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Interpolation theorem for harmonic Bergman functions

By

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Abstract

In this paper, we discuss an interpolation theorem for harmonic Bergman functions on bounded smooth domains.

§ 1. Introduction

Let Ω be a bounded smooth domain in the n-dimensional Euclidean space ($n \geq 2$). For $1 \leq p < \infty$, we denote by $b^p(\Omega)$ the harmonic Bergman space on Ω, i.e., the set of all real-valued harmonic functions f on Ω such that

$$\|f\|_p := \left(\int_\Omega |f|^p dx\right)^{\frac{1}{p}} < +\infty,$$

where dx denotes the n-dimensional Lebesgue volume measure on Ω. As is well-known, $b^p(\Omega)$ is a closed subspace of $L^p = L^p(\Omega)$ and hence, $b^p(\Omega)$ is a Banach space (for example see [2]). Especially, when $p = 2$, $b^2(\Omega)$ is a Hilbert space, which has the reproducing kernel, i.e., there exists a unique symmetric function $R(\cdot,\cdot)$ on $\Omega \times \Omega$ such that for any $f \in b^2(\Omega)$ and any $x \in \Omega$,

$$f(x) = \int_\Omega R(x,y) f(y) dy.$$

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The function $R(\cdot,\cdot)$ is called the harmonic Bergman kernel of Ω. It is known that for any $1 \le p < \infty$, $f \in b^p(\Omega)$ has the following reproducing formula:

$$f(x) = \int_\Omega f(y)R(x,y)dy,$$

for any $x \in \Omega$, see Proposition 2.3 in [3].

We denote by P the corresponding integral operator, which is called the harmonic Bergman projection,

$$P\psi(x) := \int_\Omega R(x,y)\psi(y)dy$$

for $x \in \Omega$. It is known that $P : L^p(\Omega) \to b^p(\Omega)$ is bounded for $1 < p < \infty$; see Theorem 4.2 in [8].

For any $\{\lambda_i\} \subset \Omega$ and any $1 < p < \infty$, we denote by $A = A_{p,\{\lambda_i\}}$ from l^p to $b^p(\Omega)$

$$A\{a_i\}(x) = A_{p,\{\lambda_i\}}\{a_i\}(x) = \sum_{i=1}^{\infty} a_i R(x,\lambda_i) r(\lambda_i)^{(1-\frac{1}{p})n},$$

where l^p denotes the space of all sequences $\{a_i\}$ such that $\|\{a_i\}\|_{l^p} := (\sum_{i=1}^{\infty} |a_i|^p)^{\frac{1}{p}} < \infty$ and $r(x)$ denotes the distance between x and the boundary of Ω. The author obtained in [9] the following representation theorem.

Theorem 1.1. *Let $1 < p < \infty$ and let Ω be a smooth bounded domain. Then we can choose a sequence $\{\lambda_i\}$ in Ω such that the operator $A_{p,\{\lambda_i\}}$ from l^p to $b^p(\Omega)$ is bounded and onto.*

In this paper, we consider the adjoint operator and discuss the conditions these operators are onto. We denote by $V = V_{p,\{\lambda_j\}}$ from $b^p(\Omega)$ to l^p as follow:

$$V_{p,\{\lambda_i\}} f := \{r(\lambda_i)^{\frac{n}{p}} f(\lambda_i)\}.$$

We remark that the relation $A_p^* = V_q$, where q is the exponent conjugate to p, for detail see Theorem 4 in [9].

A main theorem in this paper is the following.

Theorem 1.2. *Let $1 < p < \infty$. Then, we can choose a sequence $\{\lambda_i\}$ in Ω such that $V : b^p(\Omega) \to l^p$ is bounded and onto.*

The above theorem is called an interpolation theorem. A sequence $\{\lambda_i\}$ given in the above theorem is called interpolating sequence.

