Periodicity on poly-Euler numbers and Vandiver type congruence for Euler numbers

By

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Abstract

Poly-Euler numbers are introduced as a generalization of classical Euler numbers. In this article, a periodic property for poly-Euler numbers and Vandiver type congruence for Euler numbers are discussed.

§1. Introduction

For every integer \( k \), we define \textit{poly-Euler numbers} \( E_n^{(k)} ( n = 0, 1, 2, \ldots ) \) by

\[
\frac{\mathrm{Li}_k(1 - e^{-4t})}{4t \cosh t} = \sum_{n=0}^{\infty} \frac{E_n^{(k)}}{n!} t^n,
\]

where

\[
\mathrm{Li}_k(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^k} \quad (|x| < 1, \ k \in \mathbb{Z})
\]

is the \( k \)-th polylogarithm. Poly-Euler numbers are a generalization of classical Euler numbers \( E_n \) defined by

\[
\frac{1}{\cosh t} = \sum_{n=0}^{\infty} \frac{E_n}{n!} t^n.
\]
Indeed, we easily see that $E_{n}^{(1)} = E_{n}$. The manner of generalization using the polylogarithm is due to Kaneko [3]. He introduced the poly-Bernoulli numbers $B_{n}^{(k)}$ by

$$
\frac{\text{Li}_{k}(1 - e^{-t})}{1 - e^{-t}} = \sum_{n=0}^{\infty} \frac{B_{n}^{(k)}}{n!} t^{n} \quad (k \in \mathbb{Z})
$$

and discovered many meaningful properties of that. Furthermore, combinatorial interpretations for poly-Bernoulli numbers were discovered by Brewbaker [2] and Launois [6] up to the present (see also [10]). In the previous research, the authors presented many properties of poly-Euler numbers ([7] and [8]), for example, explicit formulae, Clausen-von Staudt type formula, and a parity formula. Further we also found certain combinatorial interpretations for poly-Euler numbers.

We should mention that the research on poly-Euler numbers relates to that of multiple $L$ values. In fact, poly-Euler numbers are introduced as special values of a generalized Dirichlet $L$-function which relates to multiple $L$-functions (see [9] and [7, 8]). We should also mention Arakawa-Kaneko’s zeta-function. Arakawa and Kaneko [1] introduced a zeta-function which relates to the poly-Bernoulli numbers and the multiple zeta-functions. This property has been applied to the research on the duality for multiple zeta-star values (see [4] and [5]). Our $L$-function would also play a key role in the research on multiple $L$ values.

In this article, we treat a periodic property for poly-Euler numbers with negative index and the Vandiver type congruence for Euler numbers. From the numerical data (see Tables 1 and 2 below), we can find that the one’s digits of poly-Euler numbers $(n + 1)E_{n}^{(-k)} (k, n \geq 0)$ change periodically with respect to $k$ and $n$. We prove this periodicity in the next section. In Section 3, we give the Vandiver type congruence for Euler numbers. Tables 1 and 2 below are the lists of numerical values of poly-Euler numbers $(n + 1)E_{n}^{(-k)}$.

§2. Periodicity for the one’s digits of poly-Euler numbers

From the numerical data, we can observe many interesting information of poly-Euler numbers. Here, we focus on the one’s digits of poly-Euler numbers and prove the periodicity. We start with reviewing an explicit formula for poly-Euler numbers which will be used in the following sections. Hereafter, we put $\bar{E}_{n}^{(k)} := (n + 1)E_{n}^{(k)}$.

Theorem 2.1 (Theorems 3.1 and 6.1 in [7]). For any non-negative integer $n$ and any integer $k$, we have

$$
\bar{E}_{n}^{(k)} = \frac{1}{2} \sum_{m=0}^{n+1} \binom{n+1}{m} B_{n-m+1}^{(k)} 4^{n-m+1}((-1)^{m} - (-3)^{m}).
$$

(2.1)
Table 1. Poly-Euler numbers $\tilde{E}_n^{(-k)} (= (n+1)E_n^{(-k)})$ ($0 \leq k \leq 4$)

<table>
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<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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</tr>
</tbody>
</table>

Table 2. Poly-Euler numbers $\tilde{E}_n^{(-k)} (= (n+1)E_n^{(-k)})$ ($5 \leq k \leq 7$)

