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Interval exchange maps and their Diophantine conditions

By

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Abstract

An interval exchange map is a bijection on an interval given by translations on each finite number of subinterval partitions, which is a generalization of the circle rotation that corresponds the interval exchange map with 2-subintervals. Many dynamical properties of rotations of circle depend on Diophantine type of the rotation number. Diophantine condition of irrational rotation numbers have several equivalent arithmetical characterizations by the continued fraction algorithm as well as several equivalent characterizations in terms of the dynamics of the corresponding circle rotations. In this survey, we introduce the continued fraction algorithm for interval exchange maps and investigate how to generalize arithmetic and dynamical Diophantine conditions to interval exchange maps.

§ 1. Introduction

Irrational circle rotations are the prototype of quasiperiodic dynamics and can be generalized as interval exchange maps. An interval exchange map T on an interval I is a bijective map to itself which is a translation on each finite number of subinterval partition of I. The map T is an orientation preserving piecewise isometry and preserves the Lebesgue measure. Let $d \geq 2$ be the number of the subintervals on which T is a translation. If $d = 2$, the interval exchange map T corresponds the rotation of circle.

By the celebrated theorem by Dirichlet we have that a real number θ is irrational if and only if there are infinitely many rationals p/q such that

$$\left| \theta - \frac{p}{q} \right| < \frac{1}{q^2}.$$

An irrational θ is called of *Roth-type* if we cannot replace the denominator of the upper bound q^2 by a bigger exponent $q^{2+\varepsilon}$, i.e., for every $\varepsilon > 0$ there exists a positive constant C_ε such that

$$\left| \theta - \frac{p}{q} \right| \geq \frac{C_\varepsilon}{q^{2+\varepsilon}} \quad \text{for all rationals } \frac{p}{q}.$$

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The set of irrational numbers of Roth type has Lebesgue measure 1 and contains all algebraic irrational numbers. Also it is invariant under the modular group $\mathrm{SL}(2,\mathbb{Z})$.

An irrational θ is called of *bounded type* or badly approximable if there exists a constant $c > 0$ such that

$$\left|\theta - \frac{p}{q}\right| > \frac{c}{q^2} \quad \text{for all rationals } \frac{p}{q}.$$

The set of irrationals of bounded type has Lebesgue measure 0 and contains all quadratic irrational numbers. An irrational θ is of bounded type if and only if its partial quotients a_k (Section 3) of the continued expansion of θ are bounded.

Roth type condition for the irrational θ can also be given in terms of the dynamics of the associated rotation $R_\theta : x \mapsto x + \theta$ on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. One arises by considering the cohomological equation associated to the rotation R_θ (see, e.g., [26]). Another dynamical characterization of Diophantine type rotations is obtained by means of the asymptotic scaling laws of first return times. The Diophantine property of an irrational rotation can also be defined according as how its orbit is distributed. If the rotation is of Roth type, then for all $\varepsilon > 0$ there is a constant C_ε such that the minimum distance between points belonging to a finite segment of an orbit made of n iterates should be bigger than $C_\varepsilon n^{-(1+\varepsilon)}$. Similarly, for the rotation of bounded type, there is a constant c such that the minimum distance between points of n-iterated orbits is bounded below by Cn^{-1}.

The minimality condition for interval exchange maps corresponds irrationality of the circle rotation. A typical interval exchange map is minimal[13]. However, minimality condition for the interval exchange map does not imply unique ergodicity[14, 16]. But still almost every interval exchange map is uniquely ergodic[28, 32] and weakly mixing[2].

The modular group $\mathrm{SL}(2,\mathbb{Z})$ plays an important role for the study of rotation of circle with renormalization scheme associated to the continued fraction algorithm. It was generalized by Rauzy and Veech for interval exchange maps by introducing the induced map on appropriated subintervals[30, 32]. The continued fraction algorithm for interval exchange maps is ergodic on the parameter space of interval exchange maps with respect to an absolutely continuous invariant measure with infinite mass.

Zorich considered an acceleration scheme to produce an ergodic finite invariant measure on the parameter space of the interval exchange maps[37]. For the rotational case ($d = 2$), Zorich's map indeed corresponds the Gauss map which is an acceleration of the Faray map which does not have an absolutely continuous invariant probability measure.

A further acceleration of the Zorich algorithm was studied in [26] by Marmi, Moussa and Yoccoz. They considered a more accelerated algorithm which also preserves an ergodic finite absolutely continues invariant measure in the investigation of the regularity of the solutions of the cohomological equation associated to interval exchange maps. Both the accelerations by Zorich and by Marmi-Moussa-Yoccoz are reduced to the Gauss map for $d = 2$.

In the last ten years, there has been progress in the Diophanitne condition of the i.e.m. (see also [5, 6, 11, 25]) and the Roth type diophantine condition for the i.e.m. has been studied in [19, 20, 27].

The Roth type condition for the irrational rotation can be generalized to the interval exchange map in several different ways. We consider arithmetic characterization using the Roth type growth condition for the Marmi-Moussa-Yoccoz cocycle and the Roth type growth condition for Zorich cocycle. Uniform return time condition and pointwise return time condition are defined in terms of the dynamics of the map in phase space instead of its evolution in parameter space. We also consider Roth type condition for the minimal distance between discontinuities.

For interval exchange maps, bounded condition for the minimal distance between discontinuities was considered to show the unique ergodicity [4, 33]. Let $\Delta(T)$ be the minimum distance between the discontinuity points of T or the end points 0 and 1. If there is a constant $c > 0$ such that

$$\Delta(T^n) > \frac{c}{n} \quad \text{for infinitly many } n\text{'s},$$

then T is uniquely ergodic (See also [3]). In [23], the bounded geodesic interval exchange maps arise from rational polygonal billiards has the full Hausdorff dimension.

The bounded type condition for the irrational rotation can also be generalized to the interval exchange map in several ways. Arithmetic characterization using the bounded growth condition for the Marmi-Moussa-Yoccoz cocycle and the bounded condition for Zorich cocycle are considered. Bounded return time condition is defined in terms of the dynamics of the map in phase space. In a similar way, we also consider bounded condition for the minimal distance between discontinuities.

In Section 2 we introduce the basic definitions and properties of the continued fraction algorithm for the interval exchange map. The generalization of the Diophantine conditions of Roth type and bounded type to interval exchange maps are discussed in detail in Section 3. We consider examples of 3-interval exchange maps and two 4-interval exchange maps in Section 4 and Section 5.

§ 2. Continued fraction algorithms for interval exchange maps

An interval exchange map is determined by the combinatorial data of the permutation and the length data of subintervals. Let $\mathcal{A}$ be a finite set for the name of subintervals. We denote the combinatorial data by two bijections (π_t, π_b) from $\mathcal{A}$ onto $\{1, 2, \ldots, d\}$, which indicate the order of the subintervals before and after the interval exchange map. The length data, denoted by $(\lambda_\alpha)_{\alpha\in\mathcal{A}}$, give the length of the corresponding subintervals.