Interpolations theorem were studied on the various settings. In [1], E. Amar studied the holomorphic Bergman spaces on the unit disc in $\mathbb{C}^n$, and obtained an interpolation theorem for the holomorphic Bergman functions. In [6], B. R. Choe and H. Yi studied the harmonic Bergman spaces on the upper-half space in $\mathbb{R}^n$, and proved representation theorems and interpolation theorems for harmonic Bergman functions. In this paper, we achieved to prove an interpolation theorem for the harmonic Bergman spaces over bounded smooth domains.

We often abbreviate inessential constants involved in inequalities by writing $X \lesssim Y$, if there exists an absolute constant $C > 0$ such that $X \leq CY$. In the following, we fix $p \in (1, \infty)$ and denote q is the exponent conjugate to p, i.e., it is satisfied that $\frac{1}{p} + \frac{1}{q} = 1$.

§ 2. The harmonic Bergman kernels

In this section, we recall the estimates for the harmonic Bergman kernels. First, we remark the estimates for the harmonic Bergman kernels introduced in [4] and [8]. The following estimates for the harmonic Bergman kernels is shown in [8].

Lemma 2.1 (Theorem 1.1 in [8]). *Let α, β be multi-indices.*

(1) *There exists a constant $C > 0$ such that*

$$|D_x^\alpha D_y^\beta R(x, y)| \leq \frac{C}{d(x, y)^{n+|\alpha|+|\beta|}}$$

for every $x, y \in \Omega$, where $d(x, y) = r(x) + r(y) + |x - y|$.

(2) *There exists a constant $C > 0$ such that*

$$R(x, x) \geq \frac{C}{r(x)^n}$$

for every $x \in \Omega$.

From above lemma, we have easily the following corollaries. It is shown in [4].

Corollary 2.2. *There exist a constant $\delta > 0$ and constants $C_1 > 0$ and $C_2 > 0$ such that*

$$C_1 r(x)^{-n} \leq R(x, y) \leq C_2 r(x)^{-n}$$

for any $x \in \Omega$ and any $y \in E_\delta(x)$, where

$$E_\delta(x) := B(x, \delta r(x)) = \{y \in \Omega : |y - x| < \delta r(x)\}$$

Corollary 2.3. *Let $1 < p < \infty$. There exist constants $C_1 > 0$ and $C_2 > 0$ such that*

$$C_1 r(x)^{(1-\frac{1}{p})n} \leq \|R(x, \cdot)\|_{b^p} \leq C_2 r(x)^{(1-\frac{1}{p})n}$$

for every $x \in \Omega$.

We prepare a tool for calculating integration.

Lemma 2.4 (Lemma 4.1 in [8]). *Let $s > -1$ and $t < 1$. If $s + t > 0$,*

$$\int_\Omega \frac{dy}{d(x,y)^{n+s} r(y)^t} \lesssim \frac{1}{r(x)^{s+t}}$$

for every $x \in \Omega$.

Finally, by using Lemma 2.4, we immediately have the following corollary.

Corollary 2.5. *The harmonic Bergman projection $P : L^p(\Omega) \to b^p(\Omega)$ is bounded for $1 < p < \infty$.*

§ 3. Proof of Theorem 1.2

In this section, we give the proof of Theorem 1.2.

First, we discuss some properties of sequences $\{\lambda_i\} \subset \Omega$ in order to define the operators $V_{p,\{\lambda_i\}}$ and $A_{p,\{\lambda_i\}}$. We define separated sequences.

Definition 3.1 (separated sequence). For $\delta > 0$, we call a sequence $\{\lambda_i\}$ in Ω a δ-separated sequence if $E_\delta(\lambda_i) \cap E_\delta(\lambda_j) = \emptyset$ for $i \neq j$.

When a sequence $\{\lambda_i\}$ in Ω is δ-separated, then we can check the well-definedness of operators $V_{p,\{\lambda_i\}}$ and $A_{p,\{\lambda_i\}}$. These operators are important in the argument in [9].

