RIMS Kôkyûroku Bessatsu
B43 (2013), 155–170

A bilinear estimate for commutators of fractional integral operators

By

Yoshihiro Sawano*, Satoko Sugano** and Hitoshi Tanaka***

Abstract

The aim of this paper is to investigate the Morrey norm boundedness of commutators generated by $\mathrm{BMO}(\mathbb{R}^n)$-functions and the Riesz kernel. A bilinear estimate is the focus of this paper, which cannot be obtained from a mere combination of the boundedness of commutators and the Hölder inequality. As a key tool, a decomposition using dyadic cubes is employed.

§ 1. Introduction

The aim of this paper is to investigate a bilinear estimate generated by commutators. Let $a \in \mathrm{BMO}(\mathbb{R}^n)$ and $0 < \alpha < n$. Let $m \in \mathbb{N}$. The m-fold commutator $[I_\alpha, a]^{(m)}$ is given by

$$[I_\alpha, a]^{(m)} f(x) \equiv \int_{\mathbb{R}^n} \frac{(a(x)-a(y))^m}{|x-y|^{n-\alpha}} f(y)\, dy.$$

Here and below we assume that the functions are real-valued and measurable. We recall the definition of $\mathrm{BMO}(\mathbb{R}^n)$ in Section 2. As is verified in (3.4), we shall consider

$$x \mapsto \int_{\mathbb{R}^n} \frac{|a(x)-a(y)|^m}{|x-y|^{n-\alpha}} f(y)\, dy$$

Received October 24, 2012. Revised April 22, 2013.
2000 Mathematics Subject Classification(s): 42B35, 42B25, 26A33, 35J10
Key Words: fractional integral operators, Schrödinger equations, multiplication operators
The first author was supported financially by Grant-in-Aid for Young Scientists (B) (No. 24740085) Japan Society for the Promotion of Science.
The third author was supported by the Global COE program at Graduate School of Mathematical Sciences, the University of Tokyo, Grant-in-Aid for Scientific Research (C) (No. 23540187), the Japan Society for the Promotion of Science, and was supported by Fūjyukai foundation.
*Department of Mathematics and Information Sciences, Tokyo Metropolitan University, 1-1 Minami-Ohsawa, Hachioji, Tokyo 192-0397, Japan.
e-mail: ysawano@tmu.ac.jp
**Kobe City College of Technology, 8-3 Gakuen-higashimachi, Nishi-ku, Kobe 651-2194, Japan.
e-mail: sugano@kobe-kosen.ac.jp
***Graduate School of Mathematical Sciences, The University of Tokyo, Tokyo, 153-8914, Japan.
e-mail: htanaka@ms.u-tokyo.ac.jp

and hence we may assume that the integral defining $[I_\alpha, a]^{(m)} f(x)$ converges for a.e. $x \in \mathbb{R}^n$.

In the present paper we investigate the boundedness of the operator given by

$$(f, g) \mapsto g \cdot [I_\alpha, a]^{(m)} f$$

on Morrey spaces. Here, we shall adopt the following definition of the Morrey space $\mathcal{M}^{p_0}_p(\mathbb{R}^n)$, $1 < p \le p_0 < \infty$: First we define $\mathcal{D}$ as the set of all dyadic cubes (see (2.1) below). For a measurable function f we define

$$\|f\|_{\mathcal{M}^{p_0}_p} \equiv \sup_{Q \in \mathcal{D}} |Q|^{\frac{1}{p_0} - \frac{1}{p}} \left(\int_Q |f(y)|^p \, dy \right)^{\frac{1}{p}}.$$

The function space $\mathcal{M}^{p_0}_p(\mathbb{R}^n)$ is the set of all measurable functions f for which the norm $\|f\|_{\mathcal{M}^{p_0}_p}$ is finite.

Now we present our main result of the present paper.

Theorem 1.1. *Suppose we are given parameters $\alpha, p_0, p, q_0, q, r_0, r$ satisfying*

$$0 < \alpha < n,\ 1 < p \le p_0 < \infty,\ 1 < q \le q_0 < \infty,\ 1 < r \le r_0 < \infty$$

and

$$q > r,\ \frac{1}{p_0} > \frac{\alpha}{n} \ge \frac{1}{q_0}.$$

Assume in addition that

$$\frac{r}{r_0} = \frac{p}{p_0},\ \frac{1}{p_0} + \frac{1}{q_0} - \frac{\alpha}{n} = \frac{1}{r_0}.$$

Then we have

$$\|g \cdot [I_\alpha, a]^{(m)} f\|_{\mathcal{M}^{r_0}_r} \le C\, \|a\|^m_{\mathrm{BMO}} \|f\|_{\mathcal{M}^{p_0}_p} \|g\|_{\mathcal{M}^{q_0}_q} \tag{1.1}$$

for all $a \in \mathrm{BMO}(\mathbb{R}^n)$, $f \in \mathcal{M}^{p_0}_p(\mathbb{R}^n)$ and $g \in \mathcal{M}^{q_0}_q(\mathbb{R}^n)$.

The method of the proof of Theorem 1.1 also covers a classical theorem of the commutator: It corresponds to the case $q_0 = \infty$ and $g \equiv 1$.

Corollary 1.2. *Suppose we are given parameters α, p_0, p, r_0, r satisfying*

$$0 < \alpha < n,\ 1 < p \le p_0 < \infty,\ 1 < r \le r_0 < \infty.$$

Assume in addition that

$$\frac{r}{r_0} = \frac{p}{p_0},\ \frac{1}{p_0} - \frac{\alpha}{n} = \frac{1}{r_0}.$$

Then we have

$$\|[I_\alpha, a]^{(m)} f\|_{\mathcal{M}^{r_0}_r} \le C\, \|a\|^m_{\mathrm{BMO}} \|f\|_{\mathcal{M}^{p_0}_p}$$

for all $a \in \mathrm{BMO}(\mathbb{R}^n)$ and $f \in \mathcal{M}^{p_0}_p(\mathbb{R}^n)$.

The proof is left for interested readers.