<table>
<thead>
<tr>
<th>$n \setminus k$</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
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</tbody>
</table>
In particular, for poly-Euler numbers with non-positive index, the above formula can be rewritten as

\[
\bar{E}_{n}^{(-k)} = \frac{(-1)^{k}}{2} \sum_{l=0}^{k} (-1)^{l} l! \left\{ \begin{array}{l} k \\ l \end{array} \right\} \{(4l + 3)^{n+1} - (4l + 1)^{n+1}\} \quad (k \geq 0).
\]

The following theorem gives the periodicity for the one’s digits of poly-Euler numbers:

**Theorem 2.2.** For any non-negative integers \(n, n', k\) and \(k'\) with \(n \equiv n'\) and \(k \equiv k'\) (mod 4), we have

\[
\bar{E}_{n}^{(-k)} \equiv \bar{E}_{n'}^{(-k')} \pmod{10}.
\]

In particular, we have \(\bar{E}_{n}^{(-k)} \equiv 6\mathcal{E}_{n}^{(-k)} + 5\delta_{n} \pmod{10}\), where \(\delta_{n}\) takes 1 if \(n\) is even and 0 otherwise, and

\[
\mathcal{E}_{n}^{(-k)} = \begin{cases} 
-(3^{n} + 3) & \text{for } k \equiv 0 \pmod{4}, \\
2^{n} & \text{for } k \equiv 1 \pmod{4}, \\
-2^{n} + (1 + (-1)^{n}) & \text{for } k \equiv 2 \pmod{4}, \\
(2^{n} + 2)(1 + (-1)^{n}) & \text{for } k \equiv 3 \pmod{4}.
\end{cases}
\]

**Proof.** The authors had shown the parity of poly-Euler numbers in [8]. Namely, we have \(\bar{E}_{n}^{(-k)} \equiv \delta_{n} \pmod{2}\). Therefore we need to show that \(\bar{E}_{n}^{(-k)} \equiv \mathcal{E}_{n}^{(-k)} \pmod{5}\). From (2.2), we have

\[
2\bar{E}_{n}^{(-k)} \equiv (-1)^{k} \sum_{l=0}^{k} (-1)^{l} l! \left\{ \begin{array}{l} k \\ l \end{array} \right\} \alpha(n, l) \pmod{5},
\]

where \(\alpha(n, l) := (3 - l)^{n+1} - (1 - l)^{n+1}\). Note that \(\left\{ \begin{array}{l} k \\ l \end{array} \right\} = 0\) for \(0 \leq k < l\) and

\[
\left\{ \begin{array}{l} k \\ l \end{array} \right\} \equiv \left\{ \begin{array}{l} k' \\ l \end{array} \right\} \pmod{p}
\]

holds for any odd prime \(p\) and any non-negative integers \(l, k\) and \(k'\) with \(k \equiv k'\) (mod \(p - 1\)). Therefore, when we put \(k \equiv a, n \equiv b\) (mod 4) \((a, b \in \{0, 1, 2, 3\})\), the above formula becomes

\[
2\bar{E}_{n}^{(-k)} \equiv (-1)^{a} \sum_{l=0}^{a} (-1)^{l} l! \left\{ \begin{array}{l} a \\ l \end{array} \right\} \alpha(b, l) \pmod{5},
\]

which gives \(\bar{E}_{n}^{(-k)} \equiv \mathcal{E}_{n}^{(-k)} \pmod{5}\). It follows that \(\bar{E}_{n}^{(-k)} \equiv 6\mathcal{E}_{n}^{(-k)} + 5\delta_{n} \pmod{10}\) from the Chinese remainder theorem. Furthermore, we have \(\bar{E}_{n}^{(-k)} \equiv \bar{E}_{n'}^{(-k')} \pmod{10}\).
for any non-negative integers \( n, n', k \) and \( k' \) with \( n \equiv n', k \equiv k' \pmod{4} \), since \( \overline{E}_n^{(-k)} \equiv \overline{E}_n^{(-k')} \pmod{2} \) and \( \overline{E}_n^{(-k)} \equiv \overline{E}_n^{(-k')} \pmod{5} \). Thus we obtain Theorem 2.2.

\[ \square \]

§ 3. Vandiver type congruence for the Euler numbers

Kaneko [3] showed the following congruence for the Bernoulli numbers which is originally due to Vandiver from the viewpoint of the poly-Bernoulli numbers: For any odd prime \( p \) and positive integer \( n(\leq p-2) \),

\[
B_n \equiv \sum_{l=1}^{p-2} H_l (l+1)^n \pmod{p},
\]

where \( H_n := \sum_{i=1}^{n} \frac{1}{i} \) is the \( n \)-th harmonic number and \( B_n \) is the \( n \)-th Bernoulli number defined by

\[
\frac{te^t}{e^t-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n.
\]

Similarly, we obtain the following Vandiver type congruence for the Euler numbers from the viewpoint of poly-Euler numbers.