Lemma 3.2. *Let a sequence $\{\lambda_i\}$ be δ-separated. Then, for any $1 \leq p < \infty$, a operator $V_{p,\{\lambda_i\}} : b^p(\Omega) \to l^p$ is bounded.*

Proof. For any $f \in b^p(\Omega)$, by using sub-mean value property for a subharmonic function $|f|^p$ and the definition of δ-separated, we have

$$\begin{aligned}
\|Vf\|_{l^p}^p &= \sum_{i=1}^\infty |f(\lambda_i)|^p r(\lambda_i)^n \\
&\leq \sum_{i=1}^\infty \frac{1}{|E_\delta(\lambda_i)|} \int_{E_\delta(\lambda_i)} |f(x)|^p dx r(\lambda_i)^n \\
&= \sum_{i=1}^\infty \frac{1}{\delta^n |B(0,1)|} \int_{E_\delta(\lambda_i)} |f(x)|^p dx \\
&\leq \frac{1}{\delta^n |B(0,1)|} \|f\|_{b^p}^p.
\end{aligned}$$

□

We remark that the above lemma implies the boundedness of V_p for $p \geq 1$. By Lemma 14 in [9], the boundedness of V implies that of A for $1 < p < \infty$. Hence, we obtain the following lemma.

Lemma 3.3. *Let $1 < p < \infty$. If a sequence $\{\lambda_i\}$ is δ-separated, then $A_{p,\{\lambda_i\}} : l^p \to b^p(\Omega)$ is bounded.*

Second, according to F. W. Gehring and B. P. Palka [7], we define the quasi-hyperbolic metric, which plays important role in the proof of our interpolation theorem.

Definition 3.4 (Quasi-hyperbolic metric)**.** Let Ω be a smooth bounded domain in $\mathbb{R}^n$. We define the quasi-hyperbolic metric $\rho(x, y)$ as the following:

$$\rho(x, y) := \inf_{\gamma \in \Gamma_{x,y}} \int_\gamma \frac{1}{r(z)} ds(z)$$

for $x, y \in \Omega$, where ds denotes the line element in Euclidean space and $\Gamma_{x,y}$ is the set of all γ which are C^∞-curves in Ω with the initial point x and the end point y.

We can investigate an intersection of sets $\{E_\delta(\lambda_i)\}$ by using the quasi-hyperbolic metric.

Lemma 3.5. *For any $\rho_0 > 0$, there exists a constant $\delta_0 > 0$ such that $E_{\delta_0}(x) \cap E_{\delta_0}(y) = \emptyset$ for any $x, y \in \Omega$ with $\rho_0 < \rho(x, y)$.*

Proof. For any $x \in \Omega$ and $r > 0$, we denote by $D_r(x)$ the quasi-hyperbolic ball

$$D_r(x) = \{y \in \mathbb{R}^n : \rho(x, y) < r\}.$$

It is sufficient to show there exists a constant δ_0 such that $E_{\delta_0}(x) \subset D_{\frac{\rho_0}{2}}(x)$ for any $x \in \Omega$. We denote by $\gamma_{x,y}$ the line with an initial point x and an end point y. For any $y \in E_\delta(x)$, by remarking $\gamma_{x,y} \subset E_\delta(x)$, we have

$$\rho(x, y) \leq \int_{\gamma_{x,y}} \frac{1}{r(z)} ds(z) \leq |x - y| \frac{1}{(1 - \delta) r(x)} \leq \frac{\delta}{1 - \delta}.$$

Therefore, when we put δ_0 satisfying $\delta_0 < \frac{\rho_0}{2+\rho_0}$, then we have $\rho(x, y) < \frac{\rho_0}{2}$ for any $y \in E_{\delta_0}(x)$. This completes the proof. $\square$

By Lemmas 3.2, 3.3 and 3.5, the following corollary is immediately shown.

Corollary 3.6. *Let a sequence $\{\lambda_i\}$ be in Ω. If there exists a constant $\rho_0 > 0$ such that $\rho(\lambda_i, \lambda_j) > \rho_0$ for $i \neq j$, then the operator $V_{p,\{\lambda_i\}}$ and $A_{p,\{\lambda_i\}}$ are bounded for any $1 < p < \infty$.*

The following proposition is necessary in order to control a sequence $\{\lambda_i\}$ in Ω, for detail see [7].

