\nu(y)p(y)}$. If $0 < \delta < \varepsilon - \mu p(0) < (n + \nu(0)p(0))/p(0) - \mu - \beta$, then there exists a constant $C > 0$ such that

$$H_{\beta,1} \leq C|x|^{\beta - (\nu(x)p(x) + \delta + n)/p(x) + \mu} K_{\varepsilon,1} + C|x|^{\beta - (\nu(x)p(x) - \delta + n)/p(x) + \mu}$$

for all $x \in B(0,1)$ and $f$ with $\|f\|_{L^{\nu,p,\mu}(\mathbb{R}^n)} \leq 1$.

**Proof.** Let $f$ be a nonnegative measurable function on $\mathbb{R}^n$ with $\|f\|_{L^{\nu,p,\mu}(\mathbb{R}^n)} \leq 1$, and $x \in B(0,1)$. Set $E = \{y : f(y) \geq |y|^{-(\nu(y)p(y) + n)/p(y) + \mu}\}$. Noting that $|y|^{\tau(y)} \sim |y|^{\tau(0)}$ when $|y| < 1$ and $\tau \in P(\mathbb{R}^n)$ by (P1), we have

$$H_{\beta,1,1} = \int_{E \cap (B(0,1) \setminus B(0,|x|))} f(y)|y|^{-\beta - n} dy$$

$$\leq \int_{E \cap (B(0,1) \setminus B(0,|x|))} f(y)|y|^{-\beta - n} \left( \frac{f(y)}{|y|^{-(\nu(y)p(y) + n)/p(y) + \mu}} \right)^{p(y) - 1} dy$$

$$\leq \int_{B(0,1) \setminus B(0,|x|)} f(y)^{p(y)}|y|^{\nu(y)p(y) - \varepsilon}|y|^\beta + \varepsilon - \mu p(y) - (\nu(y)p(y) + n)/p(y) + \mu \, dy$$

$$\leq C \int_{B(0,1) \setminus B(0,|x|)} g(y)|y|^{-\varepsilon}|y|^\beta + \varepsilon - \mu p(0) - (\nu(0)p(0) + n)/p(0) + \mu \, dy$$

$$\leq C|x|^\beta - (\nu(0)p(0) + n)/p(0) + \mu |x|^\varepsilon - \mu p(0) \int_{B(0,1) \setminus B(0,|x|)} g(y)|y|^{-\varepsilon} dy$$

$$\leq C|x|^\beta - (\nu(x)p(x) + n)/p(x) + \mu K_{\varepsilon,1}$$

since

$$K_{\varepsilon,1}^{p(x)} \leq |x|^{\varepsilon - \mu p(x)} \sum_{j=1}^{\infty} \int_{B(0,1) \cap (B(0,2^j|x|) \setminus B(0,2^{j-1}|x|))} g(y)|y|^{-\varepsilon} dy$$

$$\leq C|x|^{\varepsilon - \mu p(x)} \sum_{j=1}^{\infty} (2^j|x|)^{\mu p(x)-\varepsilon} \leq C \sum_{j=1}^{\infty} 2^j(\mu p(0)-\varepsilon) \leq C$$

(4.1)
and $0 < \epsilon - \mu p(0) < (n + \nu(0)p(0))/p(0) - \mu - \beta$.

We next obtain by Hölder’s inequality and the fact that $|y|^{\tau(x)} \sim |y|^{\tau(y)} \sim |y|^{\tau(0)}$ when $|x| \leq |y| < 1$ and $\tau \in \mathcal{P}(\mathbb{R}^n)$ by (P1)

$$H_{\beta,1,2} \equiv \int_{F} f(y)|y|^{\beta-n} dy$$

$$\leq \left( \int_{F} |y|^{(\beta-(\nu(y)p(y)-\epsilon)/p(y)-n)p'(x)}|y|^{\nu(y)p(y)-\epsilon} dy \right)^{1/p(x)}$$

$$\leq C \left( \int_{F} |y|^{(\beta-(\nu(0)p(0)-\epsilon)/p(0)-n)p'(0)}|y|^{\nu(0)p(0)-\epsilon} dy \right)^{1/p(x)}$$