This inequality (1.1) dates back to the one obtained in [9], which deals with the operator I_α given by

$$I_\alpha f(x) \equiv \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy.$$

Proposition 1.3. *Suppose we are given parameters $\alpha, p_0, p, q_0, q, r_0, r$ satisfying*

$$0<\alpha<n,\ 1<p \le p_0<\infty,\ 1<q \le q_0<\infty,\ 1<r \le r_0<\infty$$

and

$$q>r,\ \frac{1}{p_0}>\frac{\alpha}{n} \ge \frac{1}{q_0}.$$

Assume in addition that

$$\frac{r}{r_0}=\frac{p}{p_0},\ \frac{1}{p_0}+\frac{1}{q_0}-\frac{\alpha}{n}=\frac{1}{r_0}.$$

Then we have

$$\|g \cdot I_\alpha f\|_{\mathcal{M}^{r_0}_r} \le C\, \|f\|_{\mathcal{M}^{p_0}_p} \|g\|_{\mathcal{M}^{q_0}_q} \tag{1.2}$$

for all $f \in \mathcal{M}^{p_0}_p(\mathbb{R}^n)$ and $g \in \mathcal{M}^{q_0}_q(\mathbb{R}^n)$.

As was discussed in [10, p.7], Proposition 1.3 is not an immediate consequence of the well-known boundedness of I_α due to Adams [1] and the Hölder inequality. The same can be said for Theorem 1.1; it cannot be deduced directly from Corollary 1.2 and the Hölder inequality. The inequality (1.2) is called the Olsen inequality and investigated initially in [7].

Several people have tried to extend the original results (see [3, 15] for more details). In [9] we proved that the condition on $q>r$ is sharp. Although we mean (1.2) by the Olsen inequality (see [3, 9, 10, 17], for example), we overlooked the original paper [2]. In [2], on $\mathbb{R}^3$, Conlon and Redondo considered the following equation:

$$\begin{cases} (-\Delta - \mathbf{b}(x)\cdot\nabla)u(x)=f(x) & (|x|<R), \\ u(x)=0 & (|x|=R), \end{cases}$$

for $R>0$. If $\mathbf{b}$ is smooth, then as is described in [12], we have an expression of the solution;

$$u(x)=E_x\left[\int_0^\tau f(X_{\mathbf{b}}(t))\,dt\right],$$

where $X_{\mathbf{b}}(t)$ is a Brownian motion starting from $x \in \{|y|<R\}$ with drift $\mathbf{b}$, E_x denotes the expectation with respect to $X_{\mathbf{b}}(t)$ and τ is the first hitting time on the boundary $|x|=R$. Conlon and Redondo proved Proposition 1.3 with $n=3$ essentially.

One of the reasons why (1.2) holds is that in the Adams theorem I_α is not surjective. Indeed, we have;

Proposition 1.4. *Suppose $0 < \alpha < n$, $1 < p \le p_0 < \infty$, $1 < r \le r_0 < \infty$. Assume that*

$$\frac{r}{r_0} = \frac{p}{p_0}, \ \frac{1}{r_0} = \frac{1}{p_0} - \frac{\alpha}{n}.$$

Then, I_α is bounded from $\mathcal{M}^{p_0}_p(\mathbb{R}^n)$ to $\mathcal{M}^{r_0}_r(\mathbb{R}^n)$.

Proposition 1.5. *In Proposition 1.4, I_α is not surjective from $\mathcal{M}^{p_0}_p(\mathbb{R}^n)$ to $\mathcal{M}^{r_0}_r(\mathbb{R}^n)$.*

Proposition 1.5 was proven as [11, Corollary 3.6]. However, in Section 4, we give an alternative proof. Recently, in [13, 14] an inequality dealing with I_α and intersection of Morrey spaces was considered. In this note, by using this new type of inequality, we reprove Proposition 1.5.

Seemingly, Theorem 1.1 is a consequence of Corollary 1.2 and the following lemma:

Lemma 1.6. *Let $1 < q_1 \le p_1 < \infty$ and $1 < q_2 \le p_2 < \infty$. Define*

$$p \equiv \frac{p_1 p_2}{p_1 + p_2}, \quad q \equiv \frac{q_1 q_2}{q_1 + q_2}.$$

Then

$$\|f \cdot g\|_{\mathcal{M}^p_q} \le \|f\|_{\mathcal{M}^{p_1}_{q_1}} \|g\|_{\mathcal{M}^{p_2}_{q_2}}$$

for all $f \in \mathcal{M}^{p_1}_{q_1}(\mathbb{R}^n)$ and $g \in \mathcal{M}^{p_2}_{q_2}(\mathbb{R}^n)$.

However, this is not the case; a mere combination of Proposition 1.4 and Lemma 1.6 does not give Theorem 1.1. Indeed, Morrey spaces are nested:

$$\mathcal{M}^{p_0}_{p_1}(\mathbb{R}^n) \subset \mathcal{M}^{p_0}_{p_2}(\mathbb{R}^n)$$

for all $1 < p_2 \le p_1 < \infty$. The following example shows that the inclusion is strict:

Example 1.7. For $r < 1/2$, and $\vec{e} \in \{0,1\}^n$, we define

$$S_{r,\vec{e}}(x) = rx + (1-r)\vec{e} \quad (x \in \mathbb{R}^n).$$

Define inductively $\{E_j\}_{j=0}^\infty$ by

$$E_0 = [0,1]^n, \quad E_j = \bigcup_{\vec{e} \in \{0,1\}^n} S_{r,\vec{e}}(E_{j-1}) \quad (j = 1, 2, \cdots).$$

Then we have

$$\|\chi_{E_j}\|_{\mathcal{M}^p_q} \simeq \max(\|\chi_{[0,r]^n}\|_{L^p}, \|\chi_{E_j}\|_{L^q})$$

for all j, where the implicit constants in $\simeq$ do not depend upon j and r.

A detailed calculation in [10, p.6] shows that the case when $\frac{p}{p_0}r_0 < q < \frac{p}{p_0}q_0$ is beyond the reach of the combination of Corollary 1.2 and Lemma 1.6.