We set

$$\lambda^* := \sum_{\alpha\in\mathcal{A}} \lambda_A, \quad I := [0, \lambda^*)$$

and

$$p_\alpha := \sum_{\pi_t(\beta)<\pi_t(\alpha)} \lambda_\beta, \quad I_\alpha := [p_\alpha, p_\alpha + \lambda_\alpha), \quad I = \bigsqcup_{\alpha\in\mathcal{A}} I_\alpha.$$

Then the interval exchange map T associated to the combinatorial data (π_t, π_b) and the length data $(\lambda_\alpha)_{\alpha\in\mathcal{A}}$ is a bijective map on I given by

$$T(x) = x + \sum_{\pi_b(\beta)<\pi_b(\alpha)} \lambda_\beta - \sum_{\pi_t(\beta)<\pi_t(\alpha)} \lambda_\beta \quad \text{for } x \in I_\alpha.$$

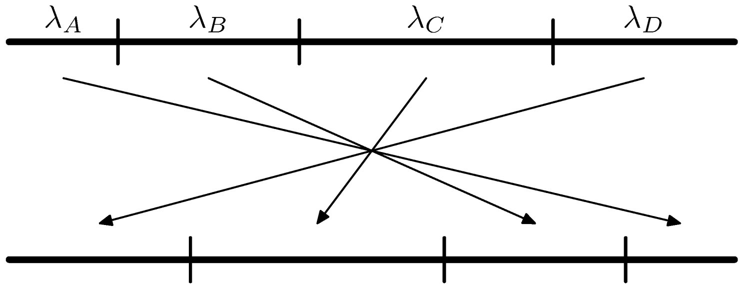


Figure 1: An interval exchange map

Note that T is discontinuous at p_α with $\pi_t(\alpha) > 1$.

We will consider only combinatorial data (π_t, π_b) which are *admissible*, in the sense that for all $k = 1, 2, \ldots, d-1$, we have

$$\pi_t^{-1}(\{1, \ldots, k\}) \neq \pi_b^{-1}(\{1, \ldots, k\}).$$

A picture of 4-interval exchange map ($d = 4$) is presented in Figure 1. Its permutation data is given by

$$\pi_t(A) = 1, \pi_t(B) = 2, \pi_t(C) = 3, \pi_t(D) = 4,$$
$$\pi_b(A) = 4, \pi_b(B) = 3, \pi_b(C) = 2, \pi_b(D) = 1,$$

which we simply denote by $\pi = \left(\begin{smallmatrix} A & B & C & D \\ D & C & B & A \end{smallmatrix}\right)$.

An interval exchange map T is said to have the *Keane property* if there exist no $\alpha, \beta \in \mathcal{A}$ and positive integer m such that $T^m(p_\alpha) = p_\beta$ and $\pi_t(\beta) > 1$. An admissible interval exchange map with rationally independent length data has the Keane property and an interval exchange map with Keane's property is minimal[13]. Thus Keane's property corresponds to the notion of irrationality for interval exchange maps.

For admissible interval exchange maps with the Keane property we can introduce the generalization of continued fraction algorithm to interval exchange maps due to the work of Rauzy [30], Veech [32] and Zorich [37, 38]. We refer to [29, 34, 35, 36, 39] and references therein for the detailed discussions and proofs.

Let (π_t, π_b) be an admissible pair. We define two new admissible pairs $\mathcal{R}_t(\pi_t, \pi_b)$ and $\mathcal{R}_b(\pi_t, \pi_b)$ as follows: let α_t and α_b be the (distinct) elements of $\mathcal{A}$ such that $\pi_t(\alpha_t) = \pi_b(\alpha_b) = d$; one has

$$\mathcal{R}_t(\pi_t, \pi_b) = (\pi_t, \hat{\pi}_b),$$
$$\mathcal{R}_b(\pi_t, \pi_b) = (\hat{\pi}_t, \pi_b),$$

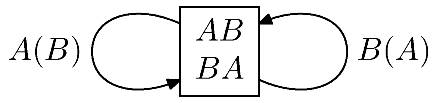


Figure 2: Rauzy diagram for $d = 2$

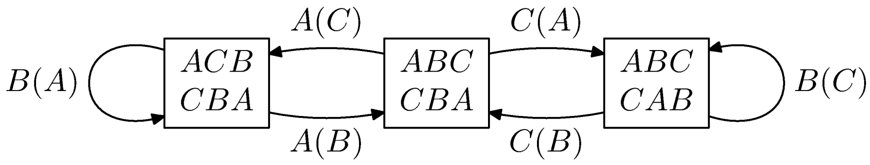


Figure 3: Rauzy diagram for $d = 3$

where

$$\hat{\pi}_b(\alpha) = \begin{cases} \pi_b(\alpha) & \text{if } \pi_b(\alpha) \leq \pi_b(\alpha_t), \\ \pi_b(\alpha) + 1 & \text{if } \pi_b(\alpha_t) < \pi_b(\alpha) < d, \\ \pi_b(\alpha_t) + 1 & \text{if } \alpha = \alpha_b,\ (\pi_b(\alpha_b) = d); \end{cases}$$

$$\hat{\pi}_t(\alpha) = \begin{cases} \pi_t(\alpha) & \text{if } \pi_t(\alpha) \leq \pi_t(\alpha_b), \\ \pi_t(\alpha) + 1 & \text{if } \pi_t(\alpha_b) < \pi_t(\alpha) < d, \\ \pi_t(\alpha_b) + 1 & \text{if } \alpha = \alpha_t,\ (\pi_t(\alpha_t) = d). \end{cases}$$

The admissible pairs $\mathcal{R}_t$ and $\mathcal{R}_b$ corresponds to the permutation data of the induced interval exchange map $\mathcal{V}(T)$ which is defined below (Figure 5).

The *Rauzy class* of (π_t, π_b) is the set of admissible pairs obtained by saturation of (π_t, π_b) under the action of $\mathcal{R}_t$ and $\mathcal{R}_b$. The *Rauzy diagram* has for vertices the elements of the Rauzy class, each vertex (π_t, π_b) being the origin of two arrows joining (π_t, π_b) to $\mathcal{R}_t(\pi_t, \pi_b)$, $\mathcal{R}_b(\pi_t, \pi_b)$. For an arrow joining (π_t, π_b) to $\mathcal{R}_t(\pi_t, \pi_b)$ (respectively $\mathcal{R}_b(\pi_t, \pi_b)$) the element $\alpha_t \in \mathcal{A}$ (respectively $\alpha_b \in \mathcal{A}$) is called the *winner* and the element $\alpha_b \in \mathcal{A}$ (respectively $\alpha_t \in \mathcal{A}$) is called the *loser*. See Figure 2, 3 and 4 for the Rauzy diagrams of 2, 3 and 4-interval exchange maps. Denote an arrow of the Rauzy diagram by $\alpha(\beta)$, where α is the winner and β is the loser of the arrow.