$$\leq C|x|^{(\beta-(\nu(x)p(x)-\epsilon+n)/p(x))p'(0)/p'(x)}(\int_{F} f(y)^{p(x)}|y|^{\nu(x)p(x)-\epsilon} dy)^{1/p(x)}$$

since $\beta - (\nu(0)p(0) - \epsilon + n)/p(0) < (\mu - \epsilon/p(0))(p(0) - 1)$, where

$$F = (B(0, 1) \setminus B(0, |x|)) \setminus E.$$ 

Here, since $0 < \delta < \epsilon - \mu p(0)$, we see that

$$\left( \int_{F} f(y)^{p(x)}|y|^{\nu(x)p(x)-\epsilon} dy \right)^{1/p(x)}$$

$$= \left( \int_{F} (f(y)|y|^{(\nu(x)p(x)+n-\mu p(x))/p(x)})^{p(x)}|y|^{-(\epsilon-\mu p(x))-n} dy \right)^{1/p(x)}$$

$$\leq C |x|^{(\delta-\epsilon)/p(x)+\mu} + C \left( \int_{F} f(y)^{p(y)}|y|^{\nu(y)p(y)-\epsilon} dy \right)^{1/p(x)},$$

since $(f(y)|y|^{(\nu(x)p(x)+n-\mu p(x))/p(x)})\leq C(f(y)|y|^{(\nu(x)p(x)+n-\mu p(x))/p(x)})p(y)$ when $|x| \leq |y| < 1$ and $|y|^{\delta/p(x)} < f(y)|y|^{(\nu(x)p(x)+n-\mu p(x))/p(x)} \leq 1$, so that

$$H_{\beta,1,2} \leq C|x|^{\beta-(\nu(x)p(x)-\delta+n)/p(x)+\mu} + C|x|^{\beta-(\nu(x)p(x)+n)/p(x)+\mu} K_{\epsilon,1},$$

which completes the proof. \(\square\)

For $p \in \mathcal{P}_1(\mathbb{R}^n)$ and $\beta \geq 0$, set

$$\frac{1}{p_\beta} = \frac{1}{p} - \frac{\beta}{n}.$$
Corollary 4.7. Let \( p \in \mathcal{P}_1(\mathbb{R}^n) \), \( \mu \geq 0 \) and \( \nu \in \mathcal{P}(\mathbb{R}^n) \) satisfy \( \beta p(0) + \mu p(0) - n < \nu(0)p(0) \). If \( 0 \leq \beta < n/p^+ \), then there exists a constant \( C > 0 \) such that
\[
\int_{B(0,t)} \left\{ H_{\beta,1}f(x)|x|^{\nu(x)} \right\}^{p_\beta(x)} \, dx \leq Ct^{\mu p_\beta(0)}
\]
for all \( t > 0 \) and \( f \geq 0 \) satisfying \( \|f\|_{L^{\nu,p,\mu}(\mathbb{R}^n)} \leq 1 \).

Proof. Let \( f \) be a nonnegative measurable function on \( \mathbb{R}^n \) satisfying
\[
\|f\|_{L^{\nu,p,\mu}(\mathbb{R}^n)} \leq 1.
\]
Take \( \delta \) and \( \varepsilon \) such that
\[
0 < \delta < \varepsilon - \mu p(0) < (n + \nu(0)p(0))/p(0) - \mu - \beta.
\]
In view of Lemma 4.6, we find
\[
H_{\beta,1}(x)|x|^{\nu(x)} \leq C|x|^\mu n/p_\beta(x) K_{\varepsilon,1}(x) + C|x|^{\delta/p(x) + \mu n/p_\beta(x)}
\]
for \( x \in B(0,1) \); and note that \( H_{\beta,1} = 0 \) outside \( B(0,1) \).

