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A periodic Schrödinger operator with two degenerate spectral gaps

By

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Abstract

In this paper, we consider the one-dimensional Schrödinger operators with periodic generalized point interactions. Let us suppose that there are 4 point interactions in the basic period cell $[0, 2\pi)$. Moreover, we assume that each point interaction is given by a rotation. Under these assumption, we investigate the coexistence problem. Especially, we construct a periodic Schrödinger operators with exactly two degenerate spectral gaps.

§ 1. Introduction and main result

In this article, we consider the spectrum of the one-dimensional Schrödinger operators with periodic point interactions. By the Floquet–Bloch theory, the spectrum of the Schrödinger operators with periodic potentials has the band structure. Namely, its spectrum consists of infinitely many closed intervals. Two consecutive closed intervals are separated by an open interval, which is called the spectral gap. Each spectral gap can be the empty set. In [9], we constructed an example of the periodic Schrödinger operators with exactly two degenerate specrtal gaps. In this article, we survey the results.

To describe the main results, we introduce notations. For an open set $I \subset \mathbf{R}$, we introduce the Sobolev space

$$H^2(I) = \{y(x) \in L^2(I) | \quad y'(x), y''(x) \in L^2(I)\}.$$

Let $0 < \kappa_1 < \kappa_2 < \kappa_3 < \kappa_4 = 2\pi$ be a partition of the interval $(0, 2\pi)$. We put $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$, where $\Gamma_j = \{\kappa_j\} + 2\pi\mathbf{Z}$ for $j = 1, 2, 3, 4$. We denote by $SL(2, \mathbf{R})$ the special linear group, and by E the 2×2 unit matrix. For

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$\theta_1, \theta_2, \theta_3, \theta_4 \in \mathbf{R}$ and $A_1, A_2, A_3, A_4 \in SL(2, \mathbf{R}) \setminus \{E, -E\}$, we define the operator $H = H(\theta_1, \theta_2, \theta_3, \theta_4, A_1, A_2, A_3, A_4)$ in $L^2(\mathbf{R})$ as follows:

$$(Hy)(x) = -y''(x), \quad x \in \mathbf{R} \setminus \Gamma,$$

$$\mathrm{Dom}(H) = \left\{ y \in H^2(\mathbf{R} \setminus \Gamma) \;\middle|\; \begin{array}{c} \begin{pmatrix} y(x+0) \\ y'(x+0) \end{pmatrix} = e^{i\theta_j} A_j \begin{pmatrix} y(x-0) \\ y'(x-0) \end{pmatrix} \\ \text{for} \quad x \in \Gamma_j, \quad j = 1, 2, 3, 4 \end{array} \right\}.$$

This is called the generalized Kronig–Penney Hamiltonian by Hughes [3]. The Schrödinger operators with periodic point-interactions play an important role in the solid state physics. The theory of the point interactions is summarized in [1, 2]. The basic spectral properties of H have been studied in our previous work [6, Proposition 1.1]. Let us quote the results. The self-adjointness of the operator H is shown in a similar way to [5, Proposition 2.1]. Since the potential of H is periodic, H has an expression of the direct integral decomposition (see [10, Section XIII-16]). Namely, H is unitary equivalent to $\int_0^{2\pi} \oplus H_\mu d\mu$, where

$$(H_\mu y)(x) = -y''(x), \qquad x \in \mathbf{R} \setminus \Gamma,$$

$$\mathrm{Dom}(H_\mu) = \left\{ y \in \mathcal{H}_\mu \;\middle|\; \begin{array}{c} y \in W^{2,2}((0, 2\pi) \setminus \{\kappa_1, \kappa_2, \kappa_3\}), \\ \begin{pmatrix} y(x+0) \\ y'(x+0) \end{pmatrix} = e^{i\theta_j} A_j \begin{pmatrix} y(x-0) \\ y'(x-0) \end{pmatrix} \\ \text{for} \quad x \in \Gamma_j, \quad j = 1, 2, 3, 4 \end{array} \right\},$$

in the Hilbert space

$$\mathcal{H}_\mu = \{u \in L^2_{\mathrm{loc}}(\mathbf{R}) | \quad u(x + 2\pi) = e^{i\mu} u(x) \quad \text{for almost every } x \in \mathbf{R}\}$$

