On the level statistics problem for the one-dimensional Schrödinger operator with random decaying potential

By

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Abstract

In this note, we review our recent work [2] on the level statistics problem of the one-dimensional Schrödinger operator with random potential decaying like $x^{-\alpha}$ at infinity. We proved that (i) (ac spectrum case) for $\alpha > \frac{1}{2}$, the point process $\xi_L$ consisting of the rescaled eigenvalues converges to a clock process, and the fluctuation of the eigenvalue spacing converges to Gaussian. (ii) (critical case) for $\alpha = \frac{1}{2}$, $\xi_L$ converges to the limit of the circular $\beta$-ensemble.

§ 1. Introduction

In this paper, we study the following Schrödinger operator

$$H := -\frac{d^2}{dt^2} + a(t)F(X_t) \quad \text{on } L^2(\mathbb{R})$$

where $a \in C^\infty$ is real valued, $a(-t) = a(t)$, non-increasing for $t \geq 0$, and satisfies

$$C_1 t^{-\alpha} \leq a(t) \leq C_2 t^{-\alpha}$$

for some positive constants $C_1, C_2$. $F$ is a real-valued, smooth, non-constant function on a compact Riemannian manifold $M$. $\{X_t\}$ is a Brownian motion on $M$. Since the potential $a(t)F(X_t)$ is $-\frac{d^2}{dt^2}$-compact, we have $\sigma_{\text{ess}}(H) = [0, \infty)$. For the nature of the spectrum of $H$ in $[0, \infty)$, Kotani-Ushiroya[4] proved that...
(1) for $\alpha < \frac{1}{2}$ : the spectrum on $[0, \infty)$ is pure point with exponentially decaying eigenfunctions,
(2) for $\alpha = \frac{1}{2}$ : the spectrum on is pure point on $[0, E_c]$ and purely singular continuous on $[E_c, \infty)$ with some explicitly computable $E_c$,
(3) for $\alpha > \frac{1}{2}$ : if we furthermore assume $\langle F \rangle := \int_M F(x)dx = 0$, the spectrum on $[0, \infty)$ is purely absolutely continuous.

In this note we report our results on the level statistics on this operator. For that purpose, let $H_L := H|_{[0,L]}$ be the local Hamiltonian with Dirichlet boundary condition and let $\{E_n(L)\}_{n=1}^{\infty}$ be its eigenvalues in the increasing order. Let $n(L) \in \mathbb{N}$ be s.t. $\{E_n(L)\}_{n \geq n(L)}$ coincides with the set of non-negative eigenvalues of $H_L$. We arbitrarily take the reference energy $E_0 > 0$ and consider the following point process

$$\xi_L := \sum_{n \geq n(L)} \delta_{L(\sqrt{E_n(L)} - \sqrt{E_0})}$$

in order to study the local fluctuation of eigenvalues near $E_0$. Our aim is to identify the limit of $\xi_L$ as $L \to \infty$.

This problem was first studied by Molchanov[7]. He proved that, when $\alpha = 0$, $\xi_L$ converges to the Poisson process. Killip-Stoiciu [3] studied the CMV matrices whose matrix elements decay like $n^{-\alpha}$. They showed that (i) for $\alpha > \frac{1}{2}$ : $\xi_L$ converges to the clock process, (ii) for $\alpha = \frac{1}{2}$ : $\xi_L$ converges to the limit of the $\beta$-ensemble, (iii) for $0 < \alpha < \frac{1}{2}$ : $\xi_L$ converges to the Poisson process. Krichevski-Valko-Virag[6] studied the one-dimensional discrete Schrödinger operator with the random potential decaying like $n^{-1/2}$, and proved that $\xi_L$ converges to the $\text{Sin}_\beta$-process. For the multidimensional Anderson-type model, we refer to [8] who proved the convergence to the Poisson process in the localized regime. The aim of our work is to do the analog of that by Killip-Stoiciu[3] for the one-dimensional Schrödinger operator in the continuum.

In section 2 (resp. section 3), we state our results for ac-case : $\alpha > \frac{1}{2}$ (resp. critical-case : $\alpha = \frac{1}{2}$).

§2. AC-case

**Definition 2.1.** Let $\mu$ be a probability measure on $[0, \pi)$. We say that $\xi$ is the clock process with spacing $\pi$ w.r.t. $\mu$ if and only if

$$\mathbb{E}[e^{-\xi(f)}] = \int_0^\pi d\mu(\phi) \exp \left( -\sum_{n \in \mathbb{Z}} f(n\pi - \phi) \right)$$

where $f \in C_c(\mathbb{R})$ and $\xi(f) := \int_{\mathbb{R}} f d\xi$.

1We have not obtained results for pp-case : $\alpha < \frac{1}{2}$. 


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We set

\[(x)_\pi \mathbb{Z} := x - [x]_\pi \mathbb{Z}, \quad [x]_\pi \mathbb{Z} := \max\{k\pi \mid k\pi \leq x, k \in \mathbb{Z}\} .\]

In order that \(\xi_L\) converge to a point process, we need to take subsequences. Thus we assume

(A)

(1) \(\alpha > \frac{1}{2}\), \(\langle F \rangle = 0\).

