Spectral properties of massless Dirac operators with real-valued potentials

By

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Abstract

We prove that a Schnol’-type theorem holds for massless Dirac operators under minimal assumptions on the potential, and apply this result to conclude that the spectrum of a certain class of such operators covers the whole real line. We also discuss embedded eigenvalues of massless Dirac operators with suitable scalar potentials.

§1. Introduction

This paper is an announcement of results on spectral properties of Dirac operators with real-valued potentials and will be followed by a complete treatment in which all proofs are given.

The Dirac operators to be considered in this paper are

(1.1) \[ H_2 = -i \sigma \cdot \nabla + q(x) \text{ in } L^2(\mathbb{R}^2; \mathbb{C}^2) \]

and

(1.2) \[ H_3 = -i \alpha \cdot \nabla + q(x) \text{ in } L^2(\mathbb{R}^3; \mathbb{C}^4). \]

Here \( \sigma = (\sigma_1, \sigma_2) \) and \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \) are given as follows:

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},
\]

and

\[
\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \quad (j \in \{1, 2, 3\}) \quad \text{with} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

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The dot products are to be read as
\[ \sigma \cdot \nabla = \sigma_1 \frac{\partial}{\partial x_1} + \sigma_2 \frac{\partial}{\partial x_2} \]
in (1.1) and
\[ \alpha \cdot \nabla = \alpha_1 \frac{\partial}{\partial x_1} + \alpha_2 \frac{\partial}{\partial x_2} + \alpha_3 \frac{\partial}{\partial x_3} \]
in (1.2). The potential \( q \) is a real-valued function on \( \mathbb{R}^d \), where \( d = 2 \) or \( d = 3 \), respectively. The operators \( H_2, H_3 \) differ from the standard Dirac operator in that they lack a mass term, usually represented by an additional anti-commuting matrix: \( \sigma_3 \) for the two-dimensional case and
\[ \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \]
for the three-dimensional case, where \( I \) is a 2 \( \times \) 2 identity matrix.

The purpose of the present paper is to show that \( \sigma(H_d) = \mathbb{R} \) under minimal assumptions on \( q \). In particular, we shall not require any restriction on the growth or decay of the potential \( q \) at infinity.

We have two motivations. First, the spectrum of the one-dimensional massless Dirac operator
\[ H_1 = -i\sigma_2 \frac{d}{dx} + q(x) \text{ in } L^2(\mathbb{R}; \mathbb{C}^2) \]
covers the whole real axis and is purely absolutely continuous whenever \( q \in L^1_{\text{loc}}(\mathbb{R}, \mathbb{R}) \). This surprising fact was first pointed out by one of the authors in [8]. By separation in spherical polar coordinates, this result also implies that \( \sigma(H_d) = \mathbb{R} \) if \( q \) is rotationally symmetric; see [9]. Second, it is believed that the energy spectrum of graphene, in which electron transport is governed by Dirac equations in two dimensions without a mass term, has no bandgap (zero bandgap); see [2], [4], [7]. For these reasons, it is natural to make an attempt to show that \( \sigma(H_d) = \mathbb{R} \) under minimal assumptions on \( q \).

\section{Embedded eigenvalues}

It is difficult to imagine that the spectra of \( H_2 \) and \( H_3 \) are always purely absolutely continuous regardless of \( q \). Actually, in the three-dimensional case, we have an example of \( q \) which gives rise to a zero mode of \( H_3 \), i.e. an example of \( q \) for which \( H_3 \) has the embedded eigenvalue 0.

\textbf{Example 1.} Let \( q(x) = -3/\langle x \rangle^2 \), where \( \langle x \rangle = \sqrt{1 + |x|^2} \). Then there exists a unique self-adjoint realization of \( H_3 \) in \( L^2(\mathbb{R}^3; \mathbb{C}^4) \) with \( \text{Dom}(H_3) = H^1(\mathbb{R}^3; \mathbb{C}^4) \), the
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Let $f(x) = \langle x \rangle^{-3}(I_4 + i \alpha \cdot x)\phi_0$ with $\phi_0$ a unit vector in $\mathbb{C}^4$, then a direct calculation shows that $H_3 f = 0$. This implies that $0 \in \sigma_p(H_3)$, because $f \in \text{Dom}(H_3)$. Thus $H_3$ has a zero mode. Since $\sigma(H_3) = \mathbb{R}$, the energy 0 is an embedded eigenvalue of $H_3$.

The potential $q$ and the zero mode $f$ in Example 1 were motivated by [6].

Example 2 below indicates that there is a difference in spectral property between $H_3$ and $H_2$. In fact, quite a similar construction in Example 2 below gives a zero resonance of $H_2$, not a zero mode of $H_2$. On the other hand, we do not know if the potential $q$ in Example 2 gives rise to a zero mode of $H_2$.