Proposition 3.7 (Quasi-hyperbolic metric)**.** *Let Ω be a smooth bounded domain. For any $x, y \in \Omega$, quasi-hyperbolic metric $\rho(x,y)$ has the following properties:*

$$\rho(x,y) \geq \log\Big(\frac{|x-y|}{\min\{r(x), r(y)\}} + 1\Big),$$

$$\rho(x,y) \geq \Big|\log\frac{r(x)}{r(y)}\Big|,$$

and

$$\rho(x,y) \leq C_3 \log\Big(\frac{|x-y|}{\min\{r(x), r(y)\}} + 1\Big) + C_4 \tag{3.1}$$

for some positive C_3 and C_4.

As the end of preparations to prove our interpolation theorem, we rewrite the above proposition.

Lemma 3.8. *There exist constants $C_3 > 0$ and $C_4 > 0$ such that*

$$\frac{1}{d(x,y)} \leq \frac{e^{-(\frac{\rho(x,y)-C_3}{C_4})}}{\min\{r(x), r(y)\}} \tag{3.2}$$

for any $x, y \in \Omega$.

Proof. By the inequality (3.1), there exist constants $C_3, C_4 > 0$ such that

$$\rho(x,y) \leq C_3 \log\Big(\frac{|x-y| + \min\{r(x), r(y)\}}{\min\{r(x), r(y)\}}\Big) + C_4 \leq C_3 \log\Big(\frac{d(x,y)}{\min\{r(x), r(y)\}}\Big) + C_4$$

for any $x, y \in \Omega$. This immediately implies the inequality (3.2). $\square$

Now, we can show Theorem 1.2.

Proof of interpolation theorem. We consider a sequence $\{\lambda_i\}$ satisfying that there exists $\rho > 0$ such that $\rho(\lambda_i, \lambda_j) > \rho$ for $i \neq j$. We take a ρ_0 in Corollary 3.6. In the following argument, we only consider the range of ρ is (ρ_0, ∞) and fix a constant $\delta > 0$ such that $E_\delta(x) \cap E_\delta(y) = \emptyset$ for any $x, y \in \Omega$ with $\rho(x,y) > \rho_0$. And we consider the following operators:

$$A\{a_i\}(x) := \sum_{i=1}^{\infty} a_i R(x, \lambda_i) r(\lambda_i)^{(1-\frac{1}{p})n}, \tag{3.3}$$

$$Vf := \{f(\lambda_i)r(\lambda_i)^{\frac{n}{p}}\}_i,$$

and

$$W\{a_i\} = W_{p,\{\lambda_i\}}\{a_i\} := V \circ A\{a_i\} = \{r(\lambda_j)^{\frac{n}{p}} \sum_{i=1}^{\infty} a_i R(\lambda_j, \lambda_i) r(\lambda_i)^{(1-\frac{1}{p})n}\}_j.$$

By Corollary 3.6, we have A and V are bounded. From Theorem 1.1, it is sufficient to show that there exists a number $\rho > 0$ such that W is bijective. To analyze W, we write $W = D + E$ where D is the diagonal part and E is the remainder, i.e.,

$$D\{a_i\} := \{a_j R(\lambda_j, \lambda_j) r(\lambda_j)^n\}_j$$

and

$$E\{a_i\} := \{r(\lambda_j)^{\frac{n}{p}} \sum_{i \neq j} a_i R(\lambda_j, \lambda_i) r(\lambda_i)^{(1-\frac{1}{p})n}\}_j.$$