We say that T is of *top type* (respectively *bottom type*) if one has $\lambda_{\alpha_t} > \lambda_{\alpha_b}$ (respectively $\lambda_{\alpha_b} > \lambda_{\alpha_t}$); we then define a new interval exchange map $\mathcal{V}(T)$ by the following data: the admissible pair $\mathcal{R}_t(\pi_t, \pi_b)$ and the lengths $(\hat{\lambda}_\alpha)_{\alpha \in \mathcal{A}}$ given by

$$\begin{cases} \hat{\lambda}_\alpha = \lambda_\alpha & \text{if } \alpha \neq \alpha_t, \\ \hat{\lambda}_{\alpha_t} = \lambda_{\alpha_t} - \lambda_{\alpha_b} & \text{otherwise} \end{cases}$$

for the top type T; the admissible pair $\mathcal{R}_b(\pi_t, \pi_b)$ and the lengths

$$\begin{cases} \hat{\lambda}_\alpha = \lambda_\alpha & \text{if } \alpha \neq \alpha_b, \\ \hat{\lambda}_{\alpha_b} = \lambda_{\alpha_b} - \lambda_{\alpha_t} & \text{otherwise} \end{cases}$$

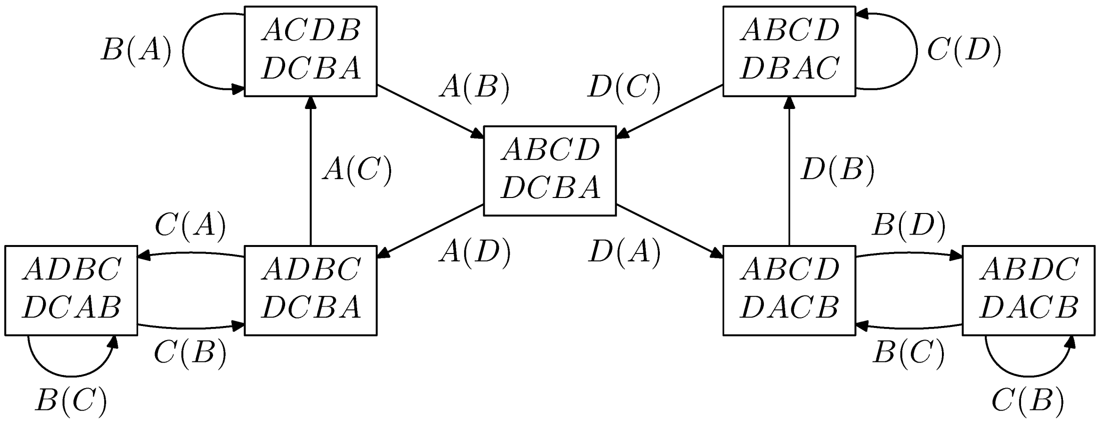
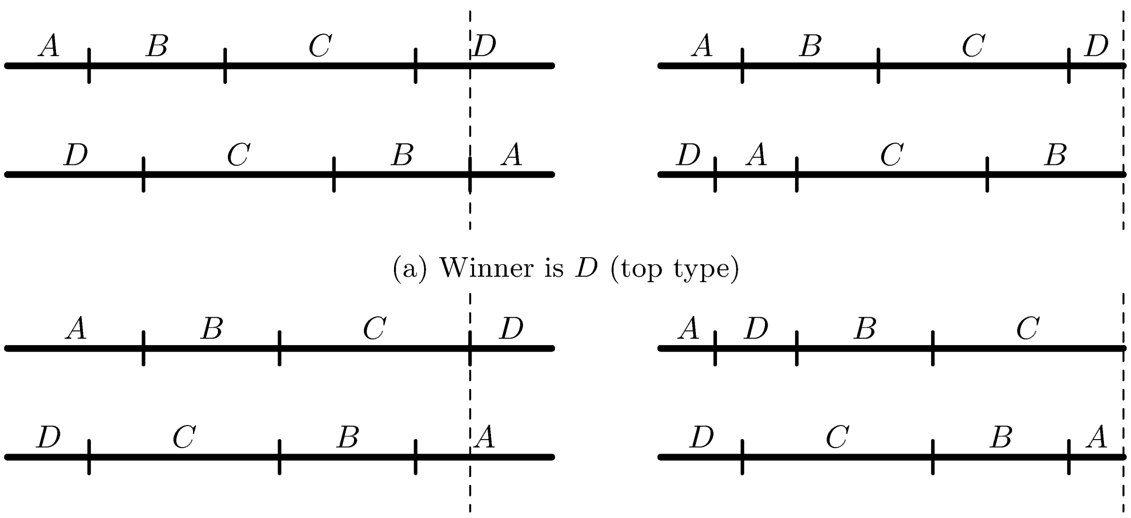


Figure 4: Rauzy diagram for $d = 4$ (first kind)



(a) Winner is D (top type)

(b) Winner is A (bottom type)

Figure 5: Induced transformations of T

for the bottom type T.

The interval exchange map $\mathcal{V}(T)$ is the first return map of T on $\left[0, \sum_\alpha \hat{\lambda}_\alpha\right)$. We also associate to T the arrow in the Rauzy diagram joining (π_t, π_b) to $\mathcal{R}_t(\pi_t, \pi_b)$ or $\mathcal{R}_b(\pi_t, \pi_b)$. The admissible pair $\mathcal{R}_t$ (respectively $\mathcal{R}_b$) is the permutation data of the new interval exchange map $\mathcal{V}(T)$ if T is of top type (respectively bottom type). In Figure 5, we present two possible pictures of $\mathcal{V}(T)$ for the permutation data $\pi = \left(\begin{smallmatrix} A & B & C & D \\ D & C & B & A \end{smallmatrix}\right)$; the permutation data of $\mathcal{V}(T)$ is either $\mathcal{R}_t(\pi) = \left(\begin{smallmatrix} A & B & C & D \\ D & A & C & B \end{smallmatrix}\right)$ or $\mathcal{R}_b(\pi) = \left(\begin{smallmatrix} A & D & B & C \\ D & C & B & A \end{smallmatrix}\right)$. See also Figure 4.

Iterating this process, we obtain a sequence of interval exchange maps $T(n) = \mathcal{V}^n(T)$, $n \geq 0$, and an infinite path in the Rauzy diagram starting from (π_t, π_b). In fact, a further property of irrational interval exchange maps (i.e., with the Keane property) is that every letter in $\mathcal{A}$ is taken as a winner infinitely many times in the infinite path (in the Rauzy diagram) associated to T. This property is fundamental in order to be able to group together several iterations of $\mathcal{V}$ to obtain the accelerated Zorich continued fraction algorithm introduced in [26].

For an arrow γ with winner α and loser β in the Rauzy diagram, let

$$B_\gamma = \mathbb{I} + E_{\beta\alpha},$$

where $\mathbb{I}$ is the identity matrix and $E_{\beta\alpha}$ is the elementary matrix with the only nonzero element at (β, α) which is equal to 1. For a finite path $\underline{\gamma} = (\gamma_1, \dots, \gamma_n)$ in the Rauzy diagram we have a $\mathrm{SL}(\mathbb{Z}^{\mathcal{A}})$ matrix with nonnegative entries

$$B_{\underline{\gamma}} = B_{\gamma_n} \cdots B_{\gamma_1}.$$

Let $\gamma^T(m,n) = \gamma(m,n)$ be the path in the Rauzy diagram from $\pi(m)$ to $\pi(n)$ for $m \le n$ and denote

$$Q(m,n) = B_{\gamma(m,n)} \text{ and } Q(n) = Q(0,n).$$

Let $\lambda(n)$ be the length data of $T(n)$. Then we have

$$\lambda(m) = \lambda(n) Q(m,n). \tag{2.1}$$

Denote $\lambda^*(n) = \sum_{\alpha\in\mathcal{A}} \lambda_\alpha(n)$.