We can pass our result to the operator given by

$$[I_\alpha,(a_1,a_2,\cdots,a_m)]f(x) \equiv \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \prod_{j=1}^m (a_j(x)-a_j(y))\,dy.$$

Theorem 1.8. *Suppose we are given parameters $\alpha, p_0, p, q_0, q, r_0, r$ satisfying*

$$0<\alpha<n,\ 1<p\le p_0<\infty,\ 1<q\le q_0<\infty,\ 1<r\le r_0<\infty$$

and

$$q>r,\ \frac{1}{p_0}>\frac{\alpha}{n}\ge\frac{1}{q_0}.$$

Assume in addition that

$$\frac{r}{r_0}=\frac{p}{p_0},\ \frac{1}{p_0}+\frac{1}{q_0}-\frac{\alpha}{n}=\frac{1}{r_0}.$$

Then we have

$$\|g\cdot[I_\alpha,(a_1,a_2,\cdots,a_m)]f\|_{\mathcal{M}^{r_0}_r} \le C\left(\prod_{j=1}^m \|a_j\|_{\mathrm{BMO}}\right)\|f\|_{\mathcal{M}^{p_0}_p}\|g\|_{\mathcal{M}^{q_0}_q}$$

for all $a_1,a_2,\cdots,a_m \in \mathrm{BMO}(\mathbb{R}^n)$, $f\in\mathcal{M}^{p_0}_p(\mathbb{R}^n)$ and $g\in\mathcal{M}^{q_0}_q(\mathbb{R}^n)$.

Theorem 1.8 follows from Theorem 1.1, a homogeneity argument and the following lemma, the proof of which will be given in the appendix:

Lemma 1.9. *For all $m\in\mathbb{N}$, the polynomial $x_1x_2\cdots x_m$ is in the linear span of the set*

$$\mathcal{V}_m \equiv \{(a_1x_1+a_2x_2+\cdots+a_mx_m)^m \,:\, a_1,a_2,\cdots,a_m\in\mathbb{R}\}.$$

§ 2. Notations and preliminaries

Here we fix some notations.

1. We define the set of all dyadic cubes as follows:

$$\mathcal{D}\equiv\left\{\prod_{j=1}^n\left[\frac{m_j}{2^\nu},\frac{m_j+1}{2^\nu}\right) \,:\, \nu\in\mathbb{Z},\ m=(m_1,m_2,\cdots,m_n)\in\mathbb{Z}^n\right\}. \tag{2.1}$$

   If a dyadic cube Q has volume $2^{-\nu n}$, then we say that Q is of the ν-th generation. We also write $\mathcal{D}_\nu$ the set of all dyadic cubes of the ν-th generation. If $Q\in\mathcal{D}_\nu$, then define $\ell(Q)=2^{-\nu}$. Observe that

$$\sum_{Q\in\mathcal{D}_\nu}\chi_Q\equiv 1. \tag{2.2}$$

2. The open ball centered at $x \in \mathbb{R}^n$ of radius $r > 0$ will be denoted by $B(x, r)$.

3. Given a function f and a dyadic cube $Q \in \mathcal{D}$, we set $m_Q(f) \equiv \dfrac{1}{|Q|} \displaystyle\int_Q f(x)\,dx$.

4. By a cube we mean a compact cube whose edges are parallel to the coordinate axes. The set $\mathcal{Q}$ denotes the totality of all cubes. For a point $x \in \mathbb{R}^n$, we write $\mathcal{Q}(x)$ for the set of all cubes in $\mathcal{Q}$ containing x.

5. The function space $\mathrm{BMO}(\mathbb{R}^n)$ is the set of all measurable functions f for which the quantity
$$\|f\|_{\mathrm{BMO}} \equiv \sup_{Q \in \mathcal{Q}} m_Q(|f - m_Q(f)|)$$
is finite.

6. The maximal operator is defined by
$$Mf(x) \equiv \sup_{Q \in \mathcal{Q}(x)} \frac{1}{|Q|} \int_Q |f(y)|\,dy.$$
There are several variants: write
$$M^{(u)}f(x) \equiv \sup_{Q \in \mathcal{Q}(x)} \left(\frac{1}{|Q|} \int_Q |f(y)|^u\,dy \right)^{\frac{1}{u}} \quad (u \in (1, \infty)),$$
$$M_\alpha f(x) \equiv \sup_{Q \in \mathcal{Q}(x)} \frac{1}{|Q|^{1-\alpha/n}} \int_Q |f(y)|\,dy \quad (\alpha \in [0, n)).$$

We shall recall some fundamental facts of the maximal operator M_α above and $\mathrm{BMO}(\mathbb{R}^n)$-functions.

Lemma 2.1. *Suppose the parameters α, p_0, p, r_0, r satisfy*
$$0 \le \alpha < n,\ 1 < p \le p_0 < \infty,\ 1 < r \le r_0 < \infty.$$
Assume in addition that
$$\frac{r}{r_0} = \frac{p}{p_0},\ \frac{1}{p_0} - \frac{\alpha}{n} = \frac{1}{r_0},\ \frac{1}{p_0} > \frac{\alpha}{n}.$$
Then we have
$$\|M_\alpha f\|_{\mathcal{M}^{r_0}_r} \le C\,\|f\|_{\mathcal{M}^{p_0}_p}.$$
Here it will be understood that M_0 denotes the Hardy-Littlewood maximal operator M.

Lemma 2.2 (The John-Nirenberg inequality). *Let $1 \le p < \infty$ and let Q be a cube. Then there exists a constant $c > 0$ such that*

$$\left(\frac{1}{|Q|}\int_Q |a(x) - m_Q(a)|^p\,dx\right)^{\frac{1}{p}} \le c\|a\|_{\mathrm{BMO}}$$

for all $a \in \mathrm{BMO}(\mathbb{R}^n)$.