First we consider the case \( \mu = 0 \). Then note that
\[
\int_{\mathbb{R}^n} g(y) \, dy \leq C.
\]
Since \( K_{\varepsilon,1}(x) < C \) by (4.1), we obtain
\[
\int_{B(0,t)} \left\{ H_{\beta,1}(x)|x|^{\nu(x)} \right\}^{p_\beta(x)} \, dx
\]
\[
\leq C \int_{B(0,t)} \{ K_{\varepsilon,1}(x) \}^{p(x)} |x|^{-n} \, dx + C \int_{B(0,t)} |x|^{\delta/p(x) + \mu n/p_\beta(x)} \, dx
\]
\[
\leq C \int_{B(0,1) \setminus B(0,|x|)} g(y)|y|^{-\varepsilon} \, dy |x|^{-n} \, dx + Ct^{\delta p_\beta(0)/p(0)}
\]
\[
\leq C \int_{\mathbb{R}^n} g(y)|y|^{-\varepsilon} \left( \int_{B(0,|y|)} |x|^{-n} \, dx \right) dy + C
\]
\[
\leq C \int_{\mathbb{R}^n} g(y) \, dy + C \leq C
\]
for \( 0 < t < 1 \).

Next we consider the case \( \mu > 0 \). By (4.1) and (4.2), we have
\[
H_{\beta,1}(x)|x|^{\nu(x)} \leq C|x|^\mu n/p_\beta(x)
\]
for \( x \in B(0, 1) \). Hence we obtain
\[
\int_{B(0,t)} \{|x|^{\nu(x)} H_{\beta,1}(x)\}^{p_{\beta}(x)} dx \leq C \int_{B(0,t)} |x|^{\mu p_{\beta}(x)-n} dx \leq C t^{\mu p_{\beta}(0)}
\]
for \( 0 < t < 1 \), as required.

In the same manner as Lemma 4.5, we can prove the following result.

**Lemma 4.8.** Let \( p \in \mathcal{P}_1(\mathbb{R}^n) \) and \( \nu \in \mathcal{P}(\mathbb{R}^n) \). Let \( p_\infty > 1 \). Suppose \( \nu(\infty) < n \left( 1 - \frac{1}{p_\infty} \right) \).

For \( t > 2 \) and a measurable function \( f \) on \( \mathbb{R}^n \), set
\[
J_{t,\infty} = \left( \frac{1}{|B(x,r)|} \int_{B(x,r)} g_{\infty,t}(y) dy \right)^{1/p_\infty},
\]
where \( g_{\infty,t}(y) = \{|f(y)||y|^{\nu(y)}\}^{p_\infty} \chi_{B(0,3t)}(y) \). Then there exists a constant \( C > 0 \) such that
\[
I \leq C |x|^{-\nu(x)} J_{t,\infty} + C H(x)
\]
for all \( x \in B(0,t) \setminus B(0,1) \), \( r > 0 \) and \( f \) such that \( f = 0 \) on \( B(0,1) \).

**Lemma 4.9.** Let \( p \in \mathcal{P}_1(\mathbb{R}^n) \), \( \mu \geq 0 \) and \( \nu \in \mathcal{P}(\mathbb{R}^n) \). For a measurable function \( f \) on \( \mathbb{R}^n \), \( \beta \geq 0 \) and \( \eta > \mu p(\infty) \), set
\[
H_{\beta} = H_{\beta} f(x) = \int_{\mathbb{R}^n \setminus B(0,|x|)} |f(y)||y|^{\beta-n} dy
\]
and
\[
K_{\eta} = K_{\eta}(x) = \left( |x|^{\eta - \mu p(\infty)} \int_{\mathbb{R}^n \setminus B(0,|x|)} g(y)|y|^{-\eta} dy \right)^{1/p(x)},
\]
where \( g(y) = |f(y)|^{p(y)}|y|^{\nu(y)p(y)} \). If \( \epsilon > 0 \) and \( 0 < \epsilon(p(\infty) - 1) + \eta - \mu p(\infty) < (n + \nu(\infty)p(\infty))/p(\infty) - \mu - \beta \), then there exists a constant \( C > 0 \) such that
\[
H_{\beta} \leq C |x|^{\beta-(\nu(x)p(x)+n)/p(x)+\mu} K_{\eta} + C |x|^{\beta-\epsilon-(\nu(x)p(x)+n)/p(x)+\mu}
\]
for all \( x \in \mathbb{R}^n \setminus B(0,1) \) and \( f \geq 0 \) satisfying \( \|f\|_{L^{\nu,p,\mu}(\mathbb{R}^n)} \leq 1 \).