(2) A sequence \(\{L_k\}_{k=1}^\infty\) satisfies \(\lim_{k \to \infty} L_k = \infty\) and

\[(\sqrt{E_0} L_k)_{\pi \mathbb{Z}} = \beta + o(1), \quad k \to \infty\]

for some \(\beta \in [0, \pi)\).

In fact, if \(a \equiv 0\), we need to take a subsequence satisfying (A)(2) in order that the limit exists, and the limit depends on \(\beta\).

**Theorem 2.2.** Assume (A). Then \(\xi_{L_k}\) converges in distribution to the clock process with spacing \(\pi\) w.r.t. a probability measure \(\mu_\beta\) on \([0, \pi)\).

**Remark 2.3.** Let \(x_t\) be the solution to the eigenvalue equation: \(H_{L} x_t = \kappa^2 x_t\). If we set

\[\begin{pmatrix} x_t \\ x'_t/\kappa \end{pmatrix} = \begin{pmatrix} r_t \sin \theta_t \\ r_t \cos \theta_t \end{pmatrix}, \quad \theta_t = \kappa t + \tilde{\theta}_t,\]

then \(\tilde{\theta}_t\) has a limit as \(t\) goes to infinity[4]: \(\lim_{t \to \infty} \tilde{\theta}_t = \tilde{\theta}_\infty, \text{ a.s.}\) \(\mu_\beta\) is the distribution of the random variable \((\beta + \tilde{\theta}_\infty(\sqrt{E_0}))_{\pi \mathbb{Z}}\).

**Remark 2.4.** We can consider point processes w.r.t. two reference energies \(E_0, E'_0\) \((E_0 \neq E'_0)\) at the same time: suppose a sequence \(\{L_k\}_{k=1}^\infty\) satisfies

\[(\sqrt{E_0} L_k)_{\pi \mathbb{Z}} = \beta + o(1), \quad (\sqrt{E'_0} L_k)_{\pi \mathbb{Z}} = \beta' + o(1), \quad k \to \infty\]

for some \(\beta, \beta' \in [0, \pi)\). We set

\[\xi_L := \sum_{n \geq n(L)} \delta_{L(\sqrt{E_n(L)} - \sqrt{E_0})}, \quad \xi'_L := \sum_{n \geq n(L)} \delta_{L(\sqrt{E_n(L)} - \sqrt{E'_0})}.\]

Then the joint distribution of \(\xi_{L_k}, \xi'_{L_k}\) converges, for \(f, g \in C_c(\mathbb{R})\),

\[\lim_{k \to \infty} E[\exp( -\xi_{L_k}(f) - \xi_{L_k}(g))] = \int_0^\pi d\mu(\phi, \phi') \exp \left(-\sum_{n \in \mathbb{Z}} (f(n\pi - \phi) + g(n\pi - \phi'))\right)\]
where $\mu(\phi, \phi')$ is the joint distribution of $(\beta + \tilde{\theta}_\infty(\sqrt{E_0}))_{\pi \mathbb{Z}}$ and $(\beta' + \tilde{\theta}_\infty(\sqrt{E_0'}))_{\pi \mathbb{Z}}$. We are unable to identify $\mu(\phi, \phi')$ but it may be possible that $\phi, \phi'$ are correlated.

**Remark 2.5.** If we rearrange eigenvalues

$$\cdots < E'_{-2}(L) < E'_{-1}(L) < E_0 < E'_1(L) < E'_2(L) < \cdots$$

near the reference energy $E_0$, then for any $j \in \mathbb{Z}$ we have

\begin{equation}
\lim_{L \to \infty} L\left(\sqrt{E'_{j+1}(L)} - \sqrt{E'_j(L)}\right) = \pi, \quad \text{a.s.}
\end{equation}

which is called the strong clock behavior [1].

We next study the finer structure of the eigenvalue spacing under the following assumption.

**(B)**

1. $\frac{1}{2} < \alpha < 1$, $\langle F \rangle = 0$,
2. A sequence $\{L_k\}_{k=1}^\infty$ satisfies $\lim_{k \to \infty} L_k = \infty$ and
   \[ \sqrt{E_0}L_k = a_k \pi + \beta + o(1), \quad k \to \infty \]
   for some $\{a_k\}_{k=1}^\infty (\subset \mathbb{N})$ and $\beta \in [0, \pi)$,
3. $a(t) = t^{-\alpha}(1 + o(1))$, $t \to \infty$.

