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Solvable models in two-solenoidal Aharonov-Bohm magnetic fields on the Euclidean plane

By

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Abstract

In 1988, Gu–Qian [Gu-Qi] give the explicit eigenfunctions of the Schrödinger operator with two pointlike magnetic fields on $\mathbb{R}^2$ by using the elliptic coordinate system and the Mathieu functions, provided that the two magnetic fluxes are equal. We point out Gu–Qian's eigenfunctions do not match on the coordinate slit, unless both the fluxes are equal to 0 or π. Moreover, we define a new family of explicitly solvable twisted Laplacians with $U(2)$-gauge, which naturally arises from the viewpoint of the Galois theory for the covering groups corresponding to the elliptic coordinate. We also consider the Dirichlet eigenvalue problem for our new operator on an ellipse whose two foci coincide with the positions of the pointlike magnetic fields, and prove the completeness of our eigenfunctions by using the multi-parameter spectral analysis.

§ 1. Introduction

We consider the magnetic Schrödinger operator on $\mathbb{R}^2$ given by

$$H_{\alpha_1,\alpha_2} = \left(\frac{1}{i}\nabla - A_{\alpha_1,\alpha_2}\right)^2,$$

$$A_{\alpha_1,\alpha_2} = (A_{\alpha_1,\alpha_2,x}, A_{\alpha_1,\alpha_2,y}) = \alpha_1 A_0(x-a,y) + \alpha_2 A_0(x+a,y),$$

$$A_0(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right),$$

where (x,y) is the cartesian coordinate in $\mathbb{R}^2$, and a, α_1, α_2 are constants with $a > 0$, $0 \le \alpha_j < 1$ $(j = 1,2)$. The corresponding magnetic field $\operatorname{curl} A_{\alpha_1,\alpha_2} = \partial_x A_{\alpha_1,\alpha_2,y} - \partial_y A_{\alpha_1,\alpha_2,x}$ satisfies

$$\operatorname{curl} A_{\alpha_1,\alpha_2} = 2\pi\alpha_1\delta_{F_1} + 2\pi\alpha_2\delta_{F_2}$$

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in the distributional sense, where $F_1(a,0)$, $F_2(-a,0)$, and δ_P denotes the Dirac δ measure concentrated on the point P.

The operator H_{α_1,α_2} is the Hamiltonian which governs the motion of the quantum particle on the Euclidean plane affected by the magnetic field enclosed in two infinitesimally thin solenoids perpendicular to the plane. This model is considered to be a generalization of Aharonov-Bohm's model [Ah-Bo], in which they consider only one solenoid. Mathematical studies for H_{α_1,α_2} are as follows. Šťovíček [St] and Kocábová–Šťovíček [Ko-St1, Ko-St2] study the Green function and the propagator. Nambu [Na] makes an attempt to obtain the scattering amplitude, but not fully succeeded. Ito–Tamura [It-Ta] study the asymptotics of the scattering amplitude as $a \to \infty$. Persson [Pe1, Pe2] studies the self-adjoint extensions and the corresponding Aharonov-Casher type formula for Pauli operators. Borg [Bo] obtains some geometric formula for the heat kernel by using the Feynman-Kac-Ito formula. Note also that the author [Mi] studies a similar operator, the Schrödinger operator with the magnetic field given by the sum of a constant field and several δ-fields.

Unfortunately, the above papers do not mention the result of Gu–Qian [Gu-Qi], in which they obtain the eigenfunctions of the operator H_{α_1,α_2} explicitly under the assumption $\alpha_1 = \alpha_2$. They use the *elliptic coordinate system* (ξ, η) defined by

$$(x,y) = (a\cosh\xi\cos\eta,\ a\sinh\xi\sin\eta), \quad \xi \geq 0,\ -\pi < \eta \leq \pi. \tag{1.1}$$

The coordinate line is illustrated in Figure 1. The elliptic coordinate system is intro-

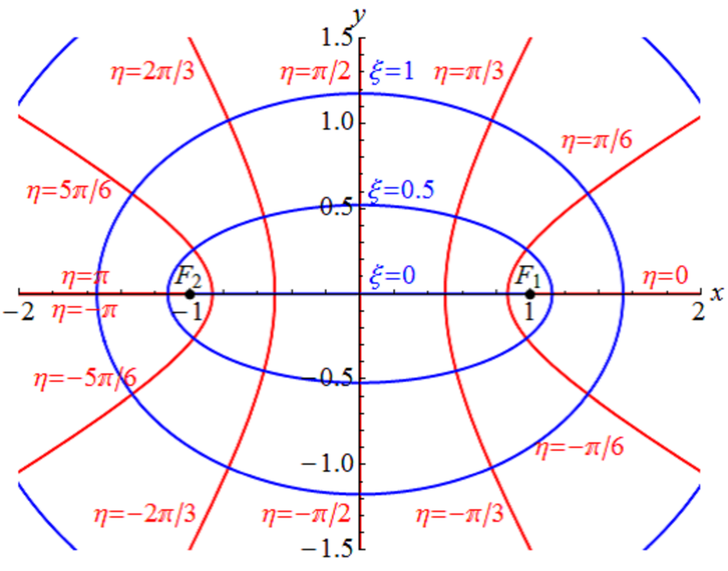

Figure 1. The elliptic coordinate system for $a = 1$. The equation $\xi =$ const. corresponds to an ellipse, and $\eta =$ const. to a hyperbola. These curves have common foci $(\pm a, 0)$.

duced by Mathieu [Ma] in order to solve the eigenvalue problem of $-\Delta$ on an ellipse by separation of variables. Gu–Qian show that Mathieu's method is also applicable for the operator H_{α_1,α_2}, by eliminating the magnetic potential via some gauge transform.

However, the solution expressed in terms of the elliptic coordinate system must satisfy some matching condition on the coordinate slit. It turns out that Gu–Qian's solutions do not satisfy the matching condition unless $\alpha_1, \alpha_2 \in \{0, 1/2\}$. We explain this fact from the viewpoint of the Galois theory for the covering manifold, by regarding the elliptic coordinate (1.1) as a covering map. This interpretation also naturally leads us a new explicitly solvable model. We define $U(2)$-representations ρ_α of the covering group corresponding to the elliptic coordinate system (see (4.2)), and consider the operator $-\Delta_{\rho_\alpha}$, which is the *Laplacian associated with the* $U(2)$*-representation* ρ_α (or the *twisted Laplacian*; see Sunada [Su1, Su2]). We also give the explicit eigenvalues and the eigenfunctions of $-\Delta_{\rho_\alpha}$ by using the Mathieu function of fractional order. We also prove the completeness of the eigenfunctions for the Dirichlet eigenvalue problem on an ellipse, by using the *multi-parameter spectral analysis* developed by Atkinson [At], Volkmer [Vo], et al. Moreover, we will see that our new operators $-\Delta_{\rho_\alpha}$ connect the four solvable Schrödinger operators $H_{0,0} = -\Delta$, $H_{1/2,0}$, $H_{0,1/2}$ and $H_{1/2,1/2}$ in the sense

$$\lim_{\alpha\to 0}(-\Delta_{\rho_\alpha}) \simeq H_{0,0} \oplus H_{1/2,1/2}, \quad \lim_{\alpha\to 1/2}(-\Delta_{\rho_\alpha}) \simeq H_{1/2,0} \oplus H_{0,1/2}. \tag{1.2}$$

Though we prove the completeness of the eigenfunctions on bounded ellipses, the eigenfunctions themselves are defined on the whole plane. Of course, our next aim is to obtain the generalized eigenfunction expansion on the whole plane, possibly by expanding the ellipse to the whole plane. We hope to study this subject in the future work.