equipped with the inner product

$$\langle u, v \rangle_{\mathcal{H}_\mu} = \int_0^{2\pi} u(x)\overline{v(x)} dx, \qquad u, v \in \mathcal{H}_\mu$$

for $\mu \in \mathbf{R}$. Since the set $\sigma(H(\theta_1, \theta_2, \theta_3, \theta_4, A_1, A_2, A_3, A_4))$ is independent of $\{\theta_j\}_{j=1}^4$, we may assume that $\theta_1 = \theta_2 = \theta_3 = \theta_4 = 0$ without any loss of generality. We abbreviate $H(0, 0, 0, 0, A_1, A_2, A_3, A_4)$ to $H(A_1, A_2, A_3, A_4)$. We denote by $\lambda_j(\mu)$ the jth eigenvalue counted with multiplicity for $j \in \mathbf{N}$. Then, $\lambda_j(\cdot)$ is continuous on $[0, 2\pi]$ and strictly monotone function on $[0, \pi]$ for each $j \in \mathbf{N}$. Every eigenvalue $\lambda_j(\mu)$ is simple if $\mu \notin \pi\mathbf{Z}$. The spectrum of $H(A_1, A_2, A_3, A_4)$ is given by these eigenvalues as follows:

$$\sigma(H(A_1, A_2, A_3, A_4)) = \bigcup_{j=1}^{\infty} \lambda_j([0, \pi]) = \bigcup_{j=1}^{\infty} \bigcup_{\mu \in [0,\pi]} \{\lambda_j(\mu)\}.$$

For $j \in \mathbf{N}$, the closed interval B_j defined by $B_j = \lambda_j([0,\pi])$ is called the jth band of the spectrum of H. We put $B_j = [\lambda_{2j-2}, \lambda_{2j-1}]$ for $j \in \mathbf{N}$. Then, two consecutive bands B_j and B_{j+1} are separated by an open interval $G_j = (\lambda_{2j-1}, \lambda_{2j})$. This is called the jth spectral gap of H. If a spectral gap is degenerate, i.e., there exists some $j \in \mathbf{N}$ satisfying $G_j = \emptyset$, then the corresponding segments B_j and B_{j+1} merge. This implies that all solutions to the equations

$$-y''(x,\lambda) = \lambda y(x,\lambda), \quad x \in \mathbf{R} \setminus \Gamma, \tag{1.1}$$

$$\begin{pmatrix} y(x+0,\lambda) \\ y'(x+0,\lambda) \end{pmatrix} = A_j \begin{pmatrix} y(x-0,\lambda) \\ y'(x-0,\lambda) \end{pmatrix}, \qquad x \in \Gamma_j, \quad j = 1,2,3,4 \tag{1.2}$$

are 2π-periodic or 2π-antiperiodic for $\lambda = \lambda_{2j} (= \lambda_{2j-1})$. In this case, one says that the periodic solutions to (1.1) and (1.2) *coexist* (see [4]). The purpose of this work is to determine whether the jth spectral gap is degenerate or not for a given $j \in \mathbf{N}$. This problem is called the coexistence problem.

§ 2. Main results

We denote by $SO(2)$ the 2-dimensional rotation group. In [9], we solved the coexistence problem in the case where $A_1, A_2, A_3, A_4 \in SO(2) \setminus \{E, -E\}$. Under this assumption, we write the conponents of A_i as

$$A_i = \begin{pmatrix} a_i & b_i \\ -b_i & a_i \end{pmatrix}.$$

Furthermore, we suppose consider the following statements (I), (II), (III), (IV), and (V).

(I) $A_4 A_3 A_2 A_1 = E$.

(II) $(A_1, A_2) \neq (A_3, A_4), \quad (A_3, -A_4), \quad (-A_3, A_4), \quad (-A_3, -A_4)$.

(III) $\kappa_1 = \kappa \in (0, \pi/2) \cup (\pi/2, \pi)$, $\kappa_2 = \pi$, $\kappa_3 = \pi + \kappa$.

(IV) $a_1 b_2 + a_3 b_4 = 0$.

(V) $(a_1 a_2 - b_1 b_2) \sinh \kappa \cosh(\pi - \kappa) + (a_3 a_4 + b_1 b_2) \cosh \kappa \sinh(\pi - \kappa) = 0$.

Then, we have the following theorems.