**Proof.** Let \( f \) be a nonnegative measurable function on \( \mathbb{R}^n \) satisfying
\[
\|f\|_{L^{\nu,p,\mu}(\mathbb{R}^n)} \leq 1,
\]
and $x \in \mathbb{R}^n \setminus B(0,1)$. Then, as in (4.1), we see that

$$|x|^{\eta - \mu p(\infty)} \int_{\mathbb{R}^n \setminus B(0,|x|)} g(y)|y|^{-\eta} dy \leq C$$

since $\eta > \mu p(\infty)$.

If $|x|^{-\epsilon} < K_\eta (< C)$, then $K_\eta^{-p(y)} \leq C K_\eta^{-p(x)}$ and $|y|^\tau(y) \sim |y|^\tau(\infty)$ when $1 < |x| \leq |y|$ and $\tau \in \mathcal{P}(\mathbb{R}^n)$ by (P2), so that

$$H_\beta \leq \int_{\mathbb{R}^n \setminus B(0,|x|)} K_\eta |y|^{-(\nu(y)p(y)+n)/p(y)+\mu} |y|^{\beta-n} dy$$

$$+ C \int_{\mathbb{R}^n \setminus B(0,|x|)} f(y)|y|^\beta-n \left( \frac{f(y)}{K_\eta |y|^{-(\nu(y)p(y)+n)/p(y)+\mu}} \right)^{p(y)-1} dy$$

$$\leq C K_\eta |x|^{\beta-(\nu(\infty)p(\infty)+n)/p(\infty)+\mu}$$

$$+ C K_\eta^{-1-p(x)} \int_{\mathbb{R}^n \setminus B(0,|x|)} g(y)|y|^{\beta-\mu p(y)-(\nu(y)p(y)+n)/p(y)+\mu} dy$$

$$\leq C K_\eta |x|^{\beta-(\nu(\infty)p(\infty)+n)/p(\infty)+\mu}$$

$$+ C K_\eta^{-1-p(x)} |x|^{\beta-(\nu(\infty)p(\infty)+n)/p(\infty)+\mu} |x|^{\eta-\mu p(x)} \int_{\mathbb{R}^n \setminus B(0,|x|)} g(y)|y|^{-\eta} dy$$

$$\leq C K_\eta |x|^{\beta-(\nu(\infty)p(\infty)+n)/p(\infty)+\mu}$$

$$+ C |x|^{\beta-(\nu(p(\infty)-1)-\mu p(y)-(\nu(y)p(y)+n)/p(y)+\mu}$$

$$\leq C |x|^{\beta-(\nu(\infty)p(\infty)+n)/p(\infty)+\mu}$$

since $\epsilon(p(\infty)-1)+\eta-\mu p(\infty) < (n+\nu(\infty)p(\infty))/p(\infty)-\mu-\beta$. Thus the proof is completed.

In the same manner as Corollary 4.7, we can prove the following result.
Corollary 4.10. Let $p \in \mathcal{P}_{1}(\mathbb{R}^{n})$, $\mu \geq 0$ and $\nu \in \mathcal{P}(\mathbb{R}^{n})$ satisfy $\beta p(\infty) + \mu p(\infty) - n < \nu(\infty)p(\infty)$. If $0 \leq \beta < n/p^{+}$, then there exists a constant $C > 0$ such that

\[
\int_{B(0,t) \setminus B(0,1)} \left\{H_{\beta}f(x)|x|^{l_{\nu}(x)}\right\}^{p_{\beta}(x)} dx \leq Ct^{\mu p_{\beta}(\infty)}
\]

for all $t > 1$ and $f \geq 0$ satisfying $\|f\|_{L^{\nu,p,\mu}(\mathbb{R}^{n})} \leq 1$.