The rest of the paper is organized as follows. In section 2, we review the classical work of Mathieu [Ma] about the eigenfunctions of $-\Delta$ on an ellipse, and the proof of the completeness by Volkmer [Vo]. In section 3, we verify Gu-Qian's result [Gu-Qi] and prove the completeness of the eigenfunctions when $\alpha_1 = \alpha_2 = 1/2$. In section 4, we define the twisted Laplacian associated with the $U(2)$-representation, give the explicit eigenfunctions and prove the completeness. In the appendix section 5, we quote definitions of various Mathieu functions and results from the theory of the multi-parameter spectral analysis, for the convenience of the readers. We recommend a reader not familiar with the Mathieu functions or the multi-parameter spectral theory to read the appendix before reading section 2. All the figures and numerical results are obtained by Mathematica 8.0.0.0.

§ 2. Eigenfunctions of $-\Delta$ on an ellipse

Let us review the Mathieu's classical work [Ma] for the eigenvalue problem of $-\Delta$ on an ellipse. For a positive constant b, put

$$E_b = \left\{ (x,y) \in \mathbb{R}^2 \mid \frac{x^2}{a^2\cosh^2 b} + \frac{y^2}{a^2\sinh^2 b} < 1 \right\}.$$

The region E_b is an ellipse with foci $F_1(a,0)$ and $F_2(-a,0)$. In the elliptic coordinate system (1.1), E_b is expressed by $0 \le \xi < b$, $-\pi < \eta \le \pi$. We consider the Dirichlet eigenvalue problem of $-\Delta$ on $L^2(E_b)$

$$-\Delta u = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u = \lambda u \quad \text{in } E_b, \tag{2.1}$$

$$u = 0 \quad \text{on } \partial E_b, \tag{2.2}$$

where ∂E_b denotes the boundary of E_b. In the elliptic coordinate system, we have the formula

$$\Delta = \frac{1}{J}\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right),$$

$$\begin{aligned} J = \det \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{pmatrix} &= a^2(\sinh^2\xi + \sin^2\eta) \\ &= \frac{a^2}{2}(\cosh 2\xi - \cos 2\eta). \end{aligned} \tag{2.3}$$

In particular,

$$J = 0 \Leftrightarrow \xi = 0,\ \eta \in \pi\mathbb{Z}.$$

Assume an eigenfunction u for (2.1), (2.2) is decomposable, that is, u is written as $u = f(\xi)g(\eta)$. Then the method of separation of variables yields

$$f''(\xi) + \left(-m + \frac{a^2\lambda}{2}\cosh 2\xi\right) f(\xi) = 0, \tag{2.4}$$

$$g''(\eta) + \left(m - \frac{a^2\lambda}{2}\cos 2\eta\right) g(\eta) = 0, \tag{2.5}$$

$$f(b) = 0, \tag{2.6}$$

where m is the *separation constant*, which is a constant independent of ξ and η. The equation (2.4) is the modified Mathieu equation for the parameters $\mu = m$ and $q =$

$a^2\lambda/4$, and (2.5) is the Mathieu equation for the same parameters (see section 5). Since the elliptic coordinate system has two coordinate slits

$$s_1 = (-a, a) \times \{0\}, \quad s_2 = (-\infty, -a) \times \{0\},$$

we need some matching conditions. Notice that the solutions $f(\xi)$ and $g(\eta)$ are defined for all $\xi, \eta \in \mathbb{R}$, since the equations (2.4) and (2.5) have no singularities. Moreover, the coordinate map (1.1) makes sense for every $(\xi, \eta) \in \mathbb{R}^2$. If we vary the value (ξ, η) along some path $\widetilde{C}$ in the $\xi\eta$-plane, the value $f(\xi)g(\eta)$ ($(\xi, \eta) \in \widetilde{C}$) gives the continuation of the solution u along the corresponding path C in the xy-plane. Especially when C is a closed path in E_b, the value $f(\xi)g(\eta)$ at the starting point of $\widetilde{C}$ and that at the terminal point must coincide, since the original equation (2.1) has no singularities in E_b. Taking C as the path turning around the center slit s_1 counterclockwise (and $\widetilde{C}$ as its lift), we have the first matching condition

$$g(\eta + 2\pi) = g(\eta). \tag{2.7}$$

Similarly, taking C as the path turning around the focus F_1 through the slit s_1, we have the second matching condition (see Figure 2)

$$f(-\xi)g(-\eta) = f(\xi)g(\eta). \tag{2.8}$$

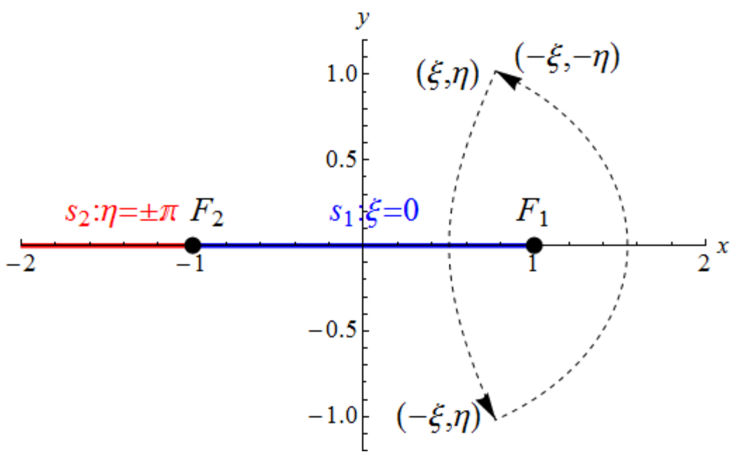

Figure 2. If we vary ξ-coordinate of the elliptic coordinate from ξ to $-\xi$, and η-coordinate from η to $-\eta$ subsequently, then the corresponding point turns around F_1.

The equations (2.5) and (2.7) imply g must be one of the Mathieu functions of integral order, so g is an even function or an odd function. The equation (2.4) implies f is a modified Mathieu function, and (2.8) implies f and g have the same parity. Thus we obtain the following result.

Theorem 1. Let (λ, u) satisfy one of the following systems of equations

$$\begin{cases} \mathrm{Ce}_n\left(b, a^2\lambda/4\right) = 0, \\ u = \mathrm{Ce}_n\left(\xi, a^2\lambda/4\right)\mathrm{ce}_n\left(\eta, a^2\lambda/4\right), \end{cases} \qquad (n = 0, 1, 2, \ldots), \tag{2.9}$$

$$\begin{cases} \mathrm{Se}_n\left(b, a^2\lambda/4\right) = 0, \\ u = \mathrm{Se}_n\left(\xi, a^2\lambda/4\right)\mathrm{se}_n\left(\eta, a^2\lambda/4\right), \end{cases} \qquad (n = 1, 2, \ldots). \tag{2.10}$$

Then, λ is an eigenvalue for the eigenvalue problem (2.1) and (2.2), and u is the corresponding eigenfunction. Moreover, the family of all the eigenfunctions u given above forms an orthogonal basis in $L^2(E_b)$.

Of course, we can obtain an orthonormal basis by normalizing the above eigenfunctions. We quote here the proof of the completeness by Volkmer [Vo] for later use.

Proof. Via the elliptic coordinate system (1.1), the space $L^2(E_b; dxdy)$ is unitarily equivalent to $\mathcal{H} = L^2([0, b] \times [-\pi, \pi]; Jd\xi d\eta)$, where the Jacobian J is given in (2.3). Then we have the direct sum decomposition

$$\mathcal{H} = \mathcal{H}_{\mathrm{even}} \oplus \mathcal{H}_{\mathrm{odd}},$$

where $\mathcal{H}_{\mathrm{even}}$ is the space of the elements of $\mathcal{H}$ even with respect to η and $\mathcal{H}_{\mathrm{odd}}$ is the odd one.