Theorem 2.1. We suppose that (I), (II), (III), (IV), (V), and $\kappa/\pi \notin \mathbf{Q}$. Then, the periodic solutions to (1.1) and (1.2) coexist if and only if $\lambda = \pm 1$. Especially, if $(A_1 A_3, A_2 A_4) = \pm(E, E)$, then the second and fourth spectral gaps are degenerate and the jth gap is non-degenerate for $j \neq 2, 4$.

Theorem 2.2. We assume that (I),(II),(III), (IV), (V), $\kappa/2\pi = q/p(\neq 1/4)$, $(p,q) \in \mathbf{N}^2$, and $\gcd(p,q) = 1$. Then, we have the following statements (A), (B), (C) and (D).

(A) Suppose that $(A_1A_3, A_2A_4) \neq \pm(E,E)$. Then the periodic solutions to (1.1) and (1.2) coexist if and only if $\lambda \in \{1\} \cup \{-1\} \cup \{p^2j^2 | \quad j \in \mathbf{N}\}$.

(B) Suppose that $(A_1A_3, A_2A_4) = \pm(E,E)$, $a_1b_2 + a_2b_1 \neq 0$ and $a_1b_2 + a_4b_3 \neq 0$. Then, we have

$$G_j = \emptyset \quad \text{if and only if} \quad j \in \{2\} \cup \{4\} \cup \{2pj+2| \quad j \in \mathbf{N}\}.$$

(C) Assume that $(A_1A_3, A_2A_4) = \pm(E,E)$ and $a_1b_2 + a_2b_1 = 0$. Then, we have

$$G_j = \emptyset \quad \text{if and only if} \quad j \in \{2\} \cup \{4\} \cup \{pj+2| \quad j \in \mathbf{N}\}.$$

(D) Assume that $(A_1A_3, A_2A_4) = \pm(E,E)$, $a_1b_2 + a_2b_1 \neq 0$ and $a_1b_2 + a_4b_3 = 0$.

- If $p = 2p'$ and $p' \not\equiv 0 (\mathrm{mod}\ 2)$, then we have

$$G_j = \emptyset \quad \text{if and only if} \quad j \in \{2\} \cup \{4\} \cup \{2pj+2| \quad j \in \mathbf{N}\} \cup \{pj - \frac{p}{2} + 2| \quad j \in \mathbf{N}\}.$$

- Otherwise, the second and fourth gaps are degenerate.

§ 3. Outline of the proof of main results

In this section, we see the outline of the proof of Theorem 2.1 and 2.2. First, let us explain the method to calculate the degenerate points of the degenerate spectral gaps of H. Let $y_1(x,\lambda)$ and $y_2(x,\lambda)$ be the solutions to (1.1) and (1.2) subject to the initial conditions

$$y_1(+0,\lambda) = 1, \quad y_1'(+0,\lambda) = 0$$

and

$$y_2(+0,\lambda) = 0, \quad y_2'(+0,\lambda) = 1,$$

respectively. The set of all degenerate points $\mathcal{B} = \cup_{j=1}^{\infty} B_j \cap B_{j+1}$ is characterized by the monodromy matrix

$$M(\lambda) = \begin{pmatrix} m_{11}(\lambda) & m_{12}(\lambda) \\ m_{21}(\lambda) & m_{22}(\lambda) \end{pmatrix} = \begin{pmatrix} y_1(2\pi+0,\lambda) & y_2(2\pi+0,\lambda) \\ y_1'(2\pi+0,\lambda) & y_2'(2\pi+0,\lambda) \end{pmatrix}.$$

In fact, since all solutions to (1.1) and (1.2) for $\lambda \in \mathcal{B}$ are 2π-periodic or 2π-antiperiodic, we have

$$\mathcal{B} = \{\lambda \in \mathbf{R} | \quad M(\lambda) = E \quad \text{or} \quad M(\lambda) = -E\}.$$

We notice that every component of this matrix can be obtained explicitly. K. Yoshitomi [11] utilized this relationship in order to solve the coexistence peroblem for the one-dimensional Schrödinger operators with two periodic δ and δ'-interactions in the basic period cell $[0, 2\pi)$. He reduced the problem to a system of algebraic equations

$$y_1(2\pi+0,\lambda) - y_2'(2\pi+0,\lambda) = y_2(2\pi+0,\lambda) = y_1'(2\pi+0,\lambda) = 0. \tag{3.1}$$