Proof of Theorem 4.1. We now show the boundedness of the maximal operator. For this purpose, take $r_{0}, R_{0}$ and $p_{0}, p_{\infty}$ such that $0 < 2r_{0} < 1 < R_{0} < \infty, 1 < p_{0} < p(0), 1 < p_{\infty} < p(\infty), \nu(0) < n(1 - 1/p_{0})$ and $\nu(\infty) < n(1 - 1/p_{\infty})$. Let $f$ be a nonnegative measurable function on $\mathbb{R}^{n}$ satisfying $\|f\|_{L^{\nu,p,\mu}(\mathbb{R}^{n})} \leq 1$, and write

\[
f(y) = f(y)\chi_{B(0,r_{0})}(y) + f(y)\chi_{B(0,2R_{0}) \setminus B(0,r_{0})}(y) + f(y)\chi_{\mathbb{R}^{n} \setminus B(0,2R_{0})}(y) = f_{0,r_{0}}(y) + f_{r_{0},2R_{0}}(y) + f_{2R_{0},\infty}(y).
\]

Case 1: $0 < t \leq 2r_{0}$. If $0 < t \leq 2r_{0}$ and $x \in B(0, t)$, then we have by Lemma 4.5

\[
|x|^{\nu(x)} Mf_{0,r_{0}}(x) \leq C\{M(g_{0,t})(x)\}^{1/p_{0}} + C|x|^{\nu(x)} Hf(x),
\]

where $g_{0,t}(y) = \{f_{0,r_{0}}(y)|y|^{\nu(y)}\}^{p_{0}} \chi_{B(0,3t)}(y)$. Here note from Lemma 4.3 that

\[
\|\{M(g_{0,t})\}^{1/p_{0}}\|_{L^{p(\cdot)}(B(0,t))} \leq \|\{M(g_{0,t})\}^{1/p_{0}}\|_{L^{p(\cdot)}(B(0,6r_{0}))} \leq C\|\{(g_{0,t})^{1/p_{0}}\}_{L^{p(\cdot)}(B(0,6r_{0}))} \leq Ct^{\mu}.
\]

Therefore we obtain by Corollary 4.7

\[
\|\{M(g_{0,t})\}^{1/p_{0}}\|_{L^{p(\cdot)}(B(0,t))} \leq C\{M(g_{0,t})(x)\}^{1/p_{0}}\|_{L^{p(\cdot)}(B(0,t))} + C\|f(x)|^{\nu(x)} Hf(x)\|_{L^{p(\cdot)}(B(0,t))} \leq Ct^{\mu}.
\]

Now it follows from Lemma 4.4 and Corollary 4.7 that

\[
\|\{M_{\nu}(x)Mf(x)\|_{L^{p(\cdot)}(B(0,t))} \leq Ct^{\mu},
\]

since $M_{f_{2R_{0},\infty}}(x) \leq C(Hf(x))$ for $x \in B(0,R_{0}/2)$.

Case 2: $2r_{0} < t \leq R_{0}$. For $x \in \mathbb{R}^{n} \setminus B(0,2r_{0})$ note that

\[
Mf_{0,r_{0}}(x) \leq C|x|^{-n} \int_{\mathbb{R}^{n}} f_{0,r_{0}}(y) dy \leq C|x|^{-n} \left( \int_{B(0,r_{0})} |y|^{-\nu(y)p'(y)} dy + \int_{B(0,r_{0})} f(y)^{p(y)} |y|^{\nu(y)p(y)} dy \right) \leq C|x|^{-n}
\]
and
\[ M_{f_{2r_{0},\infty}}(x) \leq CHf(x) \]
for \( x \in B(0, R_{0}) \). Hence we find by Lemma 4.4 and Corollaries 4.7 and 4.10
\[
\| |x|^{\nu(x)} Mf(x) \|_{L^{p(\cdot)}(B(0,t) \setminus B(0,2r_{0}))} \\
\leq C\| |x|^{\nu(x)-n} \|_{L^{p(\cdot)}(B(0,t) \setminus B(0,2r_{0}))} + \| |x|^{\nu(x)} Mf_{r_{0},2R_{0}}(x) \|_{L^{p(\cdot)}(B(0,t) \setminus B(0,2r_{0}))} \\
+ C\| |x|^{\nu(x)} Hf(x) \|_{L^{p(\cdot)}(B(0,1) \setminus B(0,2r_{0}))} + C\| |x|^{\nu(x)} Hf(x) \|_{L^{p(\cdot)}(B(0,t) \setminus B(0,1))} \\
\leq Ct^{\mu},
\]
so that the first case gives
\[
\| |x|^{\nu(x)} Mf(x) \|_{L^{p(\cdot)}(B(0,t))} \leq Ct^{\mu}.
\]