Consider a multi-parameter eigenvalue problem on $[0, b] \times [0, \pi]$

$$\begin{cases} f''(\xi) + \left((a^2\cosh 2\xi/2)\lambda - m\right) f(\xi) = 0, \\ f'(0) = 0,\ f(b) = 0, \end{cases} \tag{2.11}$$

$$\begin{cases} g''(\eta) + \left(-(a^2\cos 2\eta/2)\lambda + m\right) g(\eta) = 0, \\ g'(0) = 0,\ g'(\pi) = 0, \end{cases} \tag{2.12}$$

with the eigenvalue (λ, m). The coefficients satisfy

$$\det\begin{pmatrix} a^2\cosh 2\xi/2 & -1 \\ -a^2\cos 2\eta/2 & 1 \end{pmatrix} = J > 0$$

on $(0, b) \times (0, \pi)$. This means the multi-parameter eigenvalue problem (2.11), (2.12) is right-definite. Then the multi-parameter eigenfunction expansion theorem (Theorem 6 in section 5) implies there exists an orthogonal basis of $L^2([0, b] \times [0, \pi]; Jd\xi d\eta)$ consisting of decomposable eigenfunctions $f(\xi)g(\eta)$ for the problem (2.11), (2.12). Since the coefficients of the differential equation in (2.12) is an even analytic function with period 2π, the boundary conditions $g'(0) = g'(\pi) = 0$ imply the solution $g(\eta)$ of (2.12) can be extended as an even analytic function with period 2π. This means $g(\eta)$ is an even Mathieu function of integral order, and the orthogonal basis given above is also an orthogonal basis of $\mathcal{H}_{\mathrm{even}}$. Similarly, the solution $f(\xi)$ of (2.11) can be extended as an even analytic function, so $f(\xi)$ is a modified even Mathieu function. Conversely, we already know any non-trivial function u satisfying (2.9) is a decomposable eigenfunction of the problem (2.11), (2.12). Thus the family of the solutions (2.9) forms an orthogonal basis of $\mathcal{H}_{\mathrm{even}}$. Similarly we can prove the family of the solutions (2.10) forms an orthogonal basis of $\mathcal{H}_{\mathrm{odd}}$. □

We illustrate the numerical result for $a = 1$ and $b = 1$, in Figure 3-7.

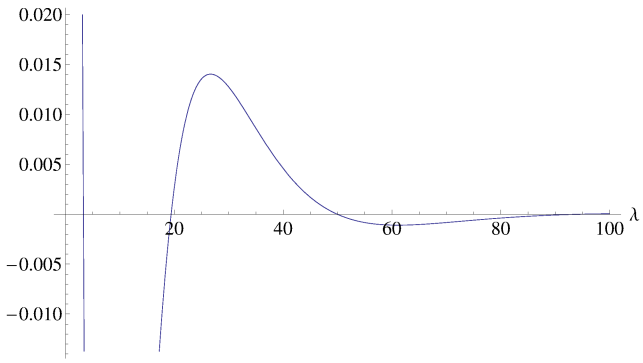

Figure 3. Graph of $\mathrm{Ce}_0(1, \lambda/4)$. The zeros $\lambda = 3.30141, 19.4171, 49.9539, 94.7943, \ldots$ are the eigenvalues of the problem (2.1), (2.2).

§ 3. Eigenvalue problem of H_{α_1,α_2} on an ellipse

Next we consider the eigenvalue problem of H_{α_1,α_2} on $L^2(E_b)$

$$H_{\alpha_1,\alpha_2} u = \lambda u \quad \text{in } E_b, \tag{3.1}$$

$$u = 0 \quad \text{on } \partial E_b, \tag{3.2}$$

$$\limsup_{(x,y)\to(\pm a,0)} |u(x,y)| < \infty. \tag{3.3}$$

The additional condition (3.3) is necessary for the self-adjointness of this problem.

Gu–Qian [Gu-Qi] solve the above eigenvalue problem in the following manner.[1] They use the well-known formula

$$H_{\alpha_1,\alpha_2} u = e^{i(\alpha_1\theta_1+\alpha_2\theta_2)}(-\Delta)e^{-i(\alpha_1\theta_1+\alpha_2\theta_2)}u,$$

[1] Gu–Qian do not consider the eigenvalue problem on an ellipse, but study the asymptotic behavior of the eigenfunctions at infinity for calculating the scattering amplitude.

Dirichlet eigenfunctions of $-\Delta$ on an ellipse

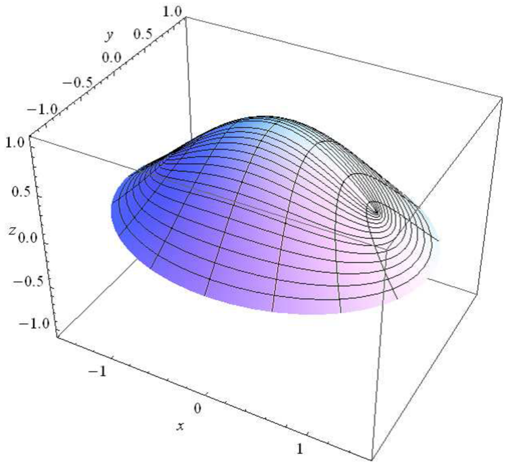

Figure 4. Eigenvalue 3.30141.

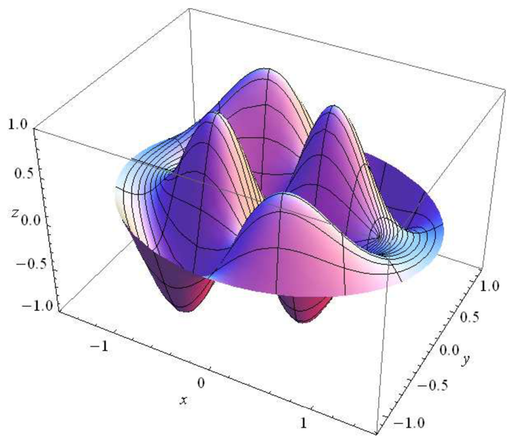

Figure 5. Eigenvalue 42.3785.

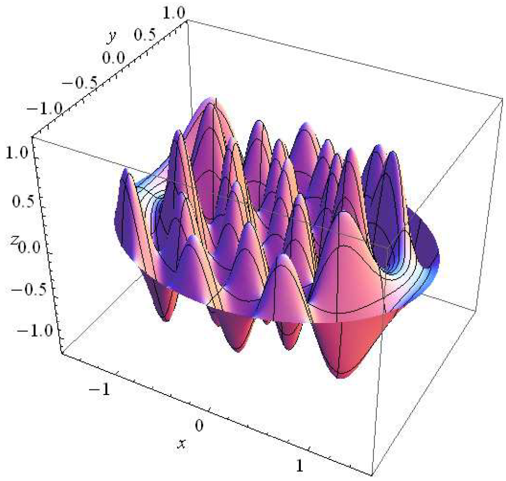

Figure 6. Eigenvalue 187.995.

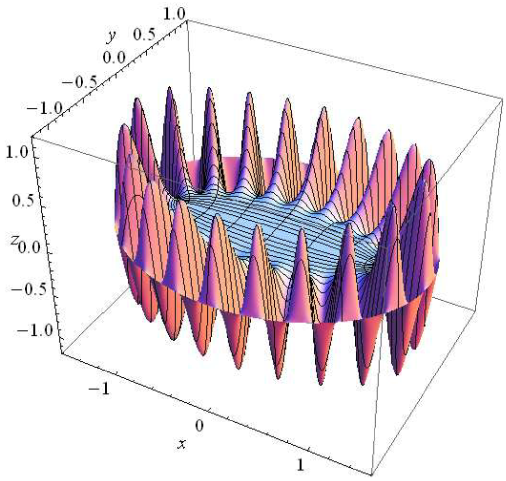

Figure 7. Eigenvalue 344.554.

where $\theta_1 = \arg(x - a + yi)$, $\theta_2 = \arg(x + a + yi)$. Put $v = e^{-i(\alpha_1\theta_1+\alpha_2\theta_2)}u$, then v is a multi-valued function unless $\alpha_1, \alpha_2 \in \mathbb{Z}$. The eigenvalue problem (3.1)-(3.3) is translated into

$$-\Delta v = \lambda v \quad \text{in } E_b, \tag{3.4}$$

$$v = 0 \quad \text{on } \partial E_b, \tag{3.5}$$

$$\limsup_{(x,y)\to(\pm a,0)} |v(x,y)| < \infty.$$

Since the equations (3.4) and (3.5) are the same as (2.1) and (2.2), we can again put $v = f(\xi)g(\eta)$ and use the method of separation of variables. In this way Gu–Qian obtain the eigenfunctions for H_{α_1,α_2}.