For an irrational rotation, there are two arrows in the Rauzy diagram (Figure 2). For each arrow γ, the corresponding matrix B_γ is either

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \text{ if } A \text{ is winner, or } \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \text{ if } B \text{ is winner.}$$

For a 2-interval exchange map $T = T(0)$, the rotation angle corresponds $\frac{\lambda_B(0)}{\lambda^*(0)}$. If A is winner of the first arrow $\gamma(0,1)$, i.e., $\lambda_B(0)/\lambda^*(0) < 1/2$, then

$$[\lambda_A(0), \lambda_B(0)] = [\lambda_A(1), \lambda_B(1)] \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = [\lambda_A(1) + \lambda_B(1), \lambda_B(1)],$$

$$\frac{\lambda_B(1)}{\lambda^*(1)} = \frac{\lambda_B(0)/\lambda^*(0)}{1 - \lambda_B(0)/\lambda^*(0)}.$$

If B is winner of the first arrow $\gamma(0,1)$, i.e., $\lambda_B(0)/\lambda^*(0) > 1/2$, then

$$[\lambda_A(0), \lambda_B(0)] = [\lambda_A(1), \lambda_B(1)] \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = [\lambda_A(1), \lambda_A(1) + \lambda_B(1)],$$

$$\frac{\lambda_A(1)}{\lambda^*(1)} = \frac{1 - \lambda_B(0)/\lambda^*(0)}{\lambda_B(0)/\lambda^*(0)}.$$

In this case, one considers $\frac{\lambda_A(1)}{\lambda^*(1)}$ as the rotation angel for $T(1)$. Therefore, we have the Farey map for the rotation angel $x \in [0,1)$ given by

$$F(x) = \begin{cases} \dfrac{x}{1-x}, & \text{if } 0 \le x < \dfrac{1}{2}, \\ \dfrac{1-x}{x}, & \text{if } \dfrac{1}{2} \le x < 1. \end{cases}$$

For an irrational number $0 < \theta < 1$, we have a unique continued fraction expansion;

$$\theta = [a_1, a_2, \cdots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}, \qquad a_n \in \mathbb{N} \text{ for all } n \geq 1.$$

Put $p_0 = 0$ and $q_0 = 1$. Choose p_n and q_n for $n \geq 1$ such that $(p_n, q_n) = 1$ and

$$\frac{p_n}{q_n} = [a_1, a_2, \cdots, a_n] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + 1/a_n}}}.$$

We call each a_n the n-th partial quotient and p_n/q_n the n-th principal convergent.

The Farey map acts as

$$F([a_1, a_2, \cdots]) = [a_1 - 1, a_2, \cdots] \text{ for } a_1 > 1 \text{ and } F([1, a_2, \cdots]) = [a_2, a_3, \cdots].$$

Therefore, if the rotation angle has the expansion $\frac{\lambda_B(0)}{\lambda_A(0)+\lambda_B(0)} = [a_1, a_2, \cdots]$ for a 2-interval exchange map, the infinite path in the Rauzy diagram denoted by the winner of each arrow is

$$\underbrace{AAA\cdots A}_{a_1-1}\underbrace{BBB\cdots B}_{a_2}\underbrace{AAA\cdots A}_{a_3}\underbrace{BBB\cdots B}_{a_4}\cdots. \tag{2.2}$$

Note that the Farey map does not have absolutely continuous invariant probability measure.

For $m \leq n$, $T(n)$ is the induced map of $T(m)$ on $I(n) = [0, \lambda^*(n))$; the return time on $I_\beta(n)$ to $I(n)$ under the iteration $T(m)$ is

$$Q_\beta(m, n) := \sum_\alpha Q_{\beta\alpha}(m, n)$$

and the time spent in $I_\alpha(m)$ is $Q_{\beta\alpha}(m, n)$. By (2.1) we have

$$\lambda^* = \sum_{\alpha\beta} \lambda_\beta(n) Q_{\beta\alpha}(n) = \sum_\beta \lambda_\beta(n) Q_\beta(n). \tag{2.3}$$

Moreover, we have

$$[0, \lambda^*) = \bigsqcup_{\alpha \in \mathcal{A}} \left(\bigsqcup_{i=0}^{Q_\alpha(n)-1} T^i(I_\alpha(n)) \right).$$

Zorich's accelerated continued fraction algorithm is obtained by considering $(\mathcal{V}^{n_k})_{k\geq 0}$ where $(n_k)_{k\geq 0}$ is the following sequence: $n_0 = 0$ and $n_{k+1} > n_k$ is chosen so as to assure that $\gamma(n_k, n_{k+1})$ is the longest path whose arrows have the same winner. The Zorich accelerated algorithm $\mathcal{V}^{n_k}$ has an absolutely continuous invariant finite measure on the parameter space of the interval exchange maps. For the irrational rotation case ($d = 2$), the Zorich algorithm corresponds the Gauss map

$$G(x) = \frac{1}{x} \pmod 1.$$

Note that $G(x) = F^{a_1}(x)$ for $x = [a_1, a_2 \dots]$ and $G(x)$ has an absolutely continuous invariant probability measure.

The further acceleration algorithm by Marmi-Moussa-Yoccoz, which was introduced in [26], is obtained by considering $(\mathcal{V}^{m_k})_{k\geq 0}$ where $(m_k)_{k\geq 0}$ is defined as follows: $m_0 = 0$ and $m_{k+1} > m_k$ is the largest integer such that not all letters in $\mathcal{A}$ are taken as winner by arrows in $\gamma(m_k, m_{k+1})$.

Let

Zorich cocycle	$Z(k) = Q(0, n_k)$,	$Z(k,\ell) = Q(n_k, n_\ell)$,
Marmi-Moussa-Yoccoz cocycle	$A(k) = Q(0, m_k)$,	$A(k,\ell) = Q(m_k, m_\ell)$.

For an irrational rotation, the Zorich acceleration and Marmi-Moussa-Yoccoz acceleration are equivalent.

$$Z(1) = A(1) = \begin{bmatrix} 1 & 0 \\ a_1 - 1 & 1 \end{bmatrix}, \quad Z(k-1,k) = A(k-1,k) = \begin{bmatrix} 1 & 0 \\ a_k & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & a_k \\ 0 & 1 \end{bmatrix}$$

and

$$Z(k) = A(k) = \begin{bmatrix} q_{k-1} - p_{k-1} & p_{k-1} \\ q_k - p_k & p_k \end{bmatrix} \text{ or } \begin{bmatrix} q_k - p_k & p_k \\ q_{k-1} - p_{k-1} & p_{k-1} \end{bmatrix},$$

depending on k is odd or even.