We also need a decomposition result about cubes. Let Q_0 be a cube and let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. We set

$$\mathcal{D}(Q_0) \equiv \{Q \in \mathcal{D} :\ Q \subset Q_0\}.$$

We write $3Q_0$ for its triple, that is, the unique cube concentric to Q_0 and having the volume $3^n|Q_0|$. Letting $\gamma_0 \equiv m_{3Q_0}(f)$ and $A = 2\cdot 18^n$, we set, for $k = 1, 2, \ldots$,

$$D_k \equiv \bigcup \left\{Q\ :\ Q \in \mathcal{D}(Q_0),\, m_{3Q}(f) > \gamma_0 A^k\right\}.$$

Considering the maximal cubes with respect to inclusion, we can write

$$D_k = \bigcup_j Q_{k,j},$$

where the cubes $\{Q_{k,j}\}_j \subset \mathcal{D}(Q_0)$ are nonoverlapping. That is, $\{Q_{k,j}\}_j$ is a family of cubes satisfying

$$\sum_j \chi_{Q_{k,j}} \le \chi_{Q_0}$$

for almost everywhere. By the maximality of $Q_{k,j}$ we see that

$$\gamma_0 A^k < m_{3Q_{k,j}}(f) \le 2^n \gamma_0 A^k. \tag{2.3}$$

Let

$$E_0 \equiv Q_0 \setminus D_1, \quad E_{k,j} \equiv Q_{k,j} \setminus D_{k+1}.$$

We need the following properties:

Lemma 2.3. ([5]) *The set $\{E_0\} \cup \{E_{k,j}\}$ forms a disjoint family of sets, which decomposes Q_0, and satisfies*

$$|Q_0| \le 2|E_0|, \quad |Q_{k,j}| \le 2|E_{k,j}|. \tag{2.4}$$

For the sake of completeness we recall the proof here.

Proof. By (2.3) we see that

$$Q_{k,j} \cap D_{k+1} \subset \{x \in Q_{k,j} :\ M[\chi_{3Q_{k,j}} f](x) > \gamma_0 A^{k+1}\}.$$

Using the weak-$(1,1)$ boundedness of M, we have

$$|Q_{k,j} \cap D_{k+1}| \le \frac{3^n}{\gamma_0 A^{k+1}} \int_{3Q_{k,j}} |f(y)|\, dy \le \frac{6^n}{A}|3Q_{k,j}| = \frac{18^n}{A}|Q_{k,j}| = \frac{1}{2}|Q_{k,j}|, \tag{2.5}$$

where we have used again (2.3). Similarly, we see that

$$|D_1| \le \frac{1}{2}|Q_0|. \tag{2.6}$$

Clearly, (2.5) and (2.6) imply (2.4). □

§ 3. Proof of Theorem 1.1

We depend on the method of Li and Perez [6, 8]. Here and below we can assume that f and g are positive.

§ 3.1. Set up

We set

$$C_1[f,g](x) \equiv g(x) \sum_{\nu \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_\nu} 2^{\nu(n-\alpha)} \chi_Q(x) \int_{3Q} |m_Q(a) - a(y)|^m f(y)\, dy$$

$$C_2[f,g](x) \equiv g(x) \sum_{\nu \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_\nu} 2^{\nu(n-\alpha)} \chi_Q(x) |a(x) - m_Q(a)|^m \int_{3Q} f(y)\, dy.$$

We decompose $C_2[f,g]$ according to Q_0: We write

$$C_{21}[f,g](x) \equiv 3^n g(x) \sum_{\nu \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_\nu,\, Q \supset Q_0} \ell(Q)^\alpha \chi_Q(x) |a(x) - m_Q(a)|^m m_{3Q}(f)$$

$$C_{22}[f,g](x) \equiv 3^n g(x) \sum_{\nu \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_\nu,\, Q \subsetneq Q_0} \ell(Q)^\alpha \chi_Q(x) |a(x) - m_Q(a)|^m m_{3Q}(f).$$

Note that $C_2[f,g] = C_{21}[f,g] + C_{22}[f,g]$.

Let us recall the notation of Lemma 2.3. We set

$$\mathcal{D}_0(Q_0) \equiv \{Q \in \mathcal{D}(Q_0) :\ m_{3Q}(f) \le \gamma_0 A\}$$

and

$$\mathcal{D}_{k,j}(Q_0) \equiv \left\{Q \in \mathcal{D}(Q_0) :\ Q \subset Q_{k,j},\, \gamma_0 A^k < m_{3Q}(f) \le \gamma_0 A^{k+1}\right\}.$$

Then we obtain

$$\mathcal{D}(Q_0) = \mathcal{D}_0(Q_0) \cup \bigcup_{k,j} \mathcal{D}_{k,j}(Q_0). \tag{3.1}$$

Next, we shall choose $\theta \in (1, p)$ and $s \in (r, q)$ so that

$$s\theta < q \tag{3.2}$$

and that

$$s'\theta < r. \tag{3.3}$$

Write

$$\theta' \equiv \frac{\theta}{\theta - 1}.$$

§ 3.2. Decomposition of the operator $[I_\alpha, a]^{(m)} f(x)$

We first obtain a pointwise estimate of $[I_\alpha, a]^{(m)} f(x)$; by using

$$|[I_\alpha, a]^{(m)} f(x)| \le \int_{\mathbb{R}^n} \frac{|a(x) - a(y)|^m}{|x - y|^{n-\alpha}} f(y)\, dy \tag{3.4}$$

we obtain

$$\begin{aligned} |[I_\alpha, a]^{(m)} f(x)| &\le \sum_{\nu \in \mathbb{Z}} \int_{2^{-\nu-1} < |x-y| < 2^{-\nu}} \frac{|a(x) - a(y)|^m}{|x - y|^{n-\alpha}} f(y)\, dy \\ &\le C \sum_{\nu \in \mathbb{Z}} 2^{\nu(n-\alpha)} \int_{2^{-\nu-1} < |x-y| < 2^{-\nu}} |a(x) - a(y)|^m f(y)\, dy. \end{aligned}$$

Now that $\mathcal{D}_\nu$ partitions $\mathbb{R}^n$ according to (2.2), we have

$$\begin{aligned} |[I_\alpha, a]^{(m)} f(x)| &\le C \sum_{\nu \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_\nu} 2^{\nu(n-\alpha)} \chi_Q(x) \int_{2^{-\nu-1} < |x-y| < 2^{-\nu}} |a(x) - a(y)|^m f(y)\, dy \\ &\le C \sum_{\nu \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_\nu} 2^{\nu(n-\alpha)} \chi_Q(x) \int_{B(x, 2^{-\nu})} |a(x) - a(y)|^m f(y)\, dy. \end{aligned}$$