In the case where the number of point interactions in $[0, 2\pi)$ is 2 or 3, this idea can be utilized for various types of point interactions (see [5, 7, 8]). However, not only the equation (3.1) for H is quite complicated but also it is hard to determine every solution $\lambda \in \mathbf{R}$ satisfying (3.1). To overcome this difficulty, we make use of the factrization of the monodromy matrix. We put $\tau = \pi - \kappa$. The monodromy matrix can be factrize as follows:

$$M(\lambda) = A_4T_4(\lambda)A_3T_3(\lambda)A_2T_2(\lambda)A_1T_1(\lambda), \tag{3.2}$$

where

$$T_1(\lambda) = T_3(\lambda) = \begin{pmatrix} \cos\kappa\sqrt{\lambda} & \frac{1}{\sqrt{\lambda}}\sin\kappa\sqrt{\lambda} \\ -\sqrt{\lambda}\sin\kappa\sqrt{\lambda} & \cos\kappa\sqrt{\lambda} \end{pmatrix},$$
$$T_2(\lambda) = T_4(\lambda) = \begin{pmatrix} \cos\tau\sqrt{\lambda} & \frac{1}{\sqrt{\lambda}}\sin\tau\sqrt{\lambda} \\ -\sqrt{\lambda}\sin\tau\sqrt{\lambda} & \cos\tau\sqrt{\lambda} \end{pmatrix}.$$

Due to this factrization, we reduce the monodromy equations $M(\lambda) = E$ and $M(\lambda) = -E$ to $A_2T_2(\lambda)A_1T_1(\lambda) = (A_4T_4(\lambda)A_3T_3(\lambda))^{-1}$ and $A_2T_2(\lambda)A_1T_1(\lambda) = -(A_4T_4(\lambda)A_3T_3(\lambda))^{-1}$, respectively. We put

$$v(\lambda) = \begin{pmatrix} \cos\kappa\sqrt{\lambda}\cos\tau\sqrt{\lambda} \\ \frac{\sin\kappa\sqrt{\lambda}\cos\tau\sqrt{\lambda}}{\sqrt{\lambda}} \\ \frac{\cos\kappa\sqrt{\lambda}\sin\tau\sqrt{\lambda}}{\sqrt{\lambda}} \\ \frac{\sin\kappa\sqrt{\lambda}\sin\tau\sqrt{\lambda}}{\lambda} \end{pmatrix}.$$

We note that there exist 4×4 matrices $\Phi_+(\lambda)$ and $\Phi_-(\lambda)$ such that $A_2T_2(\lambda)A_1T_1(\lambda) = (A_4T_4(\lambda)A_3T_3(\lambda))^{-1}$ and $A_2T_2(\lambda)A_1T_1(\lambda) = -(A_4T_4(\lambda)A_3T_3(\lambda))^{-1}$ are equivalent to

$$\Phi_+(\lambda)v(\lambda) = 0 \quad \text{and} \quad \Phi_-(\lambda)v(\lambda) = 0,$$

respectively. Solving these two equations, we obtain the set $\mathcal{B}$ explicitly. The explicit formula of $\Phi_{\pm}(\lambda)$ is given by $\Phi_{\pm}(\lambda) = (\varphi_1^{\pm}(\lambda) \quad \varphi_2^{\pm}(\lambda) \quad \varphi_3^{\pm}(\lambda) \quad \varphi_4^{\pm}(\lambda))$, where

$$\varphi_1^{\pm}(\lambda) = \begin{pmatrix} a_1a_2 - b_1b_2 \pm (b_3b_4 - a_3a_4) \\ -a_1b_2 - a_2b_1 \mp (a_3b_4 + a_4b_3) \\ a_2b_1 + a_1b_2 \pm (a_4b_3 + a_3b_4) \\ -b_1b_2 + a_1a_2 \pm (b_3b_4 - a_3a_4) \end{pmatrix},$$

$$\varphi_2^{\pm}(\lambda) = \begin{pmatrix} a_3b_4 + a_4b_3 \mp (a_2b_1 + a_1b_2)\lambda \\ -(-b_1b_2 + a_1a_2 \mp (b_3b_4 - a_3a_4))\lambda \\ a_1a_2 - b_1b_2 \pm (a_3a_4 - b_3b_4) \\ -a_1b_2 - a_2b_1 \pm (a_4b_3 + a_3b_4)\lambda \end{pmatrix},$$