The most important virtue of the Marmi-Moussa-Yoccoz cocycle is the following[26, Lemma 1.2.4]: Let $r \geq \max(2d-3, 2)$. Then we have

$$A_{\beta\alpha}(k, k+r) > 0 \text{ for all } \alpha, \beta \in \mathcal{A}.$$

The following inequality follows easily from (2.3):

$$\min_{\alpha\in\mathcal{A}} \lambda_\alpha(n) \leq \frac{\lambda^*}{\|Q(n)\|} \leq \max_{\alpha\in\mathcal{A}} \lambda_\alpha(n), \tag{2.4}$$

where the norm of a matrix B is simply the sum of the absolute values of its entries. This is the norm that we will use for matrices throughout the whole paper. We assume that $\lambda^* = 1$ unless it is specified.

§ 3. Diophantine conditions for interval exchange maps

It is well known that n-th principal convergents $(p_n/q_n)_{n\in\mathbb{N}}$ of the continued fraction expansion of an irrational θ are the best approximations in the sense that $|q_n\theta - p_n| < |q\theta - p|$ for any $1 \leq q < q_n$ and p. Therefore, an irrational θ is of Roth-type if for every $\varepsilon > 0$ there exists a positive constant C_ε such that

$$\|q_n\theta\| \geq \frac{C_\varepsilon}{q_n^{1+\varepsilon}}.$$

An irrational θ is of bounded type if there exists a constant $c > 0$ such that

$$\|q_n\theta\| > \frac{c}{q_n}.$$

Using the relations (see e.g., [17])

$$\frac{1}{q_n + q_{n+1}} < \|q_n\theta\| < \frac{1}{q_{n+1}} \quad \text{and} \quad q_{n+1} = a_{n+1}q_n + q_{n-1},$$

we characterize Diophantine conditions of irrationals by arithmetic conditions for $(q_n)_{n\in\mathbb{N}}$ and $(a_n)_{n\in\mathbb{N}}$. Roth type irrationals can also be determined by the growth conditions of $(q_n)_{n\in\mathbb{N}}$: for all $\varepsilon > 0$ there is a constant C_ε such that

$$q_{n+1} < C_\varepsilon q_n^{1+\varepsilon}.$$

This condition is also equivalent to the growth condition of the partial quotients $(a_n)_{n\in\mathbb{N}}$: for all $\varepsilon > 0$ there is a constant C_ε such that

$$a_{n+1} < C_\varepsilon q_n^{\varepsilon}.$$

The bounded type irrationals can also be determined by partial quotients $(a_n)_{n\in\mathbb{N}}$: an irrational θ is of bounded type if and only if $(a_n)_{n\in\mathbb{N}}$ are bounded.

Diophantine condition of the irrational rotation can be also characterized by the recurrence time. An irrational circle rotation T is of bounded type if and only if there is a constant $c > 0$ such that

$$\liminf_{n\to\infty} n \cdot \|n\theta\| = \liminf_{n\to\infty} n \cdot |T^n(x) - x| > c,$$

where $\|\cdot\|$ is the distance to its nearest integer. Roth type condition can be determined by the asymptotic rate of the return time. Let $r > 0$ and let $\tau_r(x)$ be the return time to r-neighborhood of x

$$\tau_r(x) = \min\{j \geq 1 : |T^j x - x| < r\}. \tag{3.1}$$

Then, for an irrational circle rotation the rotation number is of Roth type if and only if

$$\lim_{r\to 0^+} \frac{\log \tau_r(x)}{-\log r} = 1.$$

The notion of Roth type of an interval exchange map was introduced in [26]: this is a natural extension of Roth type irrational circle rotations and Roth type interval exchange maps form a full measure set in the parameter space of interval exchange maps. In [20] it was proved that for Roth type interval exchange maps the recurrence time has the same scaling behaviour as for irrational rotations, namely

$$\lim_{r\to 0^+} \frac{\log \tau_r(x)}{-\log r} = 1, \text{ a.e. } x.$$

If one considers the dynamics in parameter space of interval exchange maps one can introduce three slightly different Roth type Diophantine conditions:

(R-A) Roth type growth condition for the Marmi-Moussa-Yoccoz cocycle :
For any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for all $k \geq 1$ we have

$$\|A(k, k+1)\| \leq C_\varepsilon \|A(k)\|^\varepsilon.$$

(R-Z) Roth type growth condition for the Zorich cocycle :
For any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for all $k \geq 1$ we have

$$\|Z(k, k+1)\| \leq C_\varepsilon \|Z(k)\|^\varepsilon.$$

By the definitions, the Marmi-Moussa-Yoccoz cocycle is always bigger than the Zorich cycocle, thus it follows immediately that Condition (R-A) implies Condition (R-Z).

(R-D) Roth type condition for the minimal distance between discontinuities :
For any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for all $n \geq 1$ we have

$$\Delta(T^n) \geq \frac{C_\varepsilon}{n^{1+\varepsilon}}.$$

If one considers the dynamics of an interval exchange map in phase space then one can introduce two slightly different Diophantine conditions:

(R-R) Pointwise return time condition :

$$\lim_{r \to 0} \frac{\log \tau_r(x)}{-\log r} = 1 \qquad \text{for almost every } x.$$

(R-U) Uniform return time condition :

$$\lim_{r \to 0} \frac{\log \tau_r(x)}{-\log r} = 1 \qquad \text{uniformly.}$$

Here $\tau_r(x)$ is the first return time to r-neighborhood of x defined in (3.1).

Here and in what follows the matrix norm denoted by $\|Q\| = \sum_{\alpha\beta} |Q_{\alpha\beta}|$. Clearly, by the definition, Condition (R-U) implies Condition (R-R).

In the case of circle rotations (interval exchange map with $d = 2$) the three conditions in parameter space (namely (R-A), (R-Z) and (R-D)) are equivalent, as well as the two conditions in phase space ((R-R) and (R-U)). For an irrational rotation, we have

$$\|Z(k, k+1)\| = \|A(k, k+1)\| = a_{k+1} + 2, \qquad \|Z(k)\| = \|A(k)\| = q_k + q_{k-1}$$

and Condition (R-A) and (R-D) are equivalent to the statement that for any $\varepsilon > 0$ there is a positive constant C_ε such that $a_{k+1} \leq C_\varepsilon q_k^\varepsilon$, which is just the Roth type condition for the irrational rotation number. In [8] the equivalence for circle rotations between the two sets of conditions (Roth type in parameter space and the return time characterization) was proved.

In [26], It is shown that almost every interval exchange map satisfies Condition (R-A) with respect to the Lebesgue measure in the length data, but, there exists an interval exchange map with Condition (R-A) which is not uniquely ergodic.

The minimal distance Diophantine condition is related to the recurrence condition in the following sense: If $\tau_r(x) = n$, which implies that $|T^n(x) - x| < r$, then we have $\Delta(T^{2n}) < r$. The uniform recurrence Diophantine condition (R-U) is implied by the Diophantine condition of the minimal distance between discontinuities (R-D).

Theorem 3.1 ([20]). *For general interval exchange maps, Condition (R-A) implies Condition (R-R).*

Theorem 3.2 ([27], Proposition C.1 and [19], Section 4). *An interval exchange map T satisfies Condition (R-A) if and only if it also satisfies Condition (R-D).*

Theorem 3.3 ([19]). *Let T be a interval exchange map.*
(i) If T satisfies Condition (R-D), then so do Condition (R-U),
(ii) If T satisfies Condition (R-U), then so do Condition (R-Z).