First, we calculate the norm of D. By Corollary 2.2, there exist $C_1 > 0$ and $C_2 > 0$ such that for any $x \in \Omega$,

$$0 < C_1 \leq R(x, x) r(x)^n \leq C_2.$$

Hence, we have

$$0 < \frac{1}{C_2} \leq \|D^{-1}\| \leq \frac{1}{C_1}. \tag{3.4}$$

By a fundamental discussion, we should only show

$$\|E\| < \frac{1}{\|D^{-1}\|}. \tag{3.5}$$

Therefore, we calculate the norm of E. Before calculating the norm of E, we remark the following inequalities:

(3.6)

$$\Big(\sum_{i \neq j} a_i R(\lambda_j, \lambda_i) r(\lambda_i)^{\frac{n}{q}}\Big)^p \leq \sum_{i \neq j} |a_i|^p r(\lambda_i)^{\frac{1}{q}} |R(\lambda_j, \lambda_i)| \Big(\sum_{i \neq j} r(\lambda_i)^{n-\frac{1}{p}} |R(\lambda_j, \lambda_i)|\Big)^{\frac{p}{q}}.$$

And

$$\begin{aligned}
\sum_{i \neq j} r(\lambda_i)^{n-\frac{1}{p}} |R(\lambda_j, \lambda_i)| &\leq \delta^{-n} \sum_{i \neq j} r(\lambda_i)^{-\frac{1}{p}} \big(\delta r(\lambda_i)\big)^n |R(\lambda_j, \lambda_i)| \\
&\lesssim \delta^{-n} \sum_{i \neq j} \int_{E_\delta(\lambda_i)} r(\lambda_i)^{-\frac{1}{p}} |R(\lambda_j, \lambda_i)| dy \\
&\lesssim \delta^{-n} \sum_{i \neq j} \int_{E_\delta(\lambda_j)} \frac{r(y)^{-\frac{1}{p}}}{d(\lambda_j, y)^n} dy \\
&\leq \delta^{-n} \int_{\Omega} \frac{r(y)^{-\frac{1}{p}}}{d(\lambda_j, y)} dy \lesssim \delta^{-n} r(\lambda_j)^{-\frac{1}{p}}. \qquad (3.7)
\end{aligned}$$

By (3.6) and (3.7), we have

$$\left(\sum_{i\neq j}|a_i||R(\lambda_j,\lambda_i)|r(\lambda_i)^{\frac{n}{q}}\right)^p \leq C\delta^{-n(p-1)}r(\lambda_j)^{-\frac{1}{q}}\sum_{i\neq j}|a_i|^p r(\lambda_i)^{\frac{1}{q}}|R(\lambda_j,\lambda_i)| \tag{3.8}$$

By (3.8), we have

$$\begin{aligned}\|E\{a_i\}\|_{l^p} &= \Big(\sum_{j=1}^{\infty}\Big|r(\lambda_j)^{\frac{n}{p}}\sum_{i\neq j}a_i R(\lambda_j,\lambda_i)r(\lambda_i)^{\frac{n}{q}}\Big|^p\Big)^{\frac{1}{p}} \\ &\leq C\delta^{\frac{-n(p-1)}{p}}\Big(\sum_{j=1}^{\infty}r(\lambda_j)^n r(\lambda_j)^{-\frac{1}{q}}\sum_{i\neq j}|a_i|^p r(\lambda_i)^{\frac{1}{q}}|R(\lambda_j,\lambda_i)|\Big)^{\frac{1}{p}} \\ &= \delta^{\frac{-n(p-1)}{p}}\Big(\sum_{i=1}^{\infty}|a_i|^p r(\lambda_i)^{\frac{1}{q}}\sum_{j\neq i}r(\lambda_j)^{n-\frac{1}{q}}|R(\lambda_j,\lambda_i)|\Big)^{\frac{1}{p}}.\end{aligned}$$