$$\varphi_3^{\pm}(\lambda) = \begin{pmatrix} -a_2b_1 \pm a_3b_4 + (\pm a_4b_3 - a_1b_2)\lambda \\ b_1b_2 \pm b_3b_4 - (a_1a_2 \pm a_3a_4)\lambda \\ a_1a_2 \pm a_3a_4 - (b_1b_2 \pm b_3b_4)\lambda \\ -a_1b_2 \pm a_4b_3 + (\pm a_3b_4 - a_2b_1)\lambda \end{pmatrix},$$

$$\varphi_4^{\pm}(\lambda) = \begin{pmatrix} b_1b_2\lambda^2 + (\pm a_3a_4 - a_1a_2)\lambda \mp b_3b_4 \\ (a_2b_1 \pm a_4b_3)\lambda^2 + (a_1b_2 \pm a_3b_4)\lambda \\ (-a_1b_2 \mp a_3b_4)\lambda - a_2b_1 \mp a_4b_3 \\ \mp b_3b_4\lambda^2 + (\pm a_4a_3 - a_1a_2)\lambda + b_1b_2 \end{pmatrix}.$$

Next, we give 4 classifications on the equation $\Phi_+(\lambda)v(\lambda) = 0$. There exist some quadratic polynomials $f_1(\lambda)$ and $f_2(\lambda)$ such that

$$|\Phi_+(\lambda)| = (\lambda-1)^2(a_1a_2-b_1b_2+b_3b_4-a_3a_4)f_1(\lambda)-(\lambda-1)^2(a_1b_2+a_2b_1+a_3b_4+a_4b_3)f_2(\lambda),$$

where $|\Phi_+(\lambda)|$ implies the determinant of the matrix $\Phi_+(\lambda)$. On this formula, we give the following 4 classifications:

(a) $a_1a_2 - b_1b_2 + b_3b_4 - a_3a_4 = 0, \quad a_1b_2 + a_2b_1 + a_3b_4 + a_4b_3 = 0.$
(b) $a_1a_2 - b_1b_2 + b_3b_4 - a_3a_4 \neq 0, \quad a_1b_2 + a_2b_1 + a_3b_4 + a_4b_3 = 0.$
(c) $a_1a_2 - b_1b_2 + b_3b_4 - a_3a_4 = 0, \quad a_1b_2 + a_2b_1 + a_3b_4 + a_4b_3 \neq 0.$
(d) $a_1a_2 - b_1b_2 + b_3b_4 - a_3a_4 \neq 0, \quad a_1b_2 + a_2b_1 + a_3b_4 + a_4b_3 \neq 0.$

In this work, we consider the case (a), which is equivalent to (I).

Define $\mathcal{B}_+ = \{\lambda \in \mathbf{R} | \quad M(\lambda) = E\}$ and $\mathcal{B}_- = \{\lambda \in \mathbf{R} | \quad M(\lambda) = -E\}$. On the equation $\Phi_+(\lambda)v(\lambda) = 0$, we obtain the followings.

Lemma 3.1. Assume that (I), (II), (III). Then, we have the followings.

$$\mathcal{B}_+ = \begin{cases} \{-1\} \cup \{1\} \cup S & \text{if (IV) and (V) is valid,} \\ \{1\} \cup S & \text{otherwise,} \end{cases}$$

where $S = \{\lambda \in \mathbf{R} \setminus \{0\} | \quad \sin \kappa\sqrt{\lambda} = \sin \tau\sqrt{\lambda} = 0\}$.

In order to construct an example of the Schrödinger operators with two degenerate spectral gap, we assume that (I), (II), (III), (IV) and (V) are valid.

Next, we consider the equation $\Phi_-(\lambda)v(\lambda) = 0$. Using (I), we have

$$|\Phi_-(\lambda)| = 4(a_1b_2 + a_2b_1)(a_1a_2 - a_3a_4)(a_1b_1 - a_3b_3)(\lambda - 1)^4,$$

where $|\Phi_-(\lambda)|$ means the determinant of the matrix $\Phi_-(\lambda)$. By simple calculations with (IV) and $\kappa \neq \pi/2$, we have $a_1b_1 - a_3b_3 \neq 0$. On the equation $\Phi_-(\lambda)v(\lambda) = 0$, we have the followings.

Lemma 3.2. We have the following statements.