The example with Condition (R-U) without Condition (R-D) can be constructed in 3-interval exchange maps in Section 4. The example with Condition (R-R) without Condition (R-Z) and the example with Condition (R-Z) without Condition (R-U) are presented in Section 5.

As a similar way with the recurrence time one may consider the hitting time, i.e., starting from a point, how many iterate T to enter the neighborhood of the target point. The hitting time condition for interval exchange maps was considered in [12, 20].

One can introduce three slightly different Diophantine conditions of bounded type for interval exchange maps:

(B-A) Bounded condition for the Marmi-Moussa-Yocciz cocycle:
There is $M > 0$ such that

$$\|A(k, k+1)\| \leq M.$$

(B-Z) Bounded condition for the Zorich cocycle:
There is $M > 0$ such that

$$\|Z(k, k+1)\| \leq M.$$

(B-D) Bounded condition for the minimal distance between discontinuities:
There is a constant $c > 0$ such that

$$\Delta(T^n) \geq \frac{c}{n} \quad \text{for all } n.$$

(B-U) Bounded recurrence condition:
There is a constant $c > 0$ such that for all x

$$\liminf_{n\to\infty} n \cdot |T^n(x) - x| \geq c.$$

From the definition it is immediately followed that Condition (B-A) implies Condition (B-Z). For circle rotations (interval exchange map with $d = 2$) the three conditions in parameter space (namely (B-A), (B-Z) and (B-D)) are equivalent, as well as the condition in phase space (B-U).

Theorem 3.4 ([11], [21]). *An interval exchange map T is of bounded type (B-A) if and only if T is of bounded type (B-D).*

Theorem 3.5 ([21]). *(i) The bounded type condition (B-D) implies Condition (B-U). (ii) The bounded type condition (B-U) implies Condition (B-Z).*

The example with Condition (B-U) without Condition (B-D) can be constructed in 3-interval exchange maps in Section 4. The example with Condition (B-Z) without Condition (B-U) is presented in Subsection 5.2.

By the ergodicity of the continued fraction algorithm, the set of interval exchange maps with bounded type condition (B-A) has measure zero with respect to the Lebesgue measure in length data. However, the set of such a bounded type intervals exchange map has full Hausdorff dimension.

Kurzweil[24] showed that, if and only if the irrational θ is of bounded type, then for almost every s and a monotone decreasing positive function ψ with $\sum \psi(n) = \infty$,

$$\|n\theta - s\| < \psi(n) \quad \text{for infinitely many } n \in \mathbb{N}$$

holds. This property for the bounded type irrational is generalized as follows:

Theorem 3.6 ([7], Theorem 17). *If an interval exchange T is of bounded type (B-D), then for all monotone decreasing ψ with $\sum \psi(n) = \infty$,*

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} 1_{B(y,\psi(n))}(T^n x)}{\sum_{n=1}^{N} \psi(n)} = 1$$

for almost every y.

Since Artin's work[1], the connection between the geodesic flow and continued fractions has been studied. See [9] for a general reference. Let $\mathbb{H} = \{x + iy \in \mathbb{C} \,|\, y > 0\}$ be the upper half plane with the hyperbolic Riemannian metric $\langle u, v\rangle = 1/y^2 \cdot (u, v)$. The geodesic flow in the modular surface $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})$ is bounded if the limit of the geodesic in $\partial\mathbb{H}$ is of bounded type. See [31] for the detail. For a given length data (λ_α) and suspension data (τ_α), we define the Teichmüller geodesic flow for $t \in \mathbb{R}$

$$U^t(\lambda, \tau) = (e^{t/2}\lambda, e^{-t/2}\tau).$$

The flow commutes with the basic operation of the continued fraction algorithm. For the detail refer to [34, 35, 36, 39]. A bounded Teichmüller geodesic flow, which means all saddle connections have length at least $c > 0$, implies Condition (B-D).

§ 4. 3-interval exchange maps

Let T be a 3-interval exchange map with length data $(\lambda_A, \lambda_B, \lambda_C)$. We may assume that $\pi_t(A) = 1, \pi_t(B) = 2, \pi_t(C) = 3$ and $\pi_b(C) = 3, \pi_b(B) = 2, \pi_b(A) = 1$. Let $\lambda^* = \lambda_A + \lambda_B + \lambda_C = 1$.

Define an irrational rotation $\bar{T}$ on $\bar{I} = [0, \lambda^* + \lambda_B)$ by

$$\bar{T}(x) = \begin{cases} x + \lambda_B + \lambda_C, & \text{if } x + \lambda_B + \lambda_C \in \bar{I}, \\ x + \lambda_B + \lambda_C - (\lambda^* + \lambda_B), & \text{if } x + \lambda_B + \lambda_C \notin \bar{I}. \end{cases} \tag{4.1}$$

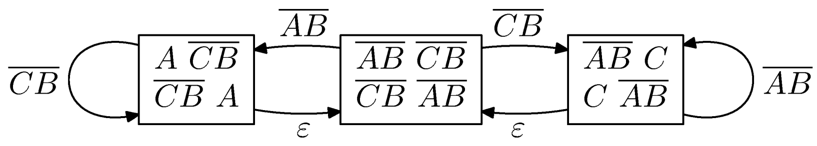


Figure 6: induced Rauzy diagram for 3-interval exchange map

Then $\bar{T}$ is a 2-interval exchange map (irrational rotation) with length data $(\lambda_{\bar{A}}, \lambda_{\bar{C}})$, where $\lambda_{\bar{A}} = \lambda_A + \lambda_B$ and $\lambda_{\bar{C}} = \lambda_A + \lambda_C$. Note that T is the induced map of $\bar{T}$ on $[0, \lambda^*)$ and T satisfies the Keane property if and only if the rotation $\bar{T}$ is irrational.

Let $\alpha = \frac{\lambda_B + \lambda_C}{\lambda^* + \lambda_B}$ be the rotation angle of $\bar{T}$ and let a_k and p_k/q_k be the partial quotients and partial convergents of α. The Diophantine condition of T is related to the Diophantine condition of α.

Lemma 4.1 (Denjoy-Koksma inequality (see [10])). *Let $\bar{T}$ be an irrational rotation by α with partial quotient denominators q_k and f be a real valued function of bounded variation on the unit interval. Then for any x we have*

$$\left| \sum_{i=0}^{q_k - 1} f(\bar{T}^i x) - q_k \int f d\mu \right| < \mathrm{var}(f).$$

For a sufficiently small r the recurrence time of an irrational rotation to r-neighborhood should be q_k. Choose $f(x) = 1_I(x)$ as the indicator function on I. Then $\mathrm{var}(f) = 2$. Let $\bar{\tau}_r$ be the first return time of $\bar{T}$. By the Denjoy-Koksma inequality, for a small $r > 0$, $\bar{\tau}_r(x) = q_k$ for some q_k

$$\left| \tau_r(x) - \bar{\tau}_r(x) \cdot \frac{\lambda^*}{\lambda^* + \lambda_B} \right| < 2.$$

For any $x \in [0, \lambda^*)$ we have

$$\lim_{r \to 0^+} \frac{\log \tau_r(x)}{-\log r} = 1 \text{ if and only if } \lim_{r \to 0^+} \frac{\log \bar{\tau}_r(x)}{-\log r} = 1.$$

Theorem 4.2 ([19]). *A 3-interval exchange map T satisfies Condition (R-U) (or (R-R)) if and only if α is of Roth type.*

By the similar way we have the followings:

Theorem 4.3 ([21]). *A 3-interval exchange map T satisfies Condition (B-U) (or (B-R)) if and only if α is of bounded type.*

Now we compare the Rauzy-Veech induction algorithm for T and $\bar{T}$. There are 6 arrows in the Rauzy diagram for a 3-interval exchange map T (see Figure 3). Each arrow in the Rauzy diagram for $\bar{T}$ corresponds to two arrows of the same loser in the Rauzy diagram for T and remaining 2 arrows of the loser B are not be mapped to any arrows in the Rauzy diagram for $\bar{T}$ (Figure 6).