We recall that we denote by $3Q$ the triple of a dyadic cube Q; $3Q$ is made up of 3^n dyadic cubes of equal size and the center of $3Q$ is that of Q. A geometric observation shows that $B(x, 2^{-\nu}) \subset 3Q$ if $x \in Q \in \mathcal{D}_\nu$. Consequently we obtain

$$|[I_\alpha, a]^{(m)} f(x)| \le C \sum_{\nu \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_\nu} 2^{\nu(n-\alpha)} \chi_Q(x) \int_{3Q} |a(x) - a(y)|^m f(y)\, dy.$$

Recall that $m_Q(a)$ denotes the average of a over a cube Q. Using $m_Q(a)$, we shall decompose

$$\begin{aligned} |g(x)[I_\alpha, a]^{(m)} f(x)| &\le C|g(x)| \sum_{\nu \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_\nu} 2^{\nu(n-\alpha)} \chi_Q(x) \int_{3Q} |m_Q(a) - a(y)|^m f(y)\, dy \\ &\quad + C|g(x)| \sum_{\nu \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_\nu} 2^{\nu(n-\alpha)} \chi_Q(x) |a(x) - m_Q(a)|^m \int_{3Q} f(y)\, dy \\ &= CC_1[f, g](x) + CC_2[f, g](x) \\ &= CC_1[f, g](x) + CC_{21}[f, g](x) + CC_{22}[f, g](x). \end{aligned}$$

Hence we have

$$|g(x)[I_\alpha, a]^{(m)} f(x)| \leq C(C_1[f,g](x) + C_{21}[f,g](x) + C_{22}[f,g](x)). \tag{3.5}$$

Thus, we are led to analyzing three operators $C_1[f,g]$, $C_{21}[f,g]$ and $C_{22}[f,g]$.

§ 3.3. Estimate for $C_1[f,g]$

The analysis of $C_1[f,g]$ depends on (1.2): First, we choose θ slightly larger than 1. By the John-Nirenberg inequality (see Lemma 2.2), we have

$$\begin{aligned}
&\frac{1}{|Q|}\int_{3Q} |m_Q(a) - a(y)|^m f(y)\,dy \\
&\leq \frac{1}{|Q|}\left(\int_{3Q} |m_Q(a) - a(y)|^{m\theta'}\,dy\right)^{\frac{1}{\theta'}}\left(\int_{3Q} f(y)^\theta\,dy\right)^{\frac{1}{\theta}} \\
&\leq C\,\|a\|_{\mathrm{BMO}}^m \inf_{y\in Q} M^{(\theta)} f(y).
\end{aligned} \tag{3.6}$$

Consequently, by inserting (3.6) to $C_1[f,g]$, we are led to a pointwise estimate:

$$C_1[f,g](x) \leq C\,\|a\|_{\mathrm{BMO}}^m g(x) I_\alpha[\,M[M^{(\theta)} f]\,](x).$$

If we use (1.2) and $\theta < p$, then we have

$$\|C_1[f,g]\|_{\mathcal{M}^{r_0}_r} \leq C\,\|a\|_{\mathrm{BMO}}^m \|f\|_{\mathcal{M}^{p_0}_p} \|g\|_{\mathcal{M}^{q_0}_q}. \tag{3.7}$$

This is an estimate we are looking for.

§ 3.4. Estimate for $C_{21}[f,g]$

We aim to estimate

$$\mathrm{II}_1 = |Q_0|^{\frac{1}{r_0}-\frac{1}{r}}\left(\int_{Q_0} |C_{21}[f,g](x)|^r\,dx\right)^{\frac{1}{r}}.$$

The estimate for $C_{21}[f,g](x)$ is simple. Let us denote by Q_k the unique cube containing Q_0 and satisfying $|Q_k| = 2^{kn}|Q_0|$. By the Hölder inequality, if we set $\nu \equiv -\log_2 |Q_k|^{\frac{1}{n}}$, we have

$$
\begin{aligned}
\mathrm{II}_1 &= |Q_0|^{\frac{1}{r_0}-\frac{1}{r}} \left(\int_{Q_0} \left\{ 2^{\nu(n-\alpha)} g(x)\chi_{Q_k}(x)|a(x)-m_{Q_k}(a)|^m \int_{3Q_k} f(y)\,dy \right\}^r dx \right)^{\frac{1}{r}} \\
&= \int_{3Q_k} f(y)\,dy \times |Q_0|^{\frac{1}{r_0}-\frac{1}{r}} \left(\int_{Q_0} \left\{ 2^{\nu(n-\alpha)} g(x)|a(x)-m_{Q_k}(a)|^m \right\}^r dx \right)^{\frac{1}{r}} \\
&\le C \int_{3Q_k} f(y)\,dy \times |Q_0|^{\frac{1}{r_0}-\frac{1}{r}} \left(\int_{Q_0} \left\{ 2^{\nu(n-\alpha)} g(x)|a(x)-m_{Q_0}(a)|^m \right\}^r dx \right)^{\frac{1}{r}} \\
&\quad + C\,|m_{Q_0}(a)-m_{Q_k}(a)|^m \int_{3Q_k} f(y)\,dy \times |Q_0|^{\frac{1}{r_0}-\frac{1}{r}} \left(\int_{Q_0} \left\{ 2^{\nu(n-\alpha)} g(x) \right\}^r dx \right)^{\frac{1}{r}} \\
&\le C\,2^{-k\left(\frac{n}{p_0}-\alpha\right)}(1+k^m)\|f\|_{\mathcal{M}^{p_0}_p}\|g\|_{\mathcal{M}^{q_0}_q}\|a\|^m_{\mathrm{BMO}}.
\end{aligned}
$$

Here for the last inequality we have invoked the fact that, for every $k \in \mathbb{Z}$, we have $|m_Q(a)-m_R(a)| \le C\,\|a\|_{\mathrm{BMO}}$, if $Q \in \mathcal{D}_k$ is engulfed by $R \in \mathcal{D}_{k-1}$. Assuming that $p_0 < \frac{n}{\alpha}$, we see that this estimate is summable over $k \in \mathbb{N}\cup\{0\}$. Hence, we have

$$
\mathrm{II}_1 = |Q_0|^{\frac{1}{r_0}-\frac{1}{r}} \left(\int_{Q_0} |C_{21}[f,g](x)|^r\,dx \right)^{\frac{1}{r}} \le C\|f\|_{\mathcal{M}^{p_0}_p}\|g\|_{\mathcal{M}^{q_0}_q}\|a\|^m_{\mathrm{BMO}}. \tag{3.8}
$$

Thus, the control of $C_{21}[f,g](x)$ is valid.