(a) If $(a_1b_2 + a_2b_1)(a_1a_2 - a_3a_4) \neq 0$, we have $\mathcal{B}_- = \emptyset$.

(b) If $a_1b_2 + a_2b_1 = 0$, then we have

$$\mathcal{B}_- = \{\lambda \in \mathbf{R} | \quad \sin \tau\sqrt{\lambda} = \cos \kappa\sqrt{\lambda} = 0\}.$$

(c) If $a_1b_2 + a_2b_1 \neq 0$ and $a_1a_2 - a_3a_4 = 0$, then we have the followings.

$$\mathcal{B}_- = \begin{cases} \emptyset & \text{if } a_1b_2 + a_4b_3 \neq 0, \\ \{\lambda \in \mathbf{R} | \quad \sin \kappa\sqrt{\lambda} = \cos \tau\sqrt{\lambda} = 0\} & \text{if } a_1b_2 + a_4b_3 = 0. \end{cases}$$

Next, we prepare two lemma. Put $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Using (IV), we have the following lemma.

Lemma 3.3. We have $(b_1b_2 + a_3a_4)(a_1a_2 - a_3a_4) = 0$. Furthermore, we have the followings.

- If $a_1a_2 - a_3a_4 = 0$, then we have $(A_1A_3, A_2A_4) = \pm(E, E)$.
- If $b_1b_2 + a_3a_4 = 0$, then we have $(A_1A_4, A_3A_2) = \pm(J, J)$.

Moreover, the following lemma follows from (IV) and (V).

Lemma 3.4. If $(A_1A_3, A_2A_4) \neq (E, E)$, then we obtain $a_1b_2 + a_2b_1 \neq 0$.

We notice that $(A_1A_3, A_2A_4) = \pm(E, E)$ implies that $a_1a_2 - a_3a_4 = 0$. Therefore, we have the followings by summarizing lemmas 3.1–3.4.

Lemma 3.5. We obtain the following statements (a), (b), (c) and (d).

(a) If$(A_1A_3, A_2A_4) \neq \pm(E, E)$,

$$\mathcal{B} = \{1\} \cup \{-1\} \cup \{\lambda \in \mathbf{R} \setminus \{0\} | \quad \sin \kappa\sqrt{\lambda} = \sin \tau\sqrt{\lambda} = 0\}.$$

(b) If $(A_1A_3, A_2A_4) = \pm(E,E)$, $a_1b_2 + a_2b_1 \neq 0$, $a_1b_2 + a_4b_3 \neq 0$, then

$$\mathcal{B} = \{1\} \cup \{-1\} \cup \{\lambda \in \mathbf{R} \setminus \{0\} | \quad \sin \kappa\sqrt{\lambda} = \sin \tau\sqrt{\lambda} = 0\}.$$

(c) If $(A_1A_3, A_2A_4) = \pm(E,E)$, $a_1b_2 + a_2b_1 = 0$, then

$$\begin{aligned} \mathcal{B} = \{1\} \cup \{-1\} \cup \{\lambda \in \mathbf{R} \setminus \{0\} | \quad & \sin \kappa\sqrt{\lambda} = \sin \tau\sqrt{\lambda} = 0\} \\ \cup\{\lambda \in \mathbf{R} \setminus \{0\} | \quad & \sin \kappa\sqrt{\lambda} = \cos \tau\sqrt{\lambda} = 0\}. \end{aligned}$$

(d) If $(A_1A_3, A_2A_4) = \pm(E,E)$, $a_1b_2 + a_2b_1 \neq 0$, $a_1b_2 + a_4b_3 = 0$, then

$$\begin{aligned} \mathcal{B} = \{1\} \cup \{-1\} \cup \{\lambda \in \mathbf{R} \setminus \{0\} | \quad & \sin \kappa\sqrt{\lambda} = \sin \tau\sqrt{\lambda} = 0\} \\ \cup\{\lambda \in \mathbf{R} \setminus \{0\} | \quad & \sin \tau\sqrt{\lambda} = \cos \kappa\sqrt{\lambda} = 0\}. \end{aligned}$$

This is why we need classifications in Theorem 2.2. Next, we enhance this lemma.

Lemma 3.6. We obtain the following statements (a) and (b).
(a) If $\kappa/\pi \notin \mathbf{Q}$, then we have the followings.