However, we do not still consider the matching condition. The multi-valuedness of v requires the following new matching conditions.

$$g(\eta + 2\pi) = e^{-2\pi(\alpha_1+\alpha_2)i}g(\eta), \tag{3.6}$$

$$f(-\xi)g(-\eta) = e^{-2\pi\alpha_1 i}f(\xi)g(\eta). \tag{3.7}$$

These conditions impose a strong restriction on the values α_1, α_2.

Proposition 2. Suppose that there exist functions f and g on $\mathbb{R}$ satisfying (3.6) and (3.7), and f and g are not identically equal to 0. Then, $\alpha_1, \alpha_2 \in (1/2)\mathbb{Z}$.

Proof. Using (3.7) twice, we have

$$\begin{aligned} f(\xi)g(\eta) &= f(-(-\xi))g(-(-\eta)) = e^{-2\pi\alpha_1 i}f(-\xi)g(-\eta) \\ &= (e^{-2\pi\alpha_1 i})^2 f(\xi)g(\eta), \end{aligned}$$

thus $e^{-2\pi\alpha_1 i} = \pm 1$ and $\alpha_1 \in (1/2)\mathbb{Z}$. Next, we have by (3.6) and (3.7)

$$\begin{aligned} f(\xi)g(\pi + \eta) &= e^{-2\pi(\alpha_1+\alpha_2)i}f(\xi)g(-\pi + \eta) \\ &= e^{-2\pi\alpha_2 i}f(-\xi)g(\pi - \eta). \end{aligned}$$

Using this equality twice, we conclude $\alpha_2 \in (1/2)\mathbb{Z}$ similarly. □

Gu–Qian take the first condition (3.6) into account, but seem to ignore the second condition (3.7). Thus their solutions have some discontinuity on the coordinate slit unless $\alpha_1, \alpha_2 \in (1/2)\mathbb{Z}$. Or they implicitly assume some discontinuous boundary conditions on the coordinate slit.

In the case $\alpha_1, \alpha_2 \in (1/2)\mathbb{Z}$, we can explicitly solve the eigenvalue problem (3.1)-(3.3). Here we only consider the case $\alpha_1 = \alpha_2 = 1/2$. Then f and g satisfy (2.4)-(2.6) and

$$g(\eta + 2\pi) = g(\eta), \tag{3.8}$$

$$f(-\xi)g(-\eta) = -f(\xi)g(\eta). \tag{3.9}$$

The conditions (2.5) and (3.8) imply g is again the Mathieu function of integral order. The conditions (2.4) and (3.9) imply f is the modified Mathieu function, but f and g have different parities. So f is a non-periodic Mathieu function (see section 5). Thus we conclude the following.

Theorem 3. Suppose $\alpha_1 = \alpha_2 = 1/2$. Let (λ, u) satisfy one of the following systems of equations

$$\begin{cases} \mathrm{Fe}_n(b, a^2\lambda/4) = 0, \\ u = e^{i(\theta_1+\theta_2)/2}\mathrm{Fe}_n(\xi, a^2\lambda/4)\mathrm{ce}_n(\eta, a^2\lambda/4), \end{cases} \qquad (n = 0, 1, 2, \dots), \tag{3.10}$$

$$\begin{cases} \mathrm{Ge}_n(b, a^2\lambda/4) = 0, \\ u = e^{i(\theta_1+\theta_2)/2}\mathrm{Ge}_n(\xi, a^2\lambda/4)\mathrm{se}_n(\eta, a^2\lambda/4), \end{cases} \qquad (n = 1, 2, \dots). \tag{3.11}$$

Then, λ is an eigenvalue for the problem (3.1)-(3.3) with $\alpha_1 = \alpha_2 = 1/2$ and u is the corresponding eigenfunction. Moreover, the family of all the eigenfunctions u given above forms an orthogonal basis in $L^2(E_b)$.

Proof. The proof of the completeness is similar to that of Theorem 1. We can prove the family of the functions $\{\mathrm{Fe}_n(\xi, a^2\lambda/4)\mathrm{ce}_n(\eta, a^2\lambda/4)\}_{(\lambda,n)}$ ((λ, n) runs over all the pairs satisfying the first equation in (3.10)) is an orthogonal basis of $\mathcal{H}_{\mathrm{even}}$, simply by replacing the boundary conditions in (2.11) with $f(0) = f(b) = 0$. Similarly, we can prove the family $\{\mathrm{Ge}_n(\xi, a^2\lambda/4)\mathrm{se}_n(\eta, a^2\lambda/4)\}_{(\lambda,n)}$ is an orthogonal basis of $\mathcal{H}_{\mathrm{odd}}$. Since the multiplication by $e^{i(\theta_1+\theta_2)/2}$ is unitary, we have the conclusion. $\square$

The numerical result for $a = 1$, $b = 1$ are illustrated in Figure 8-11. Since the eigenfunctions are complex-valued, we give the graph of the real part and that of the imaginary part separately.

§ 4. Twisted Laplacian

Let us reformulate Proposition 2 from the viewpoint of the Galois theory for the covering manifold. We regard the elliptic coordinate (1.1) as a covering map

$$p : N \ni (\xi, \eta) \mapsto (a \cosh \xi \cos \eta,\ a \sinh \xi \sin \eta) \in M,$$

Dirichlet eigenfunctions of $H_{1/2,1/2}$ on an ellipse

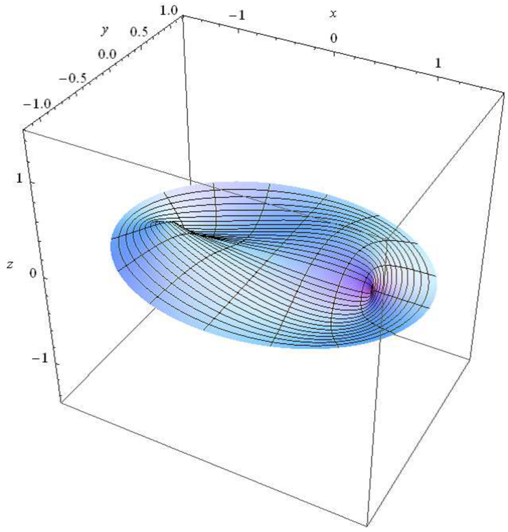

Figure 8. Eigenvalue 3.7379. Real part.

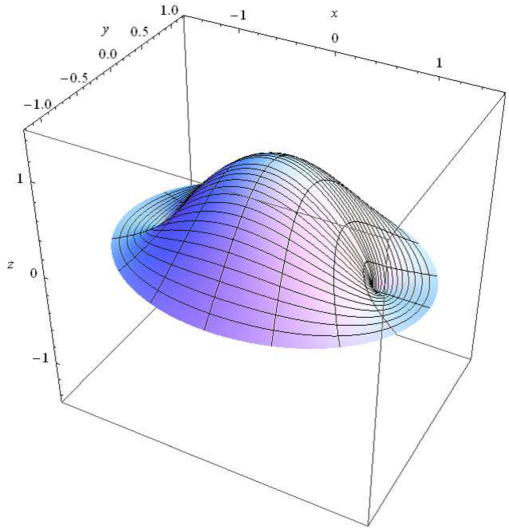

Figure 9. Eigenvalue 3.7379. Imaginary part.