Case 3: \( t > R_{0} \). If \( t > R_{0} \) and \( x \in \mathbb{R}^{n} \setminus B(0, R_{0}) \), then Lemma 4.8 gives
\[
| |x|^{\nu(x)} Mf_{2R_{0},\infty}(x) | \leq C \{ M(g_{\infty,t})(x) \}^{1/p_{\infty}} + C| |x|^{\nu(x)} H(x),
\]
where \( g_{\infty,t}(y) = \{ f_{2R_{0},\infty}(y) |y|^{\nu(y)} \}^{p_{\infty}} \chi_{B(0,3t)}(y) \). By Lemma 4.3, we find
\[
\| \{ M(g_{\infty,t}) \}^{1/p_{\infty}} \|_{L^{p(\cdot)}(B(0,t) \setminus B(0,R_{0}))} \leq \| \{ M(g_{\infty,t}) \}^{1/p_{\infty}} \|_{L^{p(\cdot)}(\mathbb{R}^{n} \setminus B(0,R_{0}))} \\
\leq C\| (g_{\infty,t})^{1/p_{\infty}} \|_{L^{p(\cdot)}(\mathbb{R}^{n} \setminus B(0,R_{0}))} \leq Ct^{\mu}
\]
and hence
\[
\| |x|^{\nu(x)} Mf_{2R_{0},\infty}(x) \|_{L^{p(\cdot)}(B(0,t) \setminus B(0,R_{0}))} \\
\leq C\| \{ M(g_{\infty,t}) \}^{1/p_{\infty}} \|_{L^{p(\cdot)}(B(0,t) \setminus B(0,R_{0}))} + C\| |x|^{\nu(x)} H(x) \|_{L^{p(\cdot)}(B(0,t) \setminus B(0,R_{0}))} \\
\leq Ct^{\mu}
\]
by Corollary 4.10. Now it follows from Lemma 4.4 that
\[
\| |x|^{\nu(x)} Mf(x) \|_{L^{p(\cdot)}(B(0,t) \setminus B(0,R_{0}))} \\
\leq C\| |x|^{\nu(x)-n} \|_{L^{p(\cdot)}(B(0,t) \setminus B(0,R_{0}))} + \| |x|^{\nu(x)} Mf_{r_{0},2R_{0}}(x) \|_{L^{p(\cdot)}(B(0,t) \setminus B(0,R_{0}))} \\
+ C\| |x|^{\nu(x)} Mf_{2R_{0},\infty}(x) \|_{L^{p(\cdot)}(B(0,t) \setminus B(0,R_{0}))} \\
\leq Ct^{\mu},
\]
and the proof is completed, with the aid of the second case. \( \square \)

§ 5. Sobolev’s inequality in Lorentz spaces

Our next aim in this paper is to establish the Sobolev type inequality for Riesz potentials.

For \( p \in \mathcal{P}_{1}(\mathbb{R}^{n}) \), set
\[
\frac{1}{p^{\#}} = \frac{1}{p} - \frac{\alpha}{n},
\]
Theorem 5.1. Let $\nu \in \mathcal{P}(\mathbb{R}^n)$, $p \in \mathcal{P}_1(\mathbb{R}^n)$ and $\mu \geq 0$. Suppose $\alpha < n/p^+$ and

\textbf{(T2)} $\alpha p(0) - n + \mu p(0) < \nu(0)p(0) < n(p(0) - 1)$ and $\alpha p(\infty) - n + \mu p(\infty) < \nu(\infty)p(\infty) < n(p(\infty) - 1)$.

Then there is a constant $C > 0$ such that

$$\|I_\alpha f\|_{L^{\nu,p,\mu}(\mathbb{R}^n)} \leq C\|f\|_{L^{\nu,p,\mu}(\mathbb{R}^n)}$$

for all $f \in L^{\nu,p,\mu}(\mathbb{R}^n)$.

With the aid of Lemma 2.4, we find the following result.