For a given 3-interval exchange map T define

$$\ell(n) := \#\left\{1 \le m \le n \mid \pi^{(m)} = \left(\begin{smallmatrix} A & C & B \\ C & B & A\end{smallmatrix}\right) \text{ or } \left(\begin{smallmatrix} A & B & C \\ C & A & B\end{smallmatrix}\right)\right\}.$$

By the mapping $\alpha(A) \mapsto \bar{C}(\bar{A})$, $\alpha(C) \mapsto \bar{A}(\bar{C})$ and $\alpha(B) \mapsto \varepsilon$, where ε is the empty arrow, the infinite sequence of arrows in the Rauzy diagram for T is mapped to the infinite sequence of arrows in the Rauzy diagram for $\bar{T}$. Denote by $\bar{Q}(m) = B_{\gamma^{\bar{T}}(0,m)}$ the continued fraction matrix for $\bar{T}$. Then for $n \ge 0$

$$\bar{Q}(\ell(n)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} Q(n) \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and}$$

$$\bar{\lambda}(\ell(n)) = \lambda(n) \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \lambda(n) \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad \lambda(n) \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

for $\pi^{(n)} = \left(\begin{smallmatrix} A & B & C \\ C & B & A\end{smallmatrix}\right)$, $\left(\begin{smallmatrix} A & C & B \\ C & B & A\end{smallmatrix}\right)$, $\left(\begin{smallmatrix} A & B & C \\ C & A & B\end{smallmatrix}\right)$, respectively.

Therefore, the size of the Zorich cocycle for T is about the same with the size of the Zorich cocycle for $\bar{T}$, which is about the same size of the partial quotients of the rotation angle α of $\bar{T}$.

Theorem 4.4. *(i) The 3-interval exchange map T satisfies Condition (R-Z) if and only if the irrational rotation α is of Roth type.*
(ii) The 3-interval exchange map T satisfies Condition (B-Z) if and only if the irrational rotation α is of bounded type.

We may construct 3-interval exchange maps satisfying Condition (B-U) (thus, (R-U) is satisfied) but not satisfying Condition (R-A) (thus, (B-A) is not satisfied). Consider a 3-interval exchange map with Rauzy-Veech induction algorithm in which there is a very long sequence of $A(C)B(A)A(B)$ followed by a much longer sequence of $C(A)B(C)C(B)$ and so on. Then the irrational rotation $\bar{T}$, which induces T, has the partial quotients $a_k = 1$. Thus, $\bar{T}$ is of bounded type, which is followed by the 3-interval exchange map T satisfies (B-U). However, if the sequences of $A(C)B(A)A(B)$ and $C(A)B(C)C(B)$ are chosen long enough that the corresponding the Marmi-Moussa-Yoccoz cocycle $A(k, k+1)$ is about the same size of $A(k)$, then T does not satisfy Condition (R-A).

§ 5. Examples

§ 5.1. Example with Condition (R-R) without Condition (R-Z)

In this subsection, we discuss an example of 4-interval exchange map which satisfies Condition (R-R) but not Condition (R-Z).

Let T be a 4-interval exchange map with the permutation data $\pi^{(0)} = \left(\begin{smallmatrix} A & B & D & C \\ D & A & C & B\end{smallmatrix}\right)$. Assume that the length data of T is determined by the infinite path in the Rauzy diagram, denoted by the winner of each arrow (see Figure 4)

$$C^{s_1} B \left(D^2 A^3 D\right)^{2+1} B \cdot C^{s_2} B \left(D^2 A^3 D\right)^{2^2+2} B \cdots C^{s_k} B \left(D^2 A^3 D\right)^{2^k+k} B \cdots .$$

Let

$$\ell_k = \sum_{i=1}^{k} (s_i + 6 \cdot 2^i + i + 2), \ \ell_0 = 0, \quad \text{and} \quad s_k = F_{2^{k+1}}.$$

The matrix associated to the path $C^{s_k} B \left(D^2 A^3 D\right)^{2^k+k} B$ is

$$Q(\ell_{k-1}, \ell_k) = \begin{bmatrix} F_{2^{k+1}+2k+1} & 0 & 0 & F_{2^{k+1}+2k} \\ F_{2^{k+1}+2k+1}-1 & 1 & F_{2^{k+1}} & F_{2^{k+1}+2k} \\ F_{2^{k+1}+2k+1}-1 & 1 & F_{2^{k+1}}+1 & F_{2^{k+1}+2k} \\ F_{2^{k+1}+2k+2}-1 & 1 & F_{2^{k+1}} & F_{2^{k+1}+2k+1} \end{bmatrix},$$

where F_n is the Fibonacci sequence: $F_{-1} = 1, F_0 = 0, F_{n+1} = F_n + F_{n-1}$. Note that $F_n = \frac{1}{\sqrt{5}}(g^n - (-g)^{-n})$, $g = \frac{\sqrt{5}+1}{2}$.

Since

$$Q(\ell_k, \ell_k + s_{k+1}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & s_{k+1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

for large k

$$\|Q(\ell_k)\|^{1/2} < \|Q(\ell_k, \ell_k + s_{k+1})\|,$$

which implies that this interval exchange map T does not satisfy Condition (R-Z).

Since $\lambda(\ell_{k+1}) Q(\ell_k, \ell_{k+1}) = \lambda(\ell_k)$, the length data of $T(\ell_k)$, $\lambda(\ell_k)$ is a vector in the simplex with the vertices

$$\begin{aligned}
&\lambda^*(\ell_{k+1})\left[F_{2^{k+2}+2k+3}, 0, 0, F_{2^{k+2}+2k+2}\right], \\
&\lambda^*(\ell_{k+1})\left[F_{2^{k+2}+2k+3}-1, 1, F_{2^{k+2}}, F_{2^{k+2}+2k+2}\right], \\
&\lambda^*(\ell_{k+1})\left[F_{2^{k+2}+2k+3}-1, 1, F_{2^{k+2}}+1, F_{2^{k+2}+2k+2}\right], \\
&\lambda^*(\ell_{k+1})\left[F_{2^{k+2}+2k+4}-1, 1, F_{2^{k+2}}, F_{2^{k+2}+2k+3}\right].
\end{aligned}$$