§ 3.5. Estimate for $C_{22}[f,g]$

The heart of the matter, as is the case with the operator $g\cdot[I_\alpha,a]^{(m)}f$, is to estimate $C_{22}[f,g]$.

Finally, we aim to estimate

$$
\mathrm{II}_2 = |Q_0|^{\frac{1}{r_0}-\frac{1}{r}} \left(\int_{Q_0} |C_{22}[f,g](x)|^r\,dx \right)^{\frac{1}{r}}.
$$

To investigate $C_{22}[f,g](x)$, we linearize the estimate: Choose a positive element $w \in L^{r'}(Q_0)$ with norm 1 so that

$$
\left(\int_{Q_0} |C_{22}[f,g](x)|^r\,dx \right)^{\frac{1}{r}} \le 2\int_{Q_0} C_{22}[f,g](x)w(x)\,dx. \tag{3.9}
$$

Using (3.1), we decompose the right-hand side of (3.9) as follows:

$$
\begin{aligned}
&\int_{Q_0} C_{22}[f,g](x)w(x)\,dx \\
&= \sum_{Q\in\mathcal{D}_0(Q_0)} \ell(Q)^\alpha \left(\int_Q |a(x)-m_Q(a)|^m g(x)w(x)\,dx \right) m_{3Q}(f) \\
&\quad + \sum_{k,j} \sum_{Q\in\mathcal{D}_{k,j}(Q_0)} \ell(Q)^\alpha \left(\int_Q |a(x)-m_Q(a)|^m g(x)w(x)\,dx \right) m_{3Q}(f).
\end{aligned} \tag{3.10}
$$

From the Hölder inequality and Lemma 2.2, we have

$$\begin{aligned}
&\int_Q |a(x)-m_Q(a)|^m g(x)w(x)\,dx\\
&\le \left(\int_Q |a(x)-m_Q(a)|^{m\theta'}\,dx\right)^{\frac{1}{\theta'}}\left(\int_Q (g(x)w(x))^\theta\,dx\right)^{\frac{1}{\theta}}\\
&=|Q|^{\frac{1}{\theta'}}\left(\frac{1}{|Q|}\int_Q |a(x)-m_Q(a)|^{m\theta'}\,dx\right)^{\frac{1}{\theta'}}\left(\int_Q (g(x)w(x))^\theta\,dx\right)^{\frac{1}{\theta}}\\
&\le C\|a\|_{\mathrm{BMO}}^m|Q|\left(\frac{1}{|Q|}\int_Q (g(x)w(x))^\theta\,dx\right)^{\frac{1}{\theta}}\le C\|a\|_{\mathrm{BMO}}^m\int_Q M^{(\theta)}[gw](x)\,dx.
\end{aligned}$$

This implies

$$\begin{aligned}
&\sum_{Q\in\mathcal{D}_{k,j}(Q_0)}\ell(Q)^\alpha\left(\int_Q |a(x)-m_Q(a)|^m g(x)w(x)\,dx\right)m_{3Q}(f)\\
&\le C\|a\|_{\mathrm{BMO}}^m\sum_{Q\in\mathcal{D}_{k,j}(Q_0)}\ell(Q)^\alpha\left(\int_Q M^{(\theta)}[gw](x)\,dx\right)m_{3Q}(f). \qquad (3.11)
\end{aligned}$$

It follows from the definition of $\mathcal{D}_{k,j}(Q_0)$, $\alpha/n<1$, support condition and (2.3) that

$$\begin{aligned}
&\sum_{Q\in\mathcal{D}_{k,j}(Q_0)}\ell(Q)^\alpha\left(\int_Q M^{(\theta)}[gw](x)\,dx\right)m_{3Q}(f)\\
&=\sum_{Q\in\mathcal{D}_{k,j}(Q_0)}|Q|^{\frac{\alpha}{n}}\left(\int_Q M^{(\theta)}[gw](x)\,dx\right)m_{3Q}(f)\\
&\le C|Q_{k,j}|^{\frac{\alpha}{n}}\left(\int_{Q_{k,j}} M^{(\theta)}[gw](x)\,dx\right)m_{3Q_{k,j}}(f)\\
&=C|Q_{k,j}|^{\frac{\alpha}{n}}m_{Q_{k,j}}(M^{(\theta)}[gw])m_{3Q_{k,j}}(f)|Q_{k,j}|\\
&\le C|Q_{k,j}|^{\frac{\alpha}{n}}m_{Q_{k,j}}(M^{(\theta)}[gw])m_{3Q_{k,j}}(f)|E_{k,j}|. \qquad (3.12)
\end{aligned}$$

The Hölder inequality gives

$$M^{(\theta)}[gw]\le M^{(s'\theta)}w\cdot M^{(s\theta)}g$$

and hence

$$m_{Q_{k,j}}(M^{(\theta)}[gw])\le\left(m_{Q_{k,j}}((M^{(s'\theta)}w)^{q'})\right)^{\frac{1}{q'}}\left(m_{Q_{k,j}}((M^{(s\theta)}g)^{q})\right)^{\frac{1}{q}}. \qquad (3.13)$$

Lemma 2.1 with $\alpha=0$, the Morrey boundedness of M enables us that, noticing (3.2),

$$|Q_{k,j}|^{\frac{1}{q_0}}\left(m_{Q_{k,j}}((M^{(s\theta)}g)^{q})\right)^{\frac{1}{q}}\le C\|g\|_{\mathcal{M}^{q_0}_q}. \qquad (3.14)$$