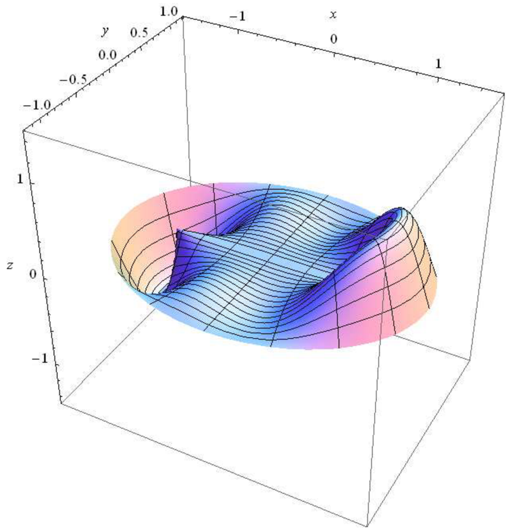

Figure 10. Eigenvalue 20.1521. Real part.

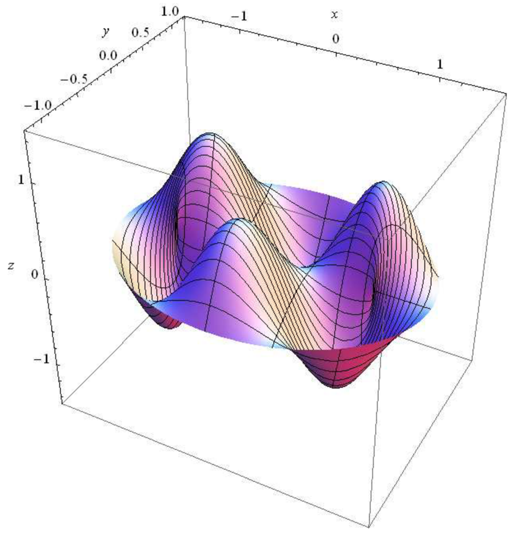

Figure 11. Eigenvalue 20.1521. Imaginary part.

where the covering manifold $N = \mathbb{R}^2 \setminus \{(0, n\pi) \mid n \in \mathbb{Z}\}$ and the base manifold $M = \mathbb{R}^2 \setminus (F_1 \cup F_2)$. Let $G = \mathrm{Gal}(N/M)$ be the covering group corresponding to the covering map $p : M \to N$, that is,

$$\mathrm{Gal}(N/M) = \{g : N \to N, \text{homeomorphism} \mid p \circ g = p\}.$$

The group $G = \mathrm{Gal}(N/M)$ is generated by two elements σ and τ, where

$$\sigma : (\xi, \eta) \mapsto (-\xi, -\eta), \quad \tau : (\xi, \eta) \mapsto (\xi, \eta + 2\pi).$$

The fundamental relations of G are

$$\sigma^2 = \mathrm{id}, \quad (\sigma\tau)^2 = \mathrm{id}. \tag{4.1}$$

Thus G is regarded as the limit of the dihedral group D_n as $n \to \infty$.

For a unitary representation $\rho : G \to U(n)$, define a Hilbert space $\mathcal{H}_\rho$ by

$$\begin{aligned}\mathcal{H}_\rho = \{v \in L^2_{\mathrm{loc}}(N; \mathbb{C}^n) \mid v \circ g = \rho(g)v \ \text{ for any } g \in G,\\ \|v\|^2_{\mathcal{H}_\rho} = \int_M |v \circ p^{-1}|^2 dxdy < \infty\}.\end{aligned}$$

Notice that $|v|$ is invariant under the action of G and the norm is well-defined. For $v \in \mathcal{H}_\rho \cap H^2_{\mathrm{loc}}(N; \mathbb{C}^n)$, we define

$$-\Delta_\rho v = -\Delta v = -\frac{1}{J}\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right) v,$$

where the act of differential operators is componentwise. The operator $-\Delta_\rho$ is called the *Laplacian associated with the unitary representation* ρ, or the *twisted Laplacian* (see [Su1, Su2]).

For a $U(1)$-representation ρ, the twisted Laplacian $-\Delta_\rho$ is unitarily equivalent to the Schrödinger operator H_{α_1,α_2}, where

$$\rho(\tau) = e^{-2\pi i(\alpha_1+\alpha_2)}, \ \rho(\sigma) = e^{-2\pi i\alpha_1}.$$

However, the group G has only four $U(1)$-representations.

Proposition 4. All the $U(1)$-representations of G are given by

$$(\rho(\sigma), \ \rho(\tau)) = (1, 1), \ (1, -1), \ (-1, 1), \ (-1, -1).$$

Proof. Since $U(1)$-representations are commutative, we have by the fundamental relations (4.1)

$$\rho(\sigma)^2 = 1, \ \rho(\sigma)^2\rho(\tau)^2 = \rho(\tau)^2 = 1.$$

This implies $\rho(\sigma) = \pm 1$ and $\rho(\tau) = \pm 1$. $\square$

Proposition 4 gives another interpretation of Proposition 2, and also suggests us the gauge group $U(1)$ is too small to represent the group G. This fact naturally leads us to consider larger gauge group $U(2)$. Actually, G has plenty of $U(2)$-representations. For example, for $\alpha \in \mathbb{R}$

$$\rho_\alpha(\sigma) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \rho_\alpha(\tau) = \begin{pmatrix} \cos 2\pi\alpha & -\sin 2\pi\alpha \\ \sin 2\pi\alpha & \cos 2\pi\alpha \end{pmatrix} \tag{4.2}$$

gives a $U(2)$-representation. The basic properties are as follows.

(i) ρ_α is irreducible $\Leftrightarrow \alpha \notin \frac{1}{2}\mathbb{Z}$.

(ii) ρ_α and $\rho_{\alpha+1}$ are the same representations.

(iii) ρ_α and $\rho_{1-\alpha}$ are equivalent representations.

(iv) ρ_0 is the direct sum of the two $U(1)$-representations $(\rho(\sigma), \rho(\tau)) = (1, 1)$, $(-1, 1)$. $\rho_{1/2}$ is the direct sum of the two $U(1)$-representations $(\rho(\sigma), \rho(\tau)) = (1, -1)$, $(-1, -1)$.

The statement (iv) explains the mathematical meaning of (1.2). Of course, there exist infinitely many representations equivalent to (4.2). We choose here (4.2) since this representation has only real components, and the eigenfunctions can also be taken real functions.

Let us consider the Dirichlet eigenvalue problem on E_b for the twisted Laplacian $-\Delta_{\rho_\alpha}$ associated with the $U(2)$-representation ρ_α given above. That is, we consider the following eigenvalue problem for the vector-valued function $v(\xi, \eta) = \begin{pmatrix} v_1(\xi, \eta) \\ v_2(\xi, \eta) \end{pmatrix}$ on N.

$$-\Delta v(\xi, \eta) = \lambda v(\xi, \eta), \tag{4.3}$$

$$v(\xi, \eta + 2\pi) = \begin{pmatrix} \cos 2\pi\alpha & -\sin 2\pi\alpha \\ \sin 2\pi\alpha & \cos 2\pi\alpha \end{pmatrix} v(\xi, \eta), \tag{4.4}$$

$$v(-\xi, -\eta) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} v(\xi, \eta), \tag{4.5}$$

$$v(b, \eta) = 0 \tag{4.6}$$

for $|\xi| < b$ and $\eta \in \mathbb{R}$, and

$$\limsup_{(\xi,\eta) \to (0, n\pi)} |v(\xi, \eta)| < \infty \tag{4.7}$$

for $n \in \mathbb{Z}$. The last condition is necessary for the self-adjointness of the problem. We can solve this problem again by applying Mathieu's method after diagonalizing the boundary conditions (4.4) and (4.5).