Hence, we focus the inside of summation with respect to i, we have

$$\begin{aligned}r(\lambda_i)^{\frac{1}{q}}\sum_{j\neq i}r(\lambda_j)^{n-\frac{1}{q}}|R(\lambda_j,\lambda_i)| &\lesssim r(\lambda_i)^{\frac{1}{q}}\sum_{j\neq i}r(\lambda_j)^{-\frac{1}{q}}\int_{E_\delta(\lambda_j)}|R(z,\lambda_i)|dz \\ &\lesssim \sum_{j\neq i}\int_{E_\delta(\lambda_j)}\frac{r(\lambda_i)^{\frac{1}{q}}r(z)^{-\frac{1}{q}}}{d(z,\lambda_i)^n}dz \\ &\leq \int_{\Omega\setminus E_\delta(\lambda_i)}\frac{r(\lambda_i)^{\frac{1}{q}}r(z)^{-\frac{1}{q}}}{d(z,\lambda_i)^n}dz.\end{aligned}$$

By using Lemma 3.8, for any $0<\varepsilon<1$ we have

$$\int_{\Omega\setminus E_\delta(\lambda_i)}\frac{r(\lambda_i)^{\frac{1}{q}}r(z)^{-\frac{1}{q}}}{d(z,\lambda_i)^n}dz \leq \int_{\Omega\setminus E_\delta(\lambda_i)}\frac{r(\lambda_i)^{\frac{1}{q}}r(z)^{-\frac{1}{q}}}{d(z,\lambda_i)^{n-\varepsilon}}\frac{e^{-\varepsilon(\frac{\rho(\lambda_i,z)-C_3}{C_4})}}{\min\{r(\lambda_i),r(z)\}^{\varepsilon}}dz$$

for some constants $C_3, C_4>0$. We put

$$\tau := e^{\frac{\rho-C_3}{C_4}}.$$

If we assume that $\{\lambda_i\}$ has the following property

$$\rho(z,\lambda_i)>\rho \text{ for any } z\in E_\delta(\lambda_j) \text{ whenever } i\neq j,$$

then from Lemma 2.4 we have

$$\int_{\Omega\setminus E_\delta(\lambda_i)}\frac{r(\lambda_i)^{\frac{1}{q}}r(z)^{-\frac{1}{q}}}{d(z,\lambda_i)^{n-\varepsilon}}\frac{e^{-\varepsilon(\frac{\rho(\lambda_i,z)-C_3}{C_4})}}{r(\lambda_i)^{\varepsilon}}dz \lesssim \tau^{-\varepsilon}$$

and

$$\int_{\Omega\setminus E_\delta(\lambda_i)}\frac{r(\lambda_i)^{\frac{1}{q}}r(z)^{-\frac{1}{q}}}{d(z,\lambda_i)^{n-\varepsilon}}\frac{e^{-\varepsilon(\frac{\rho(\lambda_i,z)-C_3}{C_4})}}{r(z)^{\varepsilon}}dz \lesssim \tau^{-\varepsilon}.$$

Hence, we have

$$\|E\{a_i\}\|_{l^p} \le C_5 \delta^{\frac{-n(p-1)}{p}} \tau^{-\frac{\epsilon}{p}} \|\{a_i\}_i\|_{l^p}$$

for some positive constant C_5. Because δ is fixed and $\tau^{-\frac{\epsilon}{p}} \to 0$ as $\rho \to \infty$, we can control the operator norm $\|E\|$. In fact, if we put

$$\rho_1 > \frac{C_4}{\varepsilon}\Big(\log C_5 - \log C_1 - \frac{n(p-1)}{p} \log \delta\Big) + C_3,$$

by direct calculation, we have

$$C_5 \delta^{\frac{-n(p-1)}{p}} \tau^{-\epsilon} < C_1.$$

Therefore, for a sequence $\{\lambda_i\}_i$ satisfying $\rho(\lambda_i, \lambda_j) > \rho_1$ for any $i \neq j$, by the inequality (3.4), we have

$$\|E\| < \frac{1}{\|D^{-1}\|}.$$

Because D is invertible and $D + E = W$, we have $\|I - WD^{-1}\| = \|ED^{-1}\|$. By the inequality (3.5), we have $\|I - WD^{-1}\| < 1$. This implies that WD^{-1} and W are invertible. Hence, we obtain that V is onto. This is the end of proof. □

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