Corollary 5.2. Let $\nu \in \mathcal{P}(\mathbb{R}^n)$, $p \in \mathcal{P}_1(\mathbb{R}^n)$, $q \in \mathcal{P}_1(\mathbb{R}^n)$ and $\mu \geq 0$. If $\alpha < n/p^+$, $\alpha + \mu < n/q(0)$ and $\alpha + \mu < n/q(\infty)$, then there is a constant $C > 0$ such that

$$\|I_\alpha f\|_{\mathcal{L}^{q,p,\mu}(\mathbb{R}^n)} \leq C\|f\|_{\mathcal{L}^{q,p,\mu}(\mathbb{R}^n)}$$

for all $f \in \mathcal{L}^{q,p,\mu}(\mathbb{R}^n)$.

§ 6. Proof of Theorem 5.1

For a measurable function $f$ on an open set $G \subset \mathbb{R}^n$, we define

$$I_{\alpha,G}f(x) = \int_G |x-y|^{\alpha-n}f(y)dy.$$

First we note the Sobolev’s inequality for Riesz potentials of functions in $L^{p(\cdot)}(\mathbb{R}^n)$.

Lemma 6.1 ([11, Theorem 6.4]). Let $p \in \mathcal{P}_1(\mathbb{R}^n)$ and $G$ be an open set in $\mathbb{R}^n$. If $1 < p^-(G) \leq p^+(G) < n/\alpha$, then there exists a constant $C > 0$ such that

$$\|I_{\alpha,G}f\|_{L^{p(\cdot)}(G)} \leq C\|f\|_{L^{p(\cdot)}(G)}.$$

For a nonnegative measurable function $f$ on $\mathbb{R}^n$, we write

$$I_\alpha f(x) = \int_{B(x,2|x|)} |x-y|^{\alpha-n}f(y)dy + \int_{\mathbb{R}^n \setminus B(x,2|x|)} |x-y|^{\alpha-n}f(y)dy$$

$$= U_\alpha f(x) + I_2(x).$$

Then note that

$$I_2(x) \leq CH_\alpha f(x).$$

For $U_\alpha f$ we have the following result in the same manner as Lemmas 4.5 and 4.8.
Lemma 6.2. Let $p \in \mathcal{P}_1(\mathbb{R}^n)$ and $\nu \in \mathcal{P}(\mathbb{R}^n)$. Let $p_0 > 1$. Suppose
\[ \nu(0) < n \left(1 - \frac{1}{p_0}\right). \]
Then there exists a constant $C > 0$ such that
\[
U_\alpha f(x) \leq C|x|^{-\nu(x)} \left[I_\alpha g_t(x) + \{I_{\alpha p_0}(g_t)^{p_0}(x)\}^{1/p_0}\right]
\]
for all $x \in B(0, t)$ with $0 < t \leq 2$ and nonnegative measurable functions $f$ on $\mathbb{R}^n$, where $g_t(y) = f(y)|y|^\nu(y)\chi_{B(0,3t)}(y)$.

Proof. Let $f$ be a nonnegative measurable function on $\mathbb{R}^n$. First note that $B(x, 2|x|) \subset B(0, 3t)$ for $x \in B(0, t)$. Set $f_t(y) = f(y)\chi_{B(0,3t)}(y)$. We have by Hölder’s inequality
\[
\int_{B(0,|x|/2)}|x-y|^{-n}f_t(y)dy \leq \left(\int_{B(0,|x|/2)}|x-y|^{-n}\{|y|^{-\nu(y)}\}^{p_0}dy\right)^{1/p_0}
\times \left(\int_{B(0,|x|/2)}|x-y|^{\alpha p_0-n}\{f_t(y)|y|^\nu(y)\}^{p_0}dy\right)^{1/p_0}
\leq C\left(|x|^{-n}\int_{B(0,|x|/2)}|y|^{-\nu(y)p_0}dy\right)^{1/p_0}\{I_{\alpha p_0}(g_t)^{p_0}(x)\}^{1/p_0}
\leq C|x|^{-\nu(x)}\{I_{\alpha p_0}(g_t)^{p_0}(x)\}^{1/p_0},
\]
as in the proof of Lemma 4.5. Moreover, if $y \in B(x, 2|x|) \setminus B(0, |x|/2)$, then $|y| \sim |x|$ and $|y|^\tau(y) \sim |x|^\tau(x)$ for $\tau \in \mathcal{P}(\mathbb{R}^n)$, so that
\[
\int_{B(x,2|x|)\setminus B(0,|x|/2)}|x-y|^{-n}f_t(y)dy
\leq C|x|^{-\nu(x)}\int_{B(x,2|x|)\setminus B(0,|x|/2)}|x-y|^{\alpha-n}f_t(y)|y|^\nu(y)dy
\leq C|x|^{-\nu(x)}I_\alpha g_t(x),
\]
as required. \qed

In the same manner as Lemma 6.2, we can prove the following result.