By the permutation data $\pi^{(\ell_k)} = \left(\begin{smallmatrix} A & B & D & C \\ D & A & C & B \end{smallmatrix}\right)$ we have

$$T(\ell_k)(x) = \begin{cases} x + \lambda_D(\ell_k) & \text{for } x \in I_A(\ell_k), \\ x - (\lambda_A(\ell_k) + \lambda_B(\ell_k)) & \text{for } x \in I_D(\ell_k), \\ x - \lambda_B(\ell_k) & \text{for } x \in I_C(\ell_k). \end{cases}$$

Let $\tilde{T}_k$ be the 2-interval exchange map on $[0, \lambda_A(\ell_k)+\lambda_B(\ell_k)+\lambda_D(\ell_k)) = [0, \lambda^*(\ell_k+s_{k+1}+1))$ with $\lambda_{\tilde{A}}(\ell_k) = \lambda_A(\ell_k) + \lambda_B(\ell_k)$ and $\lambda_{\tilde{D}}(\ell_k) = \lambda_D(\ell_k)$. Then

$$T(\ell_k)(x) = T(\ell_k + s_{k+1} + 1)(x) = \tilde{T}_k(x) \text{ on } x \in I_A(\ell_k) \cup I_D(\ell_k).$$

If

$$x \in (I_A(\ell_k) \cup I_D(\ell_k)) \setminus \left(\bigcup_{i=0}^{m} T(\ell_k)^{-i} I_B(\ell_k) \right),$$

then we have

$$T(\ell_k)^i(x) = \tilde{T}_k^i(x), \text{ for } 0 \le i < m.$$

Note that

$$\left| \frac{\lambda_{\tilde{A}}(\ell_k)}{\lambda_{\tilde{A}}(\ell_k) + \lambda_{\tilde{D}}(\ell_k)} - \frac{1}{g} \right| < \frac{1}{g^{2^{k+3}+4k+6}}.$$

Let $R_k(x)$ be the irrational rotation by $\frac{\lambda^*(\ell_k+s_{k+1}+1)}{g}$ on $[0, \lambda^*(\ell_k + s_{k+1} + 1))$. Then for each $x \in [0, \lambda^*(\ell_k + s_{k+1} + 1))$

$$\left| \tilde{T}_k^i(x) - R_k^i(x) \right| < \frac{i\lambda^*(\ell_k + s_{k+1} + 1)}{g^{2^{k+3}+4k+6}}.$$

By the Hurwitz theorem of Diophantine approximation we have for each $x \in [0, \lambda^*(\ell_k + s_{k+1} + 1))$

$$\left| R_k^i(x) - x \right| > \frac{\lambda^*(\ell_k + s_{k+1} + 1)}{2i}.$$

Hence, T satisfies Condition (R-R):

$$\lim_{r \to 0^+} \frac{\log \tau_r(x)}{-\log r} = 1 \text{ for almost every } x.$$

§ 5.2. Example with Condition (B-Z) without Condition (R-U)

In this subsection, we discuss an example of 4-interval exchange map satisfying Condition (B-Z) (thus, Condition (R-Z) is also satisfied) but not Condition (R-U) (thus, (B-U) is not satisfied).

Let T be the interval exchange map with the permutation data $\pi^{(0)} = \left(\begin{smallmatrix} A & B & D & C \\ D & A & C & B \end{smallmatrix}\right)$ and the infinite path in the Rauzy diagram denoted by the winner of each arrow

$$CB^3 \left(D^2 A^3 D\right)^{2^1} B \cdot CB^3 \left(D^2 A^3 D\right)^{2^2} B \cdots CB^3 \left(D^2 A^3 D\right)^{2^k} B \cdots .$$

Then there is no path of more than 3 arrows of the same winner. Thus, T satisfies Condition (B-Z).

Let

$$\ell_k = \sum_{i=1}^{k} \left(5 + 6 \cdot 2^i\right) = 5k + 12 \cdot (2^k - 1), \quad \ell_0 = 0.$$

Then $\gamma(\ell_{k-1}, \ell_k)$ is $CB^3 \left(D^2 A^3 D\right)^{2^k} B$ and

$$Q(\ell_{k-1}, \ell_k) = \begin{bmatrix} F_{2^{k+1}+1} & F_{2^{k+1}} & F_{2^{k+1}} & F_{2^{k+1}} \\ F_{2^{k+1}+1} - 1 & F_{2^{k+1}} + 1 & F_{2^{k+1}} + 1 & F_{2^{k+1}} \\ F_{2^{k+1}+1} - 1 & F_{2^{k+1}} + 2 & F_{2^{k+1}} + 3 & F_{2^{k+1}} \\ F_{2^{k+1}+2} - 1 & F_{2^{k+1}+1} + 1 & F_{2^{k+1}+1} + 1 & F_{2^{k+1}+1} \end{bmatrix},$$

where F_n is the Fibonacci sequence as before. Note $T(\ell_k + 3)$ has the same permutation data with $T(\ell_k)$, $\pi^{(\ell_k+3)} = \left(\begin{smallmatrix} A & B & D & C \\ D & A & C & B \end{smallmatrix}\right)$. The matrix for the path $B \left(D^2 A^3 D\right)^{2^{k+1}} B$ starting from

$\left(\begin{smallmatrix} A & B & D & C \\ D & A & C & B \end{smallmatrix}\right)$ is

$$Q(\ell_k+3,\ell_{k+1}) = \begin{bmatrix} F_{2^{k+2}+1} & 0\,0 & F_{2^{k+2}} \\ F_{2^{k+2}+1}-1 & 1\,0 & F_{2^{k+2}} \\ F_{2^{k+2}+1}-1 & 1\,1 & F_{2^{k+2}} \\ F_{2^{k+2}+2}-1 & 1\,0 & F_{2^{k+2}+1} \end{bmatrix}.$$

Since $\lambda(\ell_{k+1})Q(\ell_k+3,\ell_{k+1}) = \lambda(\ell_k+3)$, the length data $\lambda(\ell_k+3)$ is a vector in the simplex with the vertices

$$\lambda^*(\ell_{k+1})\left[F_{2^{k+2}+1},0,0,F_{2^{k+2}}\right], \quad \lambda^*(\ell_{k+1})\left[F_{2^{k+2}+1}-1,1,0,F_{2^{k+2}}\right],$$

$$\lambda^*(\ell_{k+1})\left[F_{2^{k+2}+1}-1,1,1,F_{2^{k+2}}\right], \quad \lambda^*(\ell_{k+1})\left[F_{2^{k+2}+2}-1,1,0,F_{2^{k+2}+1}\right].$$

Therefore for all $x \in I_C(\ell_k+3)$ we have

$$|T(\ell_k+3)(x) - x| = \lambda_B(\ell_k+3) < \lambda^*(\ell_{k+1}).$$

Put $r = \lambda_B(\ell_k+3)$. Then if $k \geq 4$, we have for $x \in I_C(\ell_k+3)$,

$$\frac{\log \tau_r(x)}{-\log r} < \frac{\log Q_C(\ell_k+3)}{-\log \lambda_B(\ell_k+3)} < \frac{3}{4}.$$

Hence, the asymptotic recurrence rate $\dfrac{\log \tau_r(x)}{-\log r}$ does not converges to 1 uniformly. This example does not satisfy Condition (R-U).

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