Theorem 5. Let $\alpha \in \mathbb{R} \setminus \mathbb{Z}$. Let (λ, v) satisfy one of the following systems of equations

$$\begin{cases} \mathrm{Ce}_{n+\alpha}(b, a^2\lambda/4) = 0, \\ v = \mathrm{Ce}_{n+\alpha}(\xi, a^2\lambda/4) \begin{pmatrix} \mathrm{ce}_{n+\alpha}(\eta, a^2\lambda/4) \\ \mathrm{se}_{n+\alpha}(\eta, a^2\lambda/4) \end{pmatrix} \end{cases} \quad (n \in \mathbb{Z}), \tag{4.8}$$

$$\begin{cases} \mathrm{Se}_{n+\alpha}(b, a^2\lambda/4) = 0, \\ v = \mathrm{Se}_{n+\alpha}(\xi, a^2\lambda/4) \begin{pmatrix} -\mathrm{se}_{n+\alpha}(\eta, a^2\lambda/4) \\ \mathrm{ce}_{n+\alpha}(\eta, a^2\lambda/4) \end{pmatrix} \end{cases} \quad (n \in \mathbb{Z}) \tag{4.9}$$

Then, λ is an eigenvalue for the eigenvalue problem (4.3)-(4.7) and v is the corresponding eigenfunction. Moreover, the family of all the eigenfunctions v given above forms an orthogonal basis in $\mathcal{H}_{\rho_\alpha}$.

Proof. Put

$$v = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} w, \quad w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

Then the equations (4.3)-(4.7) are equivalent to

$$-\Delta w_1(\xi, \eta) = \lambda w_1(\xi, \eta), \tag{4.10}$$

$$w_1(\xi, \eta + 2\pi) = e^{2\pi\alpha i} w_1(\xi, \eta), \tag{4.11}$$

$$w_2(\xi, \eta) = w_1(-\xi, -\eta), \tag{4.12}$$

$$w_1(b, \eta) = 0 \tag{4.13}$$

for $|\xi| < b$ and $\eta \in \mathbb{R}$, and

$$\limsup_{(\xi,\eta)\to(0,n\pi)} |w_1(\xi, \eta)| < \infty \tag{4.14}$$

for $n \in \mathbb{Z}$. Put $w_1 = f(\xi)g(\eta)$. Then using the separation of variables method in (4.10), (4.11) and (4.13), we have a multi-parameter eigenvalue problem on $[0, b] \times [-\pi, \pi]$

$$\begin{cases} f''(\xi) + \big((a^2 \cosh 2\xi/2)\lambda - m\big) f(\xi) = 0, \\ f(b) = 0, \end{cases} \tag{4.15}$$

$$\begin{cases} g''(\eta) + \big(-(a^2 \cos 2\eta/2)\lambda + m\big) g(\eta) = 0, \\ g(\pi) = e^{2\pi\alpha i} g(-\pi), \;\; g'(\pi) = e^{2\pi\alpha i} g'(-\pi). \end{cases} \tag{4.16}$$

We need additional boundary condition of f at $\xi = 0$ for the self-adjointness of the problem. First we impose

$$f'(0) = 0, \tag{4.17}$$

then the problem (4.15), (4.16) and (4.17) is a right-definite multi-parameter eigenvalue problem. The solutions are

$$f(\xi) = \mathrm{Ce}_{n+\alpha}(\xi, a^2\lambda/4), \;\; g(\eta) = \mathrm{me}_{n+\alpha}(\eta, a^2\lambda/4) \quad (n \in \mathbb{Z}),$$

where λ satisfies the boundary condition

$$\mathrm{Ce}_{n+\alpha}(b, a^2\lambda/4) = 0. \tag{4.18}$$

Then we have by (4.12)

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \mathrm{Ce}_{n+\alpha}(\xi, a^2\lambda/4) \begin{pmatrix} \mathrm{me}_{n+\alpha}(\eta, a^2\lambda/4) \\ \mathrm{me}_{n+\alpha}(-\eta, a^2\lambda/4) \end{pmatrix} \tag{4.19}$$

and

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} w_1 + w_2 \\ (w_1 - w_2)/i \end{pmatrix} = 2\mathrm{Ce}_{n+\alpha}(\xi, a^2\lambda/4) \begin{pmatrix} \mathrm{ce}_{n+\alpha}(\eta, a^2\lambda/4) \\ \mathrm{se}_{n+\alpha}(\eta, a^2\lambda/4) \end{pmatrix}.$$

Dividing these functions by 2, we obtain the solutions (4.8).

Next we impose $f(0) = 0$ instead of (4.17). Then we have similarly

$$f(\xi) = \mathrm{Se}_{n+\alpha}(\xi, a^2\lambda/4), \;\; g(\eta) = \mathrm{me}_{n+\alpha}(\eta, a^2\lambda/4) \quad (n \in \mathbb{Z}),$$

where λ satisfies the boundary condition

$$\mathrm{Se}_{n+\alpha}(b, a^2\lambda/4) = 0. \tag{4.20}$$

Then we have by (4.12)

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \mathrm{Se}_{n+\alpha}(\xi, a^2\lambda/4) \begin{pmatrix} \mathrm{me}_{n+\alpha}(\eta, a^2\lambda/4) \\ -\mathrm{me}_{n+\alpha}(-\eta, a^2\lambda/4) \end{pmatrix} \tag{4.21}$$

and

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} w_1 + w_2 \\ (w_1 - w_2)/i \end{pmatrix} = -2i\mathrm{Se}_{n+\alpha}(\xi, a^2\lambda/4) \begin{pmatrix} -\mathrm{se}_{n+\alpha}(\eta, a^2\lambda/4) \\ \mathrm{ce}_{n+\alpha}(\eta, a^2\lambda/4) \end{pmatrix}.$$

Thus we also obtain the solutions (4.9).

Lastly, we prove the completeness of the above eigenfunctions in the Hilbert space $\mathcal{H}_{\rho_\alpha} \simeq \mathcal{H} = L^2([0, b] \times [-\pi, \pi]; Jd\xi d\eta)^2$. Since the problem (4.15), (4.16), (4.17) is right-definite, we can apply Theorem 6. Then the family of the solutions w_1 given in the first component of (4.19) satisfying (4.18) forms an orthogonal basis of $L^2([0, b] \times [-\pi, \pi]; Jd\xi d\eta)$. Thus the family of the solutions (4.19) satisfying (4.18) forms an orthogonal basis of the subspace

$$\left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathcal{H} \mid y_2(\xi, \eta) = y_1(\xi, -\eta) \right\}. \tag{4.22}$$

Similarly, the family of the solutions (4.21) satisfying (4.20) forms an orthogonal basis of the subspace

$$\left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathcal{H} \mid y_2(\xi, \eta) = -y_1(\xi, -\eta) \right\}. \tag{4.23}$$

Since $\mathcal{H}$ is the direct sum of the subspaces (4.22) and (4.23) and the correspondence $w \mapsto v/\sqrt{2}$ is a unitary operator on $\mathcal{H}$, we obtain the conclusion. □

We illustrate the graphs of the components v_1, v_2 of the eigenfunction v for the lowest eigenvalue 3.3944 in the case $a = 1$, $b = 1$ and $\alpha = 1/3$, in Figure 12-15. Notice that v_1, v_2 are multi-valued functions, but $|v|^2 = v_1^2 + v_2^2$ is a single-valued function. Notice also that this is *not* an eigenfunction for the Schrödinger operator $H_{1/3,1/3}$.

We also illustrate the graph of the lowest two eigenvalues of $-\Delta_{\rho_\alpha}$ in Figure 16. This graph actually shows that the operators $-\Delta_{\rho_\alpha}$ $(0 < \alpha < 1/2)$ connect four solvable Schrödinger operators as we stated in (1.2).

Dirichlet eigenfunctions of $-\Delta_{\rho_{1/3}}$ on an ellipse

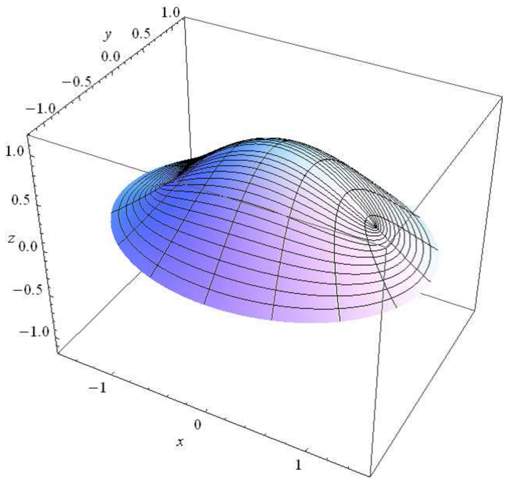

Figure 12. Eigenvalue 3.3944. Graph of v_1 on $-\pi \leq \eta \leq \pi$.