- $\{\lambda \in \mathbf{R} \setminus \{0\} | \quad \sin \kappa\sqrt{\lambda} = \sin \tau\sqrt{\lambda} = 0\} = \emptyset.$
- $\{\lambda \in \mathbf{R} \setminus \{0\} | \quad \sin \kappa\sqrt{\lambda} = \cos \tau\sqrt{\lambda} = 0\} = \emptyset.$
- $\{\lambda \in \mathbf{R} \setminus \{0\} | \quad \sin \tau\sqrt{\lambda} = \cos \kappa\sqrt{\lambda} = 0\} = \emptyset.$

(b) If $\kappa/2\pi = q/p$, $(p,q) \in \mathbf{N}^2$ and $\gcd(p,q) = 1$, then we have the followings.

- $\{\lambda \in \mathbf{R} \setminus \{0\} | \quad \sin \kappa\sqrt{\lambda} = \sin \tau\sqrt{\lambda} = 0\} = \{p^2j^2 | \quad j \in \mathbf{N}\}.$
- $\{\lambda \in \mathbf{R} \setminus \{0\} | \quad \sin \kappa\sqrt{\lambda} = \cos \tau\sqrt{\lambda} = 0\} = \left\{\frac{p^2j^2}{4} \middle| \quad j \in \mathbf{N}\right\}.$
- $\{\lambda \in \mathbf{R} \setminus \{0\} \quad \sin \tau\sqrt{\lambda} = \cos \kappa\sqrt{\lambda} = 0\} = \begin{cases} \left\{\frac{p^2(2j-1)^2}{16} \middle| \quad j \in \mathbf{N}\right\} & \text{if } \frac{p}{2} \text{ is odd,} \\ \emptyset & \text{otherwise.} \end{cases}$

Put $(p,q) \in \mathbf{N}^2$ and $\gcd(p,q) = 1$. Summarizing lemmas 3.5 and 3.6, we obtain the first statement in Theorem 2.1 and the following lemma.

Lemma 3.7. Assume that $(A_1A_3, A_2A_4) = \pm(E,E)$.
(A) If $a_1b_2 + a_2b_1 \neq 0$, $a_1b_2 + a_4b_3 \neq 0$, we have

$$\mathcal{B} = \begin{cases} \{1\} \cup \{-1\} & \text{if } \frac{\kappa}{\pi} \notin \mathbf{Q}, \\ \{1\} \cup \{-1\} \cup \{p^2j^2 | \quad j \in \mathbf{N}\} & \text{if } \frac{\kappa}{2\pi} = \frac{q}{p}. \end{cases}$$

(B) If $a_1b_2 + a_2b_1 = 0$, then we have

$$\mathcal{B} = \begin{cases} \{1\} \cup \{-1\} & \text{if } \frac{\kappa}{\pi} \notin \mathbf{Q}, \\ \{1\} \cup \{-1\} \cup \left\{ \frac{p^2 j^2}{4} \,\middle|\, j \in \mathbf{N} \right\} & \text{if } \frac{\kappa}{2\pi} = \frac{q}{p}. \end{cases}$$

(C) If $a_1b_2 + a_2b_1 \neq 0$, $a_1b_2 + a_4b_3 = 0$, then we have

$$\mathcal{B} = \begin{cases} \{1\} \cup \{-1\} \cup \{p^2 j^2 |\, j \in \mathbf{N}\} \cup \left\{ \frac{p^2(2j-1)^2}{16} \,\middle|\, j \in \mathbf{N} \right\} & \text{if } \frac{\kappa}{2\pi} = \frac{q}{p}, \frac{p}{2} \text{ is odd}, \\ \{1\} \cup \{-1\} & \text{otherwise.} \end{cases}$$

In order to determine the indices of the degenerate spectral gaps, we make use of the rotation number for H. In this article, we only prove the followings:

- $\lambda = -1$ corresponds to the 2nd degenerate spetral gap.
- $\lambda = 1$ corresponds to the 4th degenerate spectral gap.