Example 2. Let $q(x) = -2/\langle x \rangle^2$. Then there exists a unique self-adjoint realization of $H_2$ in $L^2(\mathbb{R}^2; \mathbb{C}^2)$ with $\text{Dom}(H_2) = H^1(\mathbb{R}^2; \mathbb{C}^2)$, and $\sigma(H_2) = \mathbb{R}$. If $\psi(x) = \langle x \rangle^{-2}(I_2 + i \sigma \cdot x)\phi_0$, $\phi_0$ a unit vector in $\mathbb{C}^2$, then one sees that $H_2 \psi = 0$. However, it is clear that $\psi \not\in L^2(\mathbb{R}^2; \mathbb{C}^2)$. Therefore, $\psi$ is not a zero mode of $H_2$. On the other hand, one finds that $\psi \in L^{2,-s}(\mathbb{R}^2; \mathbb{C}^2)$ for $\forall s > 0$, where $L^{2,-s}(\mathbb{R}^2; \mathbb{C}^2) = \{ \phi \mid \| \langle x \rangle^{-s} \phi \|_{L^2} < \infty \}$. This means that $\psi$ is a zero resonance of $H_2$.

It is not an easy task to clarify whether $H_d$ has embedded eigenvalues for general potentials. However, we have a good control of the embedded eigenvalues of $H_d$ if $q(x)$ is rotationally symmetric. To formulate a result, we need to introduce the definition of the limit range $R_\infty(q)$ of $q$:

$$R_\infty(q) = \bigcap_{r>0} \overline{\{q(x) \mid |x| \geq r\}},$$

where $\overline{A}$ denotes the closure of a subset $A \subset \mathbb{R}$.

Theorem 2.1 (Schmidt[9]). Let $q(x) = \eta(|x|)$ and let $\eta \in L^1_{\text{loc}}(0, \infty)$. Suppose that there exists a real number $E \in \mathbb{R} \setminus R_\infty(q)$ such that

$$\frac{1}{r(E - \eta(r)) - 1} \in BV(r_0, \infty)$$

for some $r_0 > 0$, where $BV(r_0, \infty)$ denotes the set of functions of bounded variations on the interval $(r_0, \infty)$. Then $\sigma_p(H_d) \subset R_\infty(q)$ for $d \in \{2, 3\}$. 

Theorem 2.1 is a direct consequence of [9, Corollary 1]. If q is not rotationally symmetric, we can prove the following.

Theorem 2.2. Let \( q \in C^{1}(\mathbb{R}^{d};\mathbb{R}) \), \( d \in \{2, 3\} \), and suppose that both q and \((x \cdot \nabla)q\) are bounded functions. Then \( \sigma_{p}(H_{d}) \subset [m_{q}, M_{q}] \), where

\[
m_{q} = \inf_{x}\{q(x) + (x \cdot \nabla)q(x)\}, \quad M_{q} = \sup_{x}\{q(x) + (x \cdot \nabla)q(x)\}.
\]

The proof of Theorem 2.2 is based on a virial theorem in an abstract setting; see [1, Lemma 2.1].

§3. Schrödinger's theorem for Dirac operators

We now prepare a Schnol’ theorem for Dirac operators. As for Schnol’s theorem, we refer the reader [3, p.21, Theorem 2.9] which is a characterization of the spectra of Schrödinger operators in terms of polynomially bounded eigensolutions. In the three-dimensional Dirac operators, the theorem can be stated as follows:

Theorem 3.1. Let \( q \in L^{2}_{\text{loc}}(\mathbb{R}^{3};\mathbb{R}) \), and let E be a real number. Suppose f is a polynomially bounded measurable function on \( \mathbb{R}^{3} \), not identically 0, and satisfies the equation

\[
(-i\alpha \cdot \nabla + q)f = Ef
\]

in the distribution sense. Then \( E \in \sigma(H_{3}) \) for any self-adjoint realization \( H_{3} \) such that \( \text{Dom}(H_{3}) \supset H^{1}(\mathbb{R}^{3};\mathbb{C}^{4}) \cap \text{Dom}(q) \).

Outline of the proof of Theorem 3.1. We follow the line of the proof of [3, p. 21, Theorem 2.9].

We may suppose \( f \not\in L^{2}(\mathbb{R}^{3};\mathbb{C}^{4}) \) without loss of generality. Let \( \varphi \in C_{0}^{\infty}(\mathbb{R}^{3}) \) such that \( \varphi(x) = 1 \) \( (|x| \leq 1) \) and \( \varphi(x) = 0 \) \( (|x| \geq 2) \), and put \( \varphi_{n}(x) = \varphi(x/n) \) for \( n \in \mathbb{N} \). Then introducing a monotonically increasing sequence \( (M(n))_{n \in \mathbb{N}} \) by

\[
M(n) = \int_{|x| \leq n} |f(x)|^{2} dx,
\]

and defining \( f_{n} := \varphi_{n}f/\|\varphi_{n}f\|_{L^{2}} \), we can show that there exists a positive constant \( C \) such that

\[
\| (H_{3} - E)f_{n} \|_{L^{2}}^{2} \leq C \frac{M(2n) - M(n)}{n^{2}M(n)}.
\]

for all \( n \).
On the other hand, we can deduce that
\[
\lim_{n \to \infty} \inf \frac{M(2n) - M(n)}{n^2 M(n)} = 0,
\]
which implies that there exists a subsequence \((M(n_k))_{k \in \mathbb{N}}\) such that
\[
\lim_{k \to \infty} \frac{M(2n_k) - M(n_k)}{n_k^2 M(n_k)} = 0.
\]
(3.3)