(3.11)–(3.14) yield

$$\sum_{Q\in\mathcal{D}_{k,j}(Q_0)} \ell(Q)^{\alpha}\left(\int_Q |a(x)-m_Q(a)|^m g(x)w(x)\,dx\right) m_{3Q}(f)$$

$$\leq C\|a\|_{\mathrm{BMO}}^m \|g\|_{\mathcal{M}_q^{q_0}} |Q_{k,j}|^{\frac{\alpha}{n}-\frac{1}{q_0}} \left(m_{Q_{k,j}}((M^{(s'\theta)}w)^{q'})\right)^{\frac{1}{q'}} m_{3Q_{k,j}}(f)|E_{k,j}|. \tag{3.15}$$

Similarly, we see that

$$\sum_{Q\in\mathcal{D}_{0}(Q_0)} \ell(Q)^{\alpha}\left(\int_Q |a(x)-m_Q(a)|^m g(x)w(x)\,dx\right) m_{3Q}(f)$$

$$\leq C\|a\|_{\mathrm{BMO}}^m \|g\|_{\mathcal{M}_q^{q_0}} |Q_0|^{\frac{\alpha}{n}-\frac{1}{q_0}} \left(m_{Q_0}((M^{(s'\theta)}w)^{q'})\right)^{\frac{1}{q'}} m_{3Q_0}(f)|E_0|. \tag{3.16}$$

Gathering all factors (3.10), (3.15) and (3.16) and using the fact that $\{E_0\}\cup\{E_{k,j}\}$ forms a disjoint family of sets, which decomposes Q_0, we see that the right-hand side of (3.9) is bounded by constant times

$$\begin{aligned}
&\|a\|_{\mathrm{BMO}}^m \|g\|_{\mathcal{M}_q^{q_0}} \int_{Q_0} M^{(q')}[M^{(s'\theta)}w](x) M_{\alpha-(n/q_0)}f(x)\,dx \\
&\leq C\|a\|_{\mathrm{BMO}}^m \|g\|_{\mathcal{M}_q^{q_0}} \left(\int_{Q_0} M^{(q')}[M^{(s'\theta)}w](x)^{r'}\,dx\right)^{\frac{1}{r'}} \left(\int_{Q_0} M_{\alpha-(n/q_0)}f(x)^{r}\,dx\right)^{\frac{1}{r}}.
\end{aligned}$$

Recall that θ is slightly larger than 1. Since $\dfrac{r'}{q'}>1$ and $\dfrac{r'}{s'\theta}>1$, we have

$$\left(\int_{Q_0} M^{(q')}[M^{(s'\theta)}w](x)^{r'}\,dx\right)^{\frac{1}{r'}} \leq C.$$

Thus,

$$\begin{aligned}
&|Q_0|^{\frac{1}{r_0}-\frac{1}{r}} \left(\int_{Q_0} |C_{22}[f,g](x)|^r\,dx\right)^{\frac{1}{r}} \\
&\leq C\|a\|_{\mathrm{BMO}}^m \|g\|_{\mathcal{M}_q^{q_0}} |Q_0|^{\frac{1}{r_0}-\frac{1}{r}} \left(\int_{Q_0} M_{\alpha-(n/q_0)}f(x)^{r}\,dx\right)^{\frac{1}{r}}
\end{aligned}$$

Finally, Lemma 2.1 gives

$$\mathrm{II}_2 = |Q_0|^{\frac{1}{r_0}-\frac{1}{r}} \left(\int_{Q_0} |C_{22}[f,g](x)|^r\,dx\right)^{\frac{1}{r}} \leq C\|a\|_{\mathrm{BMO}}^m \|g\|_{\mathcal{M}_q^{q_0}} \|f\|_{\mathcal{M}_p^{p_0}}. \tag{3.17}$$

From (3.5), (3.7), (3.8) and (3.17), we conclude the proof of Theorem 1.1.

§ 4. Proof of Proposition 1.5

We recall the following estimate by Hedberg:

Lemma 4.1. *Suppose the parameters α, p, s satisfy*

$$0 < \alpha < n,\ 1 < p < s < \infty,\ \frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}.$$

Let f be a positive measurable function. Then

$$I_\alpha f(x) \le C M f(x)^{\frac{p}{s}} \|f\|_{\mathcal{M}^p_1}^{1-\frac{p}{s}}.$$

Proof. The proof is well-known but for the sake of completeness, we supply it. By the Fubini theorem, we obtain

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \left(\int_{|x-y|}^\infty \frac{n-\alpha}{\ell^{n-\alpha+1}}\, d\ell \right) f(y)\, dy = (n-\alpha) \int_0^\infty \left(\frac{1}{\ell^{n-\alpha+1}} \int_{B(x,\ell)} f(y)\, dy \right) d\ell.$$

If we insert the estimates

$$\frac{1}{\ell^n} \int_{B(x,\ell)} f(y)\, dy \le C M f(x),\ \frac{1}{\ell^{n-n/p}} \int_{B(x,\ell)} f(y)\, dy \le C \|f\|_{\mathcal{M}^p_1},$$

then we obtain

$$I_\alpha f(x) \le C \int_0^\infty \min(\ell^{\alpha-1} M f(x), \ell^{\alpha-n/p-1} \|f\|_{\mathcal{M}^p_1})\, d\ell = C M f(x)^{\frac{p}{s}} \|f\|_{\mathcal{M}^p_1}^{1-\frac{p}{s}}.$$

□

Corollary 4.2. *Let $1 < t \le s < \infty$ and $1 < q \le p < \infty$ satisfy*

$$\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n},\ \frac{t}{s} = \frac{q}{p}.$$

Then

$$\|I_\alpha f\|_{\mathcal{M}^s_t} \le C \|f\|_{\mathcal{M}^p_q}^{\frac{p}{s}} \|f\|_{\mathcal{M}^p_1}^{1-\frac{p}{s}}.$$

We prove Proposition 1.5.