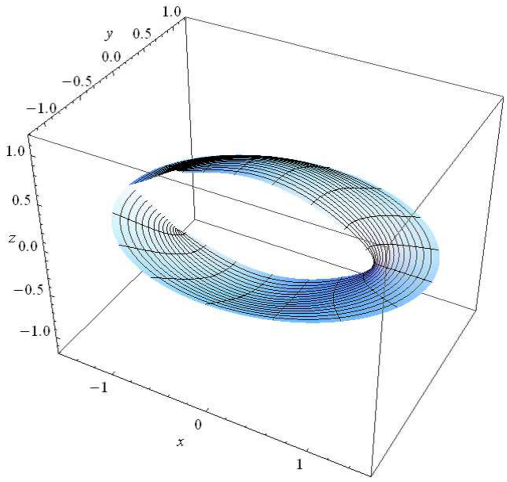

Figure 13. Eigenvalue 3.3944. Graph of v_2 on $-\pi \leq \eta \leq \pi$.

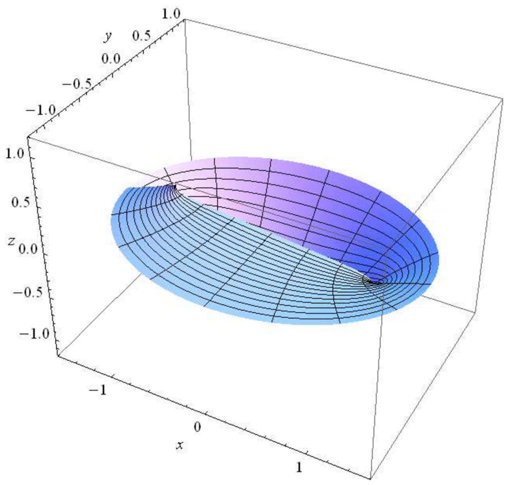

Figure 14. Eigenvalue 3.3944. Graph of v_1 on $\pi \leq \eta \leq 3\pi$.

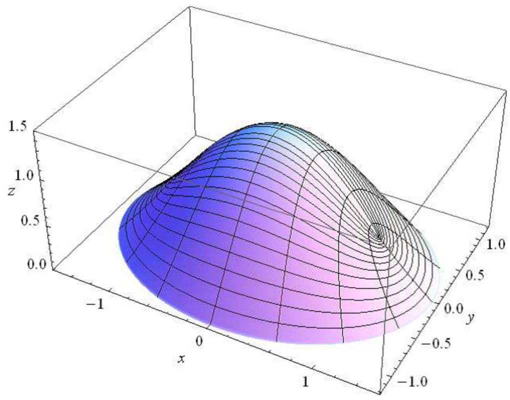

Figure 15. Eigenvalue 3.3944. Graph of $v_1^2 + v_2^2$ (single-valued).

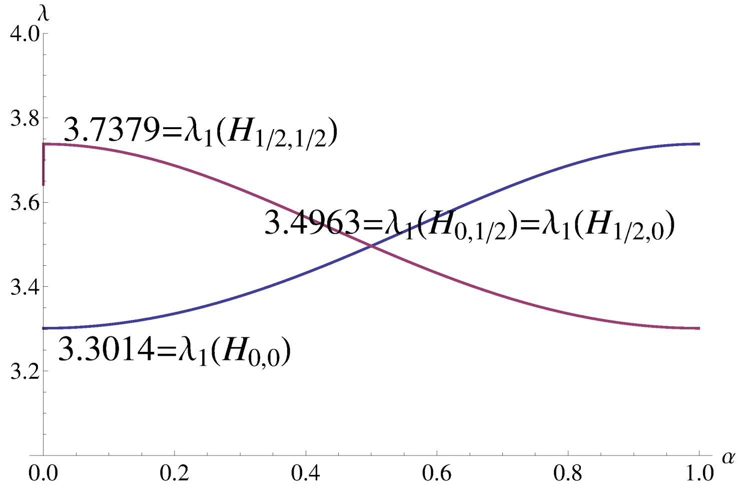

Figure 16. The graphs of the lowest two eigenvalues of $-\Delta_{\rho_\alpha}$. We see the lowest two eigenvalues of $-\Delta_{\rho_\alpha}$ connect those of $H_{0,0} \oplus H_{1/2,1/2}$ and of $H_{1/2,0} \oplus H_{0,1/2}$.

§ 5. Appendix

§ 5.1. Mathieu functions

In this section, we review the definitions of various Mathieu functions. For the reference, see McLachlan [Mc], Meixner–Schäfke–Wolf [Me-Sc-Wo], Abramowitz–Stegun [Ab-St], or Wang–Guo [Wa-Gu].

For $\mu,\ q \in \mathbb{C}$, the *Mathieu equation* is defined by

$$\frac{d^2y}{dz^2} + (\mu - 2q\cos 2z)y = 0. \tag{5.1}$$

If we replace z in (5.1) by iz, we obtain the *modified Mathieu equation*

$$\frac{d^2y}{dz^2} + (2q\cosh 2z - \mu)y = 0. \tag{5.2}$$

Since the coefficients of (5.1) and (5.2) have no singularity on the complex plane, these equations have two linearly independent solutions holomorphic on the entire complex plane.

In the sequel, we always assume $q \in \mathbb{R}$. Then the equation (5.1) can be viewed as the eigenequation for the Schrödinger operator with periodic potential

$$H_q = -\frac{d^2}{dx^2} + 2q\cos 2x \quad \text{ on } L^2(\mathbb{R}).$$

The Floquet-Bloch theory tells us the equation (5.1) has a non-trivial bounded solution

y if and only if

$$\mu \in \sigma(H_q) = \bigcup_{n=1}^{\infty} [a_{n-1}(q), b_n(q)], \quad \text{(band structure)}$$

where $\sigma(H_q)$ is the spectrum of H_q and

$$\begin{aligned} &a_0(q) < b_1(q) \le a_1(q) < b_2(q) \le \cdots \\ &\quad \cdots \le a_{n-1}(q) < b_n(q) \le a_n(q) < \cdots . \end{aligned}$$

When $q \neq 0$, all the inequalities become the strict inequalities. The values $a_n(q)$, $b_n(q)$ are called the *Mathieu characteristic values.* For $\mu = a_n(q)$ (resp. $\mu = b_n(q)$), the equation (5.1) has a non-trivial *even* (resp. *odd*) periodic solution $y = \mathrm{ce}_n(z,q)$ (resp. $y = \mathrm{se}_n(z,q)$) with period 2π. The solutions ce_n and se_n are called the *Mathieu functions of integral order.* Especially when $q = 0$, we have simply

$$\begin{aligned} a_n(0) &= n^2, \quad \mathrm{ce}_n(z,0) = \cos nz, \quad (n = 0,1,2,\ldots), \\ b_n(0) &= n^2, \quad \mathrm{se}_n(z,0) = \sin nz, \quad (n = 1,2,\ldots). \end{aligned}$$

There are two ways for the normalization of the Mathieu functions (with respect to the value at 0, or the L^2-norm on $[-\pi,\pi]$). We do not fix the normalization here, but at least we assume these functions are real for $z \in \mathbb{R}$ and continuous with respect to q.

For $\mu \in (a_{n-1}(q), b_n(q))$ for some n, we can take $r \in (0,\infty) \setminus \mathbb{Z}$ such that there exist two independent solutions $y = \mathrm{me}_{\pm r}(z,q)$ of (5.1) satisfying the *Bloch wave condition*

$$\mathrm{me}_{\pm r}(z + 2\pi, q) = e^{\pm 2\pi i r}\mathrm{me}_{\pm r}(z,q)$$

and

$$\mathrm{me}_r(-z,q) = \mathrm{me}_{-r}(z,q), \quad \mathrm{me}_r(0,q) = \mathrm{me}_{-r}(0,q) > 0.$$

The value $r = r(\mu,q)$ is uniquely determined if we assume $r(\mu,q)$ is continuous with respect to $\mu \in (a_{n-1}(q), b_n(q))$ and

$$\lim_{\mu \to a_{n-1}(q)+0} r(\mu,q) = n-1, \qquad \lim_{\mu \to b_n(q)-0} r(\mu,q) = n$$

for every $n = 1,2,\ldots$. The value $r = r(\mu,q)$ is called the *Mathieu characteristic exponent.* For $r \in \mathbb{R} \setminus \mathbb{Z}$, put

$$\begin{aligned} \mathrm{ce}_r(z,q) &= \frac{1}{2}(\mathrm{me}_r(z,q) + \mathrm{me}_{-r}(z,q)), \\ \mathrm{se}_r(z,q) &= \frac{1}{2i}(\mathrm{me}_r(z,q) - \mathrm{me}_{-r}(z,q)). \end{aligned}$$

Then ce_r is an even bounded solution of (5.1), and se_r is an odd bounded one. These functions are called the *Mathieu functions of fractional order.* Especially when $q = 0$, we have

$$r(\mu, 0) = \sqrt{\mu}, \quad \mathrm{ce}_r(z, 0) = \cos rz, \ \mathrm{se}_r(z, 0) = \sin rz.$$

When $q \neq 0$ and $\mu = a_n(q)$, the equation (5.1) has a non-trivial odd solution $y = \mathrm{fe}_n(z, q)$ which takes real values for real z and is independent of the even periodic solution $\mathrm{ce}_n(z, q)$. Similarly, when $q \neq 0$ and $\mu = b_n(q)$, the equation (5.1) has a non-trivial even solution $y = \mathrm{ge}_n(z, q)$ which takes real values for real z and is independent of the odd periodic solution $\mathrm{se}_n(z, q)$. It is known that the solutions fe_n and ge_n are non-periodic and unbounded, so these solutions are called the *non-periodic Mathieu functions.*

Further we define the *modified Mathieu functions*, which are the solutions of the modified Mathieu equation (5.2), by

$$\begin{aligned}
&\mathrm{Ce}_r(z, q) = \mathrm{ce}_r(iz, q) \quad (r = 0, 1, 2, \ldots \text{ or } r \in \mathbb{R} \setminus \mathbb{Z}),\\
&\mathrm{Se}_r(z, q) = \mathrm{se}_r(iz, q)/i \quad (r = 1, 2, \ldots \text{ or } r \in \mathbb{R} \setminus \mathbb{Z}),\\
&\mathrm{Fe}_n(z, q) = \mathrm{fe}_n(iz, q)/i \quad (n = 0, 1, 2, \ldots),\\
&\mathrm{Ge}_n(z, q) = \mathrm{ge}_n(iz, q) \quad (n = 1, 2, \ldots).
\end{aligned}$$

The functions Ce_r and Ge_n are even functions, and Se_r and Fe_n are odd functions. Notice that all these functions take real values when z is real.

§ 5.2. Multi-parameter spectral analysis

In this subsection, we quote the eigenfunction expansion theorem for the right-definite multi-parameter eigenvalue problem from the Volkmer's book [Vo]. See also Atkinson [At], Sleeman [Sl] and references therein for more detailed information.

We consider the multi-parameter eigenvalue problem

$$\frac{d^2 u_r}{d{x_r}^2} + \sum_{s=1}^{k} \lambda_s a_{rs}(x_r) u_r = 0, \quad x_r \in [b_r, c_r], \tag{5.3}$$

$$\alpha_{r1} u_r(b_r) + \alpha_{r2} u_r'(b_r) + \alpha_{r3} u_r(c_r) + \alpha_{r4} u_r'(c_r) = 0, \tag{5.4}$$

$$\beta_{r1} u_r(b_r) + \beta_{r2} u_r'(b_r) + \beta_{r3} u_r(c_r) + \beta_{r4} u_r'(c_r) = 0, \tag{5.5}$$

for $r = 1, \ldots, k$. Here $\lambda_1, \ldots, \lambda_k$ are real constants, $a_{rs}(x_r)$ are real-valued continuous functions on $[b_r, c_r]$, and α_{rj}, β_{rj} $(j = 1, \ldots, 4)$ are complex constants so that the operator

$$\begin{aligned} A_{r0}u_r &= \frac{d^2u_r}{dx_r^2}, \\ D(A_{r0}) &= \{u_r \in H^2([b_r, c_r]) \mid u_r \text{ satisfies (5.4) and (5.5)}\} \end{aligned}$$

is self-adjoint for $r = 1, \ldots, k$. Let A_{rs} be the multiplication operator by a_{rs}.

If the system of equations (5.3)-(5.5) has a non-zero solution u_r for every r, we say $\lambda = (\lambda_1, \ldots, \lambda_k)$ is an *eigenvalue*. For an eigenvalue λ, define the *eigenspace* $E(\lambda)$ by

$$E(\lambda) = E_1(\lambda) \otimes \cdots \otimes E_k(\lambda), \quad E_r(\lambda) = \mathrm{Ker}\left(A_{r0} + \sum_{s=1}^{k} \lambda_s A_{rs}\right).$$

$E(\lambda)$ is regarded as a subspace of $L^2(\Pi; dx)$, where $\Pi = [b_1, c_1] \times \cdots \times [b_k, c_k]$. We call a non-zero element of $E(\lambda)$ an *eigenvector*. We say an eigenvalue λ has *index* $i = (i_1, \ldots, i_k) \in \mathbb{N}^k$ ($\mathbb{N} = \{1, 2, \ldots\}$) if 0 is the i_r-th eigenvalue (from above, counting multiplicity) of the operator $A_{r0} + \sum_{s=1}^{k} \lambda_s A_{rs}$. We say the problem (5.3)-(5.5) is *right-definite* if

$$d_0(x) = \det(a_{rs}(x_r))_{r=1,\ldots,k,\ s=1,\ldots,k} > 0$$

for every $x = (x_1, \ldots, x_k)$ in some dense subset of Π.

Theorem 6 ([Vo]). Assume the multi-parameter eigenvalue problem (5.3)-(5.5) is right-definite. Then, there is a system $\{\lambda^i, u^i\}_{i\in\mathbb{N}^k}$ satisfying the following conditions 1.- 4.

1. $\lambda^i = (\lambda_1^i, \ldots, \lambda_k^i) \in \mathbb{R}^k$ is an eigenvalue with index i.
2. u^i is an eigenvector for the eigenvalue λ^i.
3. u^i is decomposable, that is, u^i is written as
$$u^i(x) = u_1^i(x_1) \cdots u_k^i(x_k).$$
4. $\{u^i\}_{i\in\mathbb{N}^k}$ is an orthonormal basis in $L^2(\Pi; d_0(x)dx)$.

Theorem 6 is almost included in [Vo, Theorem 6.8.3],[2] which is a corollary of more general expansion theorem [Vo, Theorem 6.5.2]. Especially, Theorem 6 implies the family of all the decomposable eigenfunctions for the right-definite problem forms a complete set in $L^2(\Pi; d_0(x)dx)$. This fact is frequently used in previous sections.

[2]In Volkmer's theorem [Vo, Theorem 6.8.3] the coefficients α_{rj}, β_{rj} are assumed to be real, but the proof works for the complex coefficients if we assume the self-adjointness of the operator A_{r0}. If the coefficients are real, we can relate the index i with the numbers of zeros of eigenfunctions u_r^i via the Strum-Liouville oscillation theory.

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