Roughly speaking, $E_{a_k}(L)$ is the eigenvalue closest to $E_0$. In view of (2.1), we set

$$X_k(n) := \left\{ \left( \sqrt{E_{a_k+n+1}(L_k)} - \sqrt{E_{a_k+n}(L_k)} \right)L_k - \pi \right\}L_k^{\alpha - \frac{1}{2}}, \quad n \in \mathbb{Z}.$$

**Theorem 2.6.** Assume (B). Then $\{X_k(n)\}_{n \in \mathbb{Z}}$ converges in distribution to the Gaussian system with covariance

$$C(n, n') = \frac{C(E_0)}{8E_0} Re \int_0^1 s^{-2\alpha} e^{i(n-n')\pi s}2(1 - \cos \pi s)ds, \quad n, n' \in \mathbb{Z},$$

where $C(E) := \int_M \left| \nabla(L + 2i\sqrt{E})^{-1}F \right|^2 dx$

$L$ is the generator of $(X_t)$.

**Remark 2.7.** Suppose we consider two reference energies $E_0, E'_0 (E_0 \neq E'_0)$ at the same time and suppose a sequence $\{L_k\}_{k=1}^\infty$ satisfies $\lim_{k \to \infty} L_k = \infty$ and

$$\sqrt{E_0}L_k = a_k \pi + \beta + o(1), \quad \sqrt{E'_0}L_k = b_k \pi + \beta' + o(1), \quad k \to \infty$$

for some $a_k, b_k \in \mathbb{N}$, and $\beta, \beta' \in [0, \pi)$. Then $\{X_k(n)\}_{n}, \{X'_k(m)\}_{m}$ converge jointly to the mutually independent Gaussian systems.
§ 3. Critical-case

Definition 3.1. The circular $\beta$-ensemble with $n$-points is given by

$$E_n^\beta[G] := \frac{1}{Z_{n, \beta}} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} G(\theta_1, \cdots, \theta_n) |\triangle(e^{i\theta_1}, \cdots, e^{i\theta_n})|^\beta$$

where $Z_{n, \beta}$ is the normalization constant, $G \in C(T^n)$ is bounded and $\triangle$ is the Vandermonde determinant. The limit $\xi_\beta$ of the circular $\beta$-ensemble is defined

$$E[e^{-\xi_\beta(f)}] = \lim_{n \to \infty} E_n^\beta \left[ \exp \left( - \sum_{j=1}^{n} f(n\theta_j) \right) \right], \quad f \in C_c(R)$$

whose existence and characterization is given by [3]. We set the following assumption.

(C) $\langle F \rangle = 0$ and $a(t) = t^{-\frac{1}{2}}(1+o(1)), \quad t \to \infty$.

Theorem 3.2. Assume (C). Writing $\xi_\beta = \sum_j \delta_{\lambda_j}$, let $\xi_\beta' := \sum_j \delta_{\lambda_j/2}$. Then $\xi_L \overset{d}{\to} \xi_\beta'$ with $\beta = \frac{8E_0}{C(E_0)^2}$.

Remark 3.3. The corresponding $\beta = \beta(E_0) = \frac{8E_0}{C(E_0)^2}$ depends on the reference energy $E_0$, so that the spacing distribution may change if we look at the different region in the spectrum. To see how $\beta$ changes, we recall some results in [4]. Let $\sigma_F(\lambda)$ be the spectral measure of the generator $L$ of $\{X_t\}$ with respect to $F$. Then

$$\gamma(E) := -\frac{1}{4E} \int_{-\infty}^{0} \frac{\lambda}{\lambda^2 + 4E} d\sigma_F(\lambda), \quad E > 0$$

is the Lyapunov exponent in the sense that any generalized eigenfunction $\psi_E$ of $H$ satisfies

$$\lim_{|t| \to \infty} (\log t)^{-1} \log \left\{ \left| \psi_E(t)^2 + \psi'_E(t)^2 \right|^{1/2} \right\} = -\gamma(E), \quad a.s.$$  

Moreover $E < E_c$ (resp. $E > E_c$) if and only if $\gamma(E) > \frac{1}{2}$ (resp. $\gamma(E) < \frac{1}{2}$) and $\gamma(E_c) = \frac{1}{2}$. Since $C(E) = 8E \cdot \gamma(E)$, we have $\beta(E) = \frac{1}{\gamma(E)}$. It then follows that $E < E_c$ (resp. $E > E_c$) if and only if $\beta(E) < 2$ (resp. $\beta(E) > 2$) and $\beta(E_c) = 2$ (Figure 1.). Similar statement also holds for discrete Hamiltonian and CMV matrices. This is consistent with our general belief that in the point spectrum (resp. in the continuous spectrum) the level repulsion is weak (resp. strong).

We note that, for $\beta = 2$, the circular $\beta$-ensemble with $n$-points coincides with the eigenvalue distribution of the unitary ensemble with the Haar measure on $U(n)$. In [9], Valkó-Virág showed that $\text{Sine}_\beta$ process has a phase transition at $\beta = 2$. 
Remark 3.4. If we consider two reference energies $E_0, E'_0$ ($E_0 \neq E'_0$), then the corresponding point process $\xi_L, \xi'_L$ converges jointly to the independent $\xi_\beta, \xi'_\beta$.

References