Lemma 6.3. Let $p \in \mathcal{P}_1(\mathbb{R}^n)$ and $\nu \in \mathcal{P}(\mathbb{R}^n)$. Let $p_\infty > 1$. Suppose
\[ \nu(\infty) < n \left(1 - \frac{1}{p_\infty}\right). \]
Then there exists a constant $C > 0$ such that
\[
U_\alpha f(x) \leq C|x|^{-\nu(x)} \left[I_\alpha g_t(x) + \{I_{\alpha p_\infty}(g_t)^{p_\infty}(x)\}^{1/p_\infty}\right]
\]
all \( x \in B(0, t) \setminus B(0, 1) \), \( t > 2 \) and nonnegative measurable functions \( f \) such that \( f = 0 \) on \( B(0, 1) \), where \( g_t(y) = f(y)|y|^{\nu(y)}\chi_{B(0, 3t)}(y) \).

**Proof of Theorem 5.1.** Take \( r_0, R_0 \) and \( p_0, p_{\infty} \) such that \( 0 < 2r_0 < 1 < R_0 < \infty, \) \( 1 < p_0 < p(0), 1 < p_{\infty} < p(\infty), \nu(0) < n(1 - 1/p_0) \) and \( \nu(\infty) < n(1 - 1/p_{\infty}) \). Let \( f \) be a nonnegative measurable function on \( \mathbb{R}^n \) satisfying \( \|f\|_{L^{\nu, p, \mu}(\mathbb{R}^n)} \leq 1 \), and write
\[
f(y) = f(y)\chi_{B(0, r_0)}(y) + f(y)\chi_{B(0, 2R_0) \setminus B(0, r_0)}(y) + f(y)\chi_{\mathbb{R}^n \setminus B(0, 2R_0)}(y)
= f_{0, r_0}(y) + f_{r_0, 2R_0}(y) + f_{2R_0, \infty}(y).
\]
As in the proof of Theorem 4.1, it suffices to show that
\[
\|x|^{\nu(x)} I_{\alpha} f_{0, r_0}(x) \|_{L^{p(\cdot)}(B(0, t))} \leq Ct^\mu \quad \text{for } 0 < t \leq 2r_0
\]
and
\[
\|x|^{\nu(x)} I_{\alpha} f_{2R_0, \infty}(x) \|_{L^{p(\cdot)}(B(0, t) \setminus B(0, R_0))} \leq Ct^\mu \quad \text{for } t \geq R_0.
\]
In the rest of the present proof, we are only concerned with \( f_{0, r_0} \). For this note from Lemmas 6.1, 6.2 and Corollary 4.7 that
\[
\|x|^{\nu(x)} I_{\alpha} f_{0, r_0}(x) \|_{L^{p(\cdot)}(B(0, t))} \leq C \|I_{\alpha} g_t\|_{L^{p(\cdot)}(B(0, 6r_0))}
+ C \|\{I_{\alpha p_0} (g_t)^{p_0}\}^{1/p_0} \|_{L^{p(\cdot)}(B(0, 6r_0))} + C \|\|x|^{\nu(x)} Hf(x)\|_{L^{p(\cdot)}(B(0, t))}
\leq C \|g_t\|_{L^{p(\cdot)}(B(0, 6r_0))} + Ct^\mu \leq Ct^\mu
\]
when \( 0 < t < 2r_0 \), where \( g_t(\cdot) = f(\cdot)|\cdot|^{\nu(\cdot)}\chi_{B(0, 3t)}(\cdot) \).

The remaining part of the proof is easily completed. \( \square \)

**References**