**Theorem 3.1.** For any odd prime \( p \) and non-negative integer \( n \) not exceeding \( p-3 \), we have

\[
(n+1)E_n \equiv \sum_{l=1}^{(p-1)/2} H_l \tilde{A}(n, l) \pmod{p},
\]

where

\[
\tilde{A}(n, l) = \begin{cases} 
0 & \text{when } n \text{ is odd,} \\
1 & \text{when } l = (p-1)/2 \text{ and } n \text{ is even,} \\
2 \sum_{j=0}^{n/2} \binom{n+1}{2j+1}(4l+2)^{n-2j} & \text{otherwise.}
\end{cases}
\]

**Proof.** In [3], Kaneko showed an explicit formula for the poly-Bernoulli numbers: For any integers \( k \) and \( n \geq 0 \), we have

\[
\mathcal{B}_n^{(k)} = (-1)^n \sum_{m=0}^{n} \frac{(-1)^m m! \binom{n}{m}}{(m+1)^k}.
\]

From this formula, we see that \( \mathcal{B}_n^{(1)} \equiv \mathcal{B}_n^{(2-p)} \pmod{p} \) for \( n = 0, 1, \ldots, p-2 \) and any odd prime \( p \). Consequently, \( \overline{E}_n^{(1)} \equiv \overline{E}_n^{(2-p)} \pmod{p} \) for \( n = 0, 1, \ldots, p-3 \) from Theorem 2.1.
Hereafter we use another explicit formula for poly-Euler numbers which is a modified version of (2.2) (see Corollary 6.6 in [7]):

\[ \tilde{E}_n^{(-k)} = (-1)^k \sum_{l=0}^{k} (-1)^l l! \left\{ \begin{array}{l} k \\ l \end{array} \right\} \sum_{m=1}^{n+1} \binom{n+1}{m} (4l+2)^{n+1-m}. \]

Since \( \tilde{E}_n^{(1)} \equiv \tilde{E}_n^{(2-p)} \pmod{p} \), we have

\[ (3.1) \quad \tilde{E}_n^{(1)} \equiv - \sum_{l=0}^{p-2} (-1)^l l! \left\{ \begin{array}{l} p-2 \\ l \end{array} \right\} A(n, l) \pmod{p}. \]

Here, we have put

\[ A(n, l) := \sum_{j=0}^{[n/2]} \binom{n+1}{2j+1} (4l+2)^{n-2j}. \]

Since \( 4(p-l-1)+2 \equiv -(4l+2) \pmod{p} \), we have

\[ A(n, l) \equiv \begin{cases} 0 \pmod{p} & \text{for } l = (p-1)/2 \text{ and } n \text{ is odd,} \\ 1 \pmod{p} & \text{for } l = (p-1)/2 \text{ and } n \text{ is even,} \\ (-1)^n A(n, p-l-1) \pmod{p} & \text{otherwise}. \end{cases} \]

Furthermore we remark that

\[ (-1)^l l! \left\{ \begin{array}{l} p-2 \\ l \end{array} \right\} \equiv (-1)^{p-l-1} (p-l-1)! \left\{ \begin{array}{l} p-2 \\ p-1 \end{array} \right\} \pmod{p} \]

holds for any positive integer \( l \) less than \( p-1 \) (see Lemma 7.6 in [7]). Hence (3.1) is rewritten as

\[ \tilde{E}_n^{(1)} \equiv - \sum_{l=1}^{(p-1)/2} (-1)^l l! \left\{ \begin{array}{l} p-2 \\ l \end{array} \right\} \tilde{A}(n, l) \pmod{p}. \]

Thus the theorem is proved by using the following lemma:

**Lemma 3.2** (Lemma 2 in [3]). Suppose \( p \) is an odd prime, and \( 1 \leq l \leq p-2 \). Then,

\[ (-1)^{l-1} l! \left\{ \begin{array}{l} p-2 \\ l \end{array} \right\} \equiv H_l \pmod{p}. \]

\[ \square \]

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References