Proof. Since I_α is known to be injective, if I_α were surjective, then by virtue of the open mapping theorem, $I_\alpha : \mathcal{M}^p_q(\mathbb{R}^n) \to \mathcal{M}^s_t(\mathbb{R}^n)$ would be isomorphic. So, we would have a constant C such that

$$C^{-1} \|f\|_{\mathcal{M}^p_q} \le \|I_\alpha f\|_{\mathcal{M}^s_t} \le C \|f\|_{\mathcal{M}^p_q}.$$

If we combine this with Corollary 4.2, then we obtain

$$C^{-1} \|f\|_{\mathcal{M}^p_q} \le C \|f\|_{\mathcal{M}^p_q}^{\frac{p}{s}} \|f\|_{\mathcal{M}^p_1}^{1-\frac{p}{s}}.$$

This implies that $\mathcal{M}_1^p(\mathbb{R}^n) \subset \mathcal{M}_q^p(\mathbb{R}^n)$. Since $\mathcal{M}_1^p(\mathbb{R}^n) \supset \mathcal{M}_q^p(\mathbb{R}^n)$ is known to hold, it follows that $\mathcal{M}_1^p(\mathbb{R}^n) = \mathcal{M}_q^p(\mathbb{R}^n)$ with norm equivalence. This is a contradiction to Example 1.7. □

§ 5. Appendix: The proof of Lemma 1.9

In what follows we prove Lemma 1.9. When $m = 2, 3$, this is true as the following identities show:

$$x_1x_2 = \frac{1}{4}(x_1+x_2)^2 - \frac{1}{4}(x_1-x_2)^2.$$

$$\begin{aligned} x_1x_2x_3 = {} & \frac{1}{24}(x_1+x_2+x_3)^3 \\ & - \frac{1}{24}(x_1+x_2-x_3)^3 - \frac{1}{24}(x_1-x_2+x_3)^3 - \frac{1}{24}(-x_1+x_2+x_3)^3. \end{aligned}$$

Suppose that Lemma 1.9 is correct for $m = m_0$. Then by the induction assumption, it suffices to prove that $\xi^{m_0}x_{m_0+1}$ is in the linear span of $\mathcal{V}_{m_0+1}$, where $\xi \equiv a_1x_1 + a_2x_2 + \cdots + a_{m_0}x_{m_0}$.

Consider an m_0+2 matrix

$$A \equiv \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 3 & \dots & m_0+2 \\ 1^2 & 2^2 & 3^2 & \dots & (m_0+2)^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1^{m_0+1} & 2^{m_0+1} & 3^{m_0+1} & \dots & (m_0+2)^{m_0+1} \end{pmatrix}.$$

Then, by virtue of the Vandermonde determinant, A becomes invertible. We now set

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_{m_0+2} \end{pmatrix} \equiv A^{-1} \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

We notice that

$$\sum_{j=1}^{m_0+2} b_j j^k = \begin{cases} 1 & (k=1), \\ 0 & (k = 0, 2, 3, \cdots m_0+1). \end{cases}$$

This implies

$$\sum_{j=1}^{m_0+2} b_j(\xi + jx_{m_0+2})^{m_0+1} = (m_0+1)\xi^{m_0}x_{m_0+2}.$$

So, Lemma 1.9 is correct for $m = m_0+1$.

Acknowledgement

The authors are thankful to the anonymous referee for his/her careful reading of the paper.

References

[1] D. R. Adams, *A note on Riesz potentials*, Duke Math. J., **42** (1975), 765–778.

[2] J. Conlon and J. Redondo, *Estimates on the solution of an elliptic equation related to Brownian motion with drift*, Rev. Mat. Iberoamericana **11** (1995), 1–65.

[3] H. Gunawan, Y. Sawano and I. Sihwaningrum, *Fractional integral operators in non-homogeneous spaces*, Bull. Aust. Math. Soc. **80** (2009), 324–334.

[4] L. Hedberg, *On certain convolution inequalities*, Proc. Amer. Math. Soc. **36** (1972), 505–510.

[5] T. Iida, E. Sato, Y. Sawano and H. Tanaka, *Weighted norm inequalities for multilinear fractional operators on Morrey spaces*, Studia Math. **205** (2011), no. 2, 139–170.

[6] W. Li, *Weighted inequalities for commutators of potential type operators*, J. Korean Math. Soc. **44** (2007), 1233–1241.

[7] P. Olsen, *Fractional integration, Morrey spaces and Schrödinger equation*, Comm. Partial Differential Equations, **20** (1995), 2005–2055.

[8] C. Pérez, *Sharp L^p-weighted Sobolev inequalities*, Ann. Inst. Fourier (Grenoble) **45** (1995), 809–824.

[9] Y. Sawano, S. Sugano and H. Tanaka, *Generalized fractional integral operators and fractional maximal operators in the framework of Morrey spaces*, Trans. Amer. Math. Soc. **363** (2011), 6481–6503.

[10] ———, *A note on generalized fractional integral operators on generalized Morrey spaces*, Bound. Value Probl. 2009, Art. ID 835865, 18 pp.

[11] ———, *Identification of the image of Morrey spaces by the fractional integral operators*, Proc. A. Razmadze Math. Inst. **149** (2009), 87–93.

[12] D. Stroock, Varadhan, and S. R. Srinivasa, *Multidimensional diffusion processes*, Grundlehren der Mathematischen Wissenschaften, **233**, Springer-Verlag, Berlin-New York, 1979. xii+338 pp.

[13] S. Sugano, *Some inequalities for generalized fractional integral operators on generalized Morrey spaces*, Math. Inequal. Appl. **14** (2011), 849–865.

[14] ———, *Some inequalities for generalized fractional integral operators on generalized Morrey spaces and their remarks*, to appear in Sci. Math. Jpn.

[15] S. Sugano and H. Tanaka, *Boundedness of fractional integral operators on generalized Morrey spaces*, Sci. Math. Jpn., **58** (2003), 531–540.

[16] H. Tanaka, *Morrey spaces and fractional operators*, J. Aust. Math. Soc., **88** (2010), 247–259.

[17] M. I. Utoyo, T. Nusantara, B. Widodo and Suhariningsih, *Fractional integral operator and Olsen inequality in the non-homogeneous classic Morrey space*, Int. J. Math. Anal. **6** (2012), no. 29-32, 1501-1511.