Combining (3.2) with (3.3), we can conclude that \(E \in \sigma(H_3)\). ■

We give an application of the Schnol’ theorem, and show that the spectra of massless Dirac operators with real-valued potentials always coincide with the whole real axis, provided that the potentials are of the form specified in Theorem 3.2 below.

**Theorem 3.2.** Let \(\eta \in C^1(\mathbb{R};\mathbb{R})\) and define \(q(x) := \eta(x \cdot k)\) on \(\mathbb{R}^d\), \(d \in \{2, 3\}\), where \(k \in \mathbb{R}^d\) is a unit vector. Then \(\sigma(H_d) = \mathbb{R}\).

**Outline of the proof of Theorem 3.2.** We only give the proof for \(d = 3\). Put
\[
\xi(t) = \int_0^t \eta(\tau) d\tau.
\]

Choose a unit vector \(\phi_0 \in \mathbb{C}^4, \neq 0\), so that \((\alpha \cdot k)\phi_0 = \phi_0\). For a given \(E \in \mathbb{R}\), define
\[
f(x) = e^{-i(\alpha \cdot k) \xi(x \cdot k)} e^{iEx \cdot k} \phi_0.
\]

Then \(f\) satisfies the equation (3.1). Moreover, \(f \in C^1(\mathbb{R}^3;\mathbb{C}^4)\), and \(|f(x)|_{\mathbb{C}^4} = 1\) for all \(x \in \mathbb{R}^3\). It follows from Theorem 3.1 that \(E \in \sigma(H_3)\). Since \(E\) is an arbitrary real number, we can conclude that \(\sigma(H_3) = \mathbb{R}\). ■

§ 4. The main result

We now state the main theorem, which greatly generalizes Theorem 3.2.

**Theorem 4.1.** Let \(q \in L_{loc}^2(\mathbb{R}^3;\mathbb{R})\). Suppose that there is a sequence \((k_n)_{n \in \mathbb{N}}\) of unit vectors in \(\mathbb{R}^3\), a sequence \((B_{r_n}(a_n))_{n \in \mathbb{N}}\) of balls with centre \(a_n \in \mathbb{R}^3\) and radius \(r_n \to \infty\) \((n \to \infty)\), and a sequence of square-integrable functions \(q_n : (-r_n, r_n) \to \mathbb{R}\) \((n \in \mathbb{N})\) such that
\[
r_n^{-3} \int_{B(a_n, r_n)} \left| q(x) - q_n((x-a_n) \cdot k_n) \right|^2 dx \to 0
\]
as \(n \to \infty\). Then \(\sigma(H_3) = \mathbb{R}\) for any self-adjoint extension \(H_3\) of
\[
(-i\alpha \cdot \nabla + q)|_{C_0^\infty(\mathbb{R}^3)^4}.
\]

The two dimensional analogue of the statements above holds true.
Outline of the proof of Theorem 4.1. For a given energy $E \in \mathbb{R}$, we shall construct a singular sequence $(h_n)_{n \in \mathbb{N}}$, thus showing that $E \in \sigma(H_3)$. To this end, we first choose a function $\tilde{\eta}_n \in C^\infty(-r_n, r_n)$ so that
\[
\frac{1}{2r_n} \int_{-r_n}^{r_n} |q_n(t) - \tilde{\eta}_n(t)|^2 dt \to 0
\]
as $n \to \infty$. Then following the idea in the proof of Theorem 3.2, we put
\[
\xi_n(t) = \int_0^t \tilde{\eta}_n(\tau) d\tau,
\]
and choose a sequence of unit vectors $\phi_n \in \mathbb{C}^4, \neq 0$, so that $(\alpha \cdot k_n)\phi_n = \phi_n$, and define a sequence of functions $(f_n)_{n \in \mathbb{N}}$ by
\[
f_n(x) = e^{-i(\alpha \cdot k_n)\xi_n((x - a_n) \cdot k_n)}e^{iEx \cdot k_n}\phi_n : B_{r_n}(a_n) \to \mathbb{C}^4.
\]
We now choose a function $\chi \in C_C^\infty(\mathbb{R}^3)$ such that $\text{supp} \chi \subset B_1(0)$ and that $\|\chi\|_{L^2} = 1$. Putting
\[
h_n(x) = r_n^{-3/2} \chi\left(\frac{x - a_n}{r_n}\right)f_n(x),
\]
we can obtain the desired singular sequence. ■

References