We quote the definition and the properties of the rotation number for H from [6]. Let $y(x, \lambda)$ be a non-trivial solution to (1.1) and (1.2). We denote by (r, ω) the polar coordinates of (y', y);

$$y'(x, \lambda) = r(x, \lambda) \cos \omega(x, \lambda) \quad \text{and} \quad y(x, \lambda) = r(x, \lambda) \sin \omega(x, \lambda).$$

We call $\omega(x, \lambda)$ the Prüfer transform of $y(x, \lambda)$. We put $a_j = \cos \alpha_j$, $b_j = \sin \alpha_j$, where $\alpha_j \in (-\pi, 0) \cup (0, \pi)$. The function $\omega(x, \lambda)$ satisfies the following equations:

$$\omega'(x, \lambda) = \cos^2 \omega(x, \lambda) + \lambda \sin^2 \omega(x, \lambda), \quad x \in \mathbf{R} \setminus \Gamma. \tag{3.3}$$

$$\omega(x + 0, \lambda) - \omega(x - 0, \lambda) = \alpha_j, \quad x \in \Gamma_j, \quad j = 1, 2, 3, 4. \tag{3.4}$$

For $\omega_0 \in \mathbf{R}$, let $\omega(x, \lambda, \omega_0)$ be the solution to (3.3) and (3.4) subject to

$$\omega(+0, \lambda) = \omega_0.$$

Definition 3.8. We define the rotation number for H .

$$\rho(\lambda) = \lim_{k \to \infty} \frac{\omega(2k\pi + 0, \lambda, \omega_0) - \omega_0}{2k\pi} \qquad (k \in \mathbf{N}). \tag{3.5}$$

Theorem 3.9 ([6]).

(a) The limit (3.5) exists.

(b) The function $\rho(\lambda)$ does not depend on ω_0.

(c) The function $\rho(\lambda)$ is continuous and non-decreasing on $\mathbf{R}$.

(d) For $j \in \mathbf{N}$, we have

$$\lambda_{2j-2} = \max\left\{\lambda \in \mathbf{R} \middle| \quad \rho(\lambda) = \frac{j-\ell-1}{2}\right\},$$

$$\lambda_{2j-1} = \min\left\{\lambda \in \mathbf{R} \middle| \quad \rho(\lambda) = \frac{j-\ell}{2}\right\},$$

where $\ell =^\sharp \{j \in \{1,2,3,4\} | \quad \alpha_j < 0\}$.

Using this theorem, we prove that $\lambda = 1$ is the 4th spectral gap. For this purpose, we assume that $(A_1A_3, A_2A_4) = \pm(E,E)$.

$$\omega(2\pi k + 0, 1, \omega_0) - \omega_0 = (2\pi + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)k, \qquad k \in \mathbf{N}.$$

Since

$$\begin{aligned}&(\alpha_1+\alpha_3, \alpha_2+\alpha_4)\\ &= \begin{cases}(0,0) & \text{if } (A_1A_3, A_2A_4) = (E,E),\\ (\pi,\pi), (\pi,-\pi), (-\pi,\pi), (-\pi,-\pi) & \text{if } (A_1A_3, A_2A_4) = (-E,-E),\end{cases}\end{aligned}$$

we have

$$\omega(2\pi k + 0, 1, 0) = \begin{cases}4\pi k & \text{if } (\alpha_1+\alpha_3, \alpha_2+\alpha_4) = (\pi,\pi),\\ 0 & \text{if } (\alpha_1+\alpha_3, \alpha_2+\alpha_4) = (-\pi,-\pi),\\ 2\pi k & \text{otherwise.}\end{cases}$$

By the definition of the rotation number, we have

$$\rho(1) = \begin{cases}2 & \text{if } (\alpha_1+\alpha_3, \alpha_2+\alpha_4) = (\pi,\pi),\\ 0 & \text{if } (\alpha_1+\alpha_3, \alpha_2+\alpha_4) = (-\pi,-\pi),\\ 1 & \text{otherwise.}\end{cases}$$

On the other hand, we have

$$\ell = \begin{cases}0 & \text{if } (\alpha_1+\alpha_3, \alpha_2+\alpha_4) = (\pi,\pi),\\ 4 & \text{if } (\alpha_1+\alpha_3, \alpha_2+\alpha_4) = (-\pi,-\pi),\\ 2 & \text{otherwise.}\end{cases} \tag{3.6}$$

This combined with Theorem 3.9 implies that $\lambda = 1$ corresponds to 4th spectral gap.

It follows by $M(-1) = E$ that $\lambda = -1$ corresponds to an even numbered spectral gap. Namely, it turns out that $\lambda = -1$ corresponds to the 2nd spectral gap.

In a similar way, we can prove the other part of Theorem